Mehanički rad i virtualni pomaci

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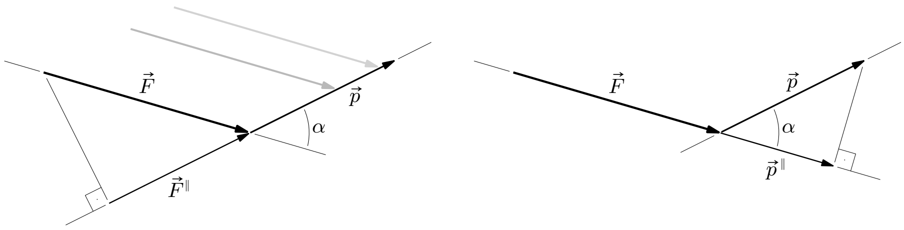
mehanički rad:

poseban slučaj: rad nepromjenjive sile na pravocrtnom putu



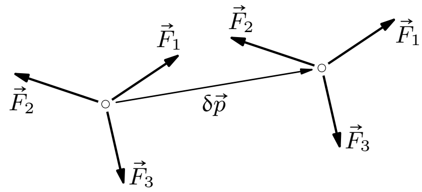
$$W \;=\; \vec{F}\cdot\vec{p} \;=\; \|\vec{F}\|\,\|\vec{p}\|\cos\alpha \;=\; |F|\,|p|\cos\alpha \;=\; F^{\parallel}\,p \;=\; F\,p^{\parallel}$$

$$W \;=\; \vec{F}\cdot\vec{p} \;=\; F_x\,p_x \;+\; F_y\,p_y \;+\; F_z\,p_z$$

drugi poseban slučaj: rad nepromjenivoga momenta na zaokretu oko osi koja se poklapa s pravcem vektora momenta

$$W \;=\; M\cdot\varphi$$

teorem o virtualnom radu za materijalnu točku:



1. *ako je materijalna točka, na koju djeluju sile, u ravnoteži, onda je zbroj radova tih sila na bilo kojemu virtualnom pomaku točke jednak nuli*:

$$\sum\nolimits_i \vec{F}_i \;=\; \vec{0} \qquad\Longrightarrow\qquad \delta W \;=\; 0 \quad \forall\delta\vec{p}$$

„dokaz”: $\delta W \;=\; \sum_i \delta W_i \;=\; \sum_i \left(\vec{F}_i\cdot\delta\vec{p}\right) \;=\; \left(\sum_i \vec{F}_i\right)\cdot\delta\vec{p} \;=\; \vec{0}\cdot\delta\vec{p} \;=\; 0$

2. i obratno, *ako je zbroj radova sila, koje djeluju na materijalnu točku, na bilo kojemu virtualnom pomaku te točke jednak nuli, onda je točka u ravnoteži*:

$$\delta W \;=\; 0 \quad \forall\delta\vec{p} \qquad\Longrightarrow\qquad \sum\nolimits_i \vec{F}_i \;=\; \vec{0}$$

dokaz:

$$\begin{aligned} 0 \;=\; \delta W \;&=\; \sum\nolimits_i \left(\vec{F}_i\cdot\delta\vec{p}\right) \;=\; \sum\nolimits_i \left(F_{i,x}\,\delta p_x \;+\; F_{i,y}\,\delta p_y \;+\; F_{i,z}\,\delta p_z\right) \\ &=\; \left(\sum\nolimits_i F_{i,x}\right)\delta p_x \;+\; \left(\sum\nolimits_i F_{i,y}\right)\delta p_y \;+\; \left(\sum\nolimits_i F_{i,z}\right)\delta p_z \end{aligned}$$

(1) $f(x)+g(y)+h(z) \;=\; 0$ *&* f, g, h nezavisne $\quad\Rightarrow\quad f=0$ *&* $g=0$ *&* $h=0$

(2) $ax \;=\; 0 \quad \forall x \quad\Rightarrow\quad a \;=\; 0,$ tako da

$$\left.\begin{aligned} \left(\sum\nolimits_i F_{i,x}\right)\delta p_x \;=\; 0 \quad \forall\delta p_x \quad&\Rightarrow\quad \sum\nolimits_i F_{i,x} \;=\; 0 \\ \left(\sum\nolimits_i F_{i,y}\right)\delta p_y \;=\; 0 \quad \forall\delta p_y \quad&\Rightarrow\quad \sum\nolimits_i F_{i,y} \;=\; 0 \\ \left(\sum\nolimits_i F_{i,z}\right)\delta p_z \;=\; 0 \quad \forall\delta p_z \quad&\Rightarrow\quad \sum\nolimits_i F_{i,z} \;=\; 0 \end{aligned}\right\} \qquad \sum\nolimits_i \vec{F}_i \;=\; \vec{0}$$

teorem o virtualnom radu za kruto tijelo:

1. *ako je kruto tijelo, na koje djeluju sile i momenti, u ravnoteži, onda je zbroj radova tih sila i momenata na bilo kojim virtualnim pomacima i zaokretima tijela jednak nuli*:

$$\sum_i \vec{F}_i = \vec{0} \quad \& \quad \sum_j \overrightarrow{M}_j = \vec{0} \qquad \Longrightarrow \qquad \delta W = 0 \quad \forall \delta\vec{p}$$

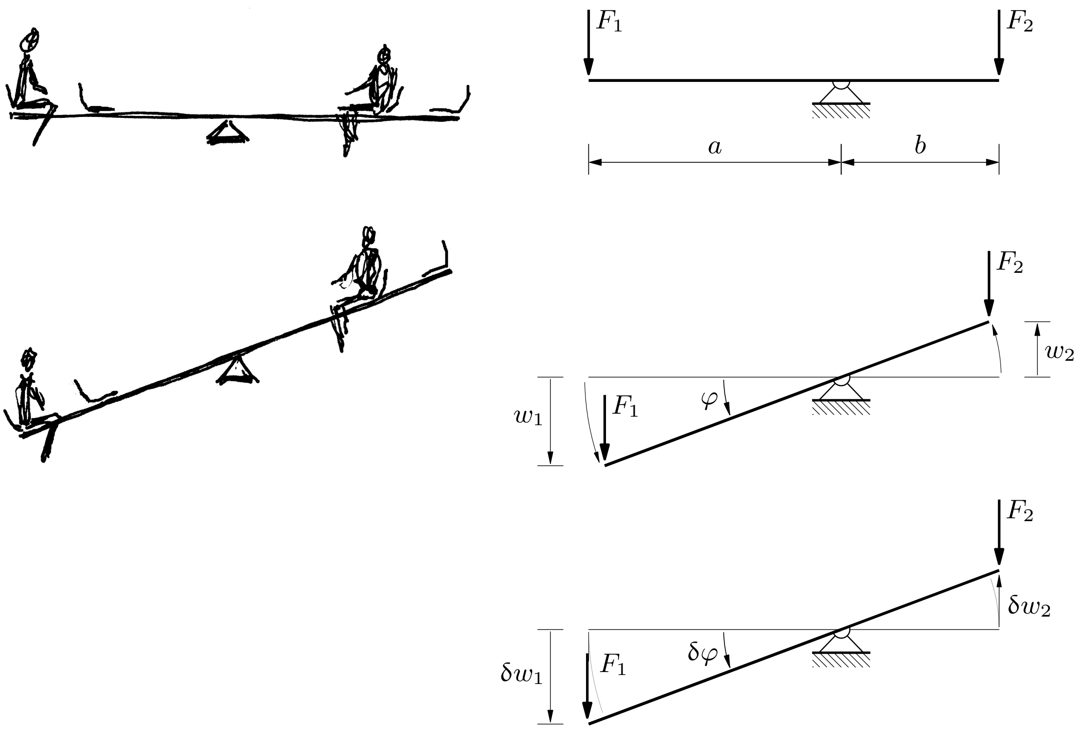
2. i obratno, *ako je zbroj radova sila i momenata na bilo kojim virtualnim pomacima i zaokretima krutoga tijela na koje djeluju jednak nuli, onda je tijelo u ravnoteži*:

$$\delta W = 0 \quad \forall \delta\vec{p} \qquad \Longrightarrow \qquad \sum_i \vec{F}_i = \vec{0} \quad \& \quad \sum_j \overrightarrow{M}_j = \vec{0}$$

ili, sažeto,

$$\sum_i \vec{F}_i = \vec{0} \quad \& \quad \sum_j \overrightarrow{M}_j = \vec{0} \qquad \Longleftrightarrow \qquad \delta W = 0 \quad \forall \delta\vec{p}$$

primjer (koji, naravno, nije dokaz):



aa.

ab.

ac.

izmjerivi (konačni) pomaci i zaokreti (sl. aa. i ab.):

ravnoteža $\Rightarrow$ nema rada:

$$\sum M_{/\text{ležaj}} = 0 : \qquad F_1\, a - F_2\, b = 0$$

$$w_1 = a \sin\varphi \qquad \& \qquad w_2 = b \sin\varphi \qquad \text{za neki (bilo koji) } \varphi$$

$$F_1\, w_1 - F_2\, w_2 = F_1\, a \sin\varphi - F_2\, b \sin\varphi = (F_1\, a - F_2\, b) \sin\varphi = 0 \cdot \sin\varphi = 0$$

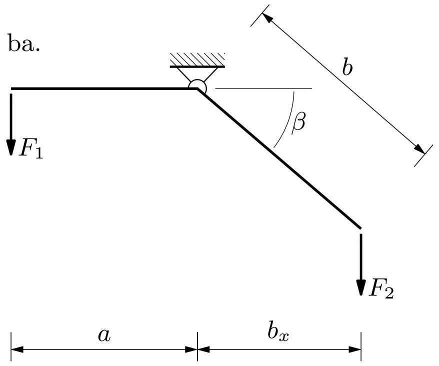
nema rada $\Rightarrow$ ravnoteža:

$$0 \;=\; F_1\, w_1 \;-\; F_2\, w_2 \;=\; F_1\, a\, \sin\varphi \;-\; F_2\, b\, \sin\varphi \;=\; (F_1\, a \;-\; F_2\, b)\, \sin\varphi$$

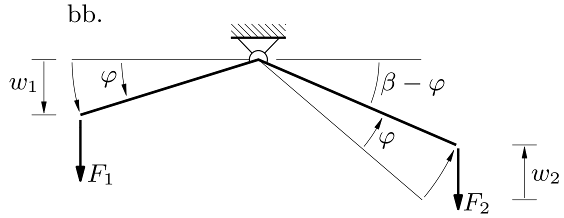
$$(F_1\, a \;-\; F_2\, b)\, \sin\varphi \;=\; 0 \quad \forall\varphi \qquad \Rightarrow \qquad F_1\, a \;-\; F_2\, b \;=\; 0$$

... ali, za izmjerive pomake teorem ne vrijedi uvijek — stoga, protuprimjer (sl. ba. i bb.):

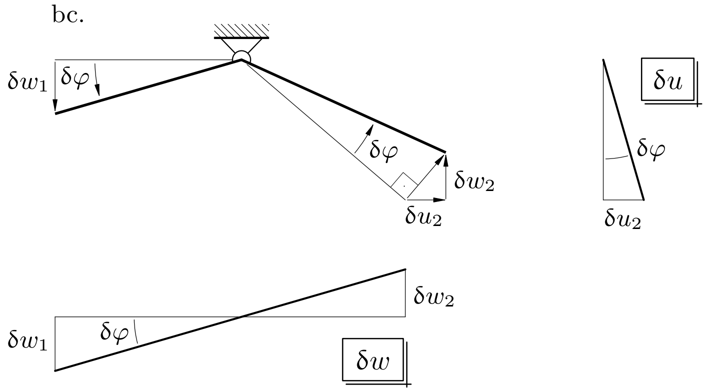
ba.



bb.



bc.



ravnoteža $\not\Rightarrow$ nema rada

$$F_1\, a - F_2\, b_x \;=\; 0, \qquad b_x \;=\; b\, \cos\beta, \qquad F_1\, a - F_2\, b\, \cos\beta \;=\; 0$$

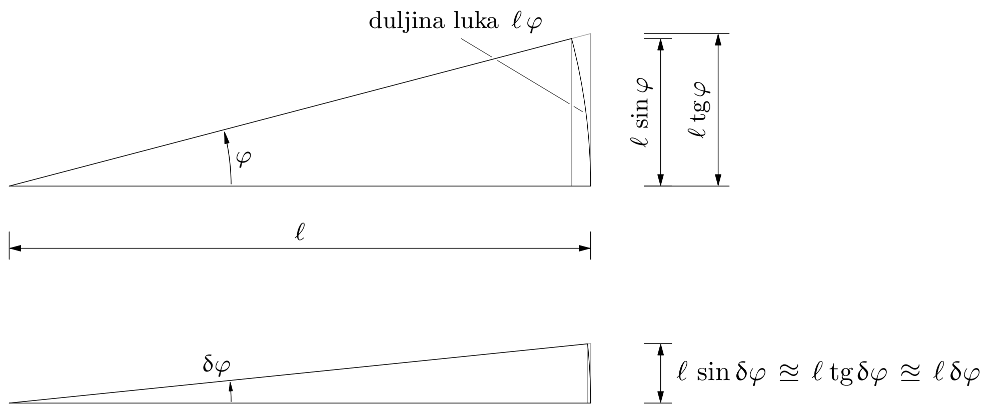
$$w_1 \;=\; a\, \sin\varphi \qquad \& \qquad w_2 \;=\; b\, \sin\beta - b\, \sin\,(\beta - \varphi)$$

$$\begin{aligned}
F_1\, w_1 \;-\; F_2\, w_2 \;&=\; F_1\, a\, \sin\varphi \;-\; F_2\, \big[b\, \sin\beta - b\, \sin\,(\beta - \varphi)\big] \\
&=\; F_1\, a\, \sin\varphi \;-\; F_2\, b\, \sin\beta \;+\; F_2\, b\, (\cos\varphi\, \sin\beta \;-\; \sin\varphi\, \cos\beta) \\
&=\; (F_1\, a \;-\; F_2\, b\, \cos\beta)\, \sin\varphi \;+\; F_2\, b\, (\cos\varphi - 1)\, \sin\beta \\
&=\; 0 \cdot \sin\varphi \;+\; F_2\, b\, (\cos\varphi - 1)\, \sin\beta \\
&=\; F_2\, b\, (\cos\varphi - 1)\, \sin\beta \;\neq\; 0
\end{aligned}$$

neizmjerno mali (infinitezimalni) pomaci i zaokreti (slika na sljedećoj stranici):

$$\ell\, \sin\delta\varphi \;\approxeq\; \ell\, \mathrm{tg}\,\delta\varphi \;\approxeq\; \ell\, \delta\varphi \qquad \& \qquad \ell\, \cos\delta\varphi \;\approxeq\; \ell$$

pa uzimamo $\quad \ell\, \sin\delta\varphi \;=\; \ell\, \mathrm{tg}\,\delta\varphi \;=\; \ell\, \delta\varphi \quad \& \quad \ell\, \cos\delta\varphi \;=\; \ell$



. . . i, protuprimjer postaje primjer (sl. ba. i bc.):

ravnoteža $\Rightarrow$ nema rada:

$$\delta w_1 \;=\; a\,\delta\varphi \qquad \& \qquad \delta w_2 \;=\; b_x\,\delta\varphi \qquad \text{za neki (bilo koji) } \delta\varphi$$

$$F_1\,\delta w_1 \;-\; F_2\,\delta w_2 \;=\; F_1\,a\,\delta\varphi \;-\; F_2\,b_x\,\delta\varphi \;=\; (F_1\,a \;-\; F_2\,b_x)\,\delta\varphi \;=\; 0$$

nema rada $\Rightarrow$ ravnoteža:

$$0 \;=\; F_1\,\delta w_1 \;-\; F_2\,\delta w_2 \;=\; F_1\,a\,\delta\varphi \;-\; F_2\,b_x\,\delta\varphi \;=\; (F_1\,a \;-\; F_2\,b_x)\,\delta\varphi$$

$$(F_1\,a \;-\; F_2\,b_x)\,\delta\varphi \;=\; 0 \quad \forall\,\delta\varphi \qquad \Rightarrow \qquad F_1\,a \;-\; F_2\,b_x \;=\; 0$$

prvi primjer, ponovo, ali s neizmjerno malim pomacima (sl. aa. i ac.):

ravnoteža $\Rightarrow$ nema rada:

$$F_1\,a - F_2\,b \;=\; 0$$

$$\delta w_1 \;=\; a\,\delta\varphi \qquad \& \qquad \delta w_2 \;=\; b\,\delta\varphi \qquad \text{za neki } \delta\varphi$$

$$F_1\,\delta w_1 \;-\; F_2\,\delta w_2 \;=\; F_1\,a\,\delta\varphi \;-\; F_2\,b\,\delta\varphi \;=\; (F_1\,a \;-\; F_2\,b)\,\delta\varphi \;=\; 0$$

nema rada $\Rightarrow$ ravnoteža:

$$0 \;=\; F_1\,\delta w_1 \;-\; F_2\,\delta w_2 \;=\; F_1\,a\,\delta\varphi \;-\; F_2\,b\,\delta\varphi \;=\; (F_1\,a \;-\; F_2\,b)\,\delta\varphi$$

$$(F_1\,a \;-\; F_2\,b)\,\delta\varphi \;=\; 0 \quad \forall\,\delta\varphi \qquad \Rightarrow \qquad F_1\,a \;-\; F_2\,b \;=\; 0$$

teorem o virtualnom radu za sistem krutih tijela:

1. *ako su tijela sistema krutih tijela, na koja djeluju sile i momenti, u ravnoteži, onda je zbroj radova tih sila i momenata na bilo kojim virtualnim pomacima i zaokretima tijelâ jednak nuli:*

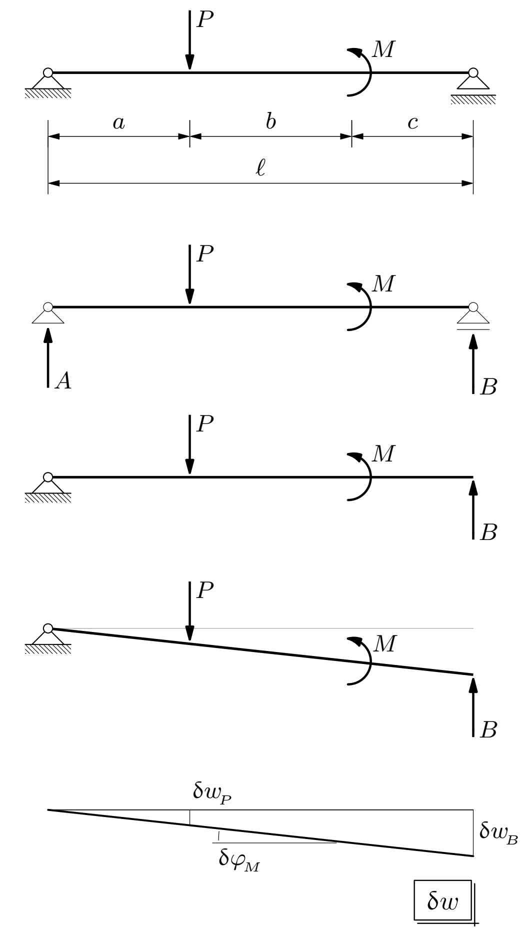
$$\sum_i \vec{F}_i \;=\; \vec{0} \quad \& \quad \sum_j \overrightarrow{M}_j \;=\; \vec{0} \qquad \Longrightarrow \qquad \delta W \;=\; 0 \quad \forall \delta\vec{p}$$

2. i obratno, *ako je zbroj radova sila i momenata na bilo kojim virtualnim pomacima i zaokretima tijelâ sistema krutih tijela na koja djeluju jednak nuli, onda su tijela u ravnoteži:*

$$\delta W \;=\; 0 \quad \forall \delta\vec{p} \qquad \Longrightarrow \qquad \sum_i \vec{F}_i \;=\; \vec{0} \quad \& \quad \sum_j \overrightarrow{M}_j \;=\; \vec{0}$$

primjene teorema o virtualnom radu za kruto tijelo:

izračunavanje vrijednostî reakcija:



vrijednost reakcije B:

rad na virtualnim pomacima izazvanima pomakom δw_B ležaja B

$$P\,\delta w_P \;-\; M\,\delta\varphi_M \;-\; B\,\delta w_B \;=\; 0$$

$$\delta\varphi_M \;=\; \frac{\delta w_B}{\ell}$$

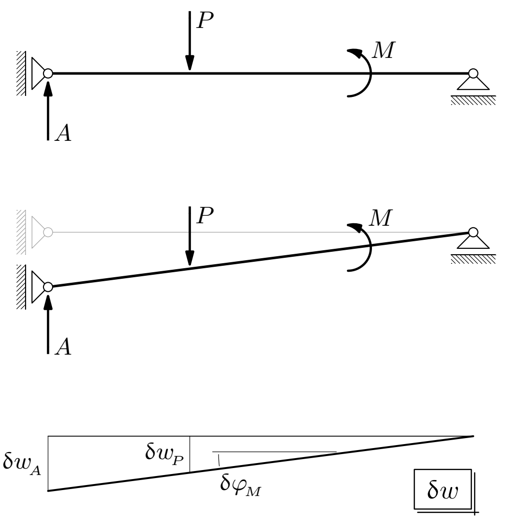
$$\delta w_P \;=\; a\,\delta\varphi_M \;=\; \frac{a}{\ell}\,\delta w_B$$

$$P\,\frac{a}{\ell}\,\delta w_B \;-\; M\,\frac{\delta w_B}{\ell} \;-\; B\,\delta w_B \;=\; 0$$

$$\left(P\,\frac{a}{\ell} \;-\; M\,\frac{1}{\ell} \;-\; B\right)\delta w_B \;=\; 0 \qquad \forall\,\delta w_B$$

$$P\,\frac{a}{\ell} \;-\; M\,\frac{1}{\ell} \;-\; B \;=\; 0$$

$$B \;=\; \frac{a}{\ell}\,P \;-\; \frac{1}{\ell}\,M$$



vrijednost reakcije A:

rad na virtualnim pomacima izazvanima pomakom δw_A ležaja A

$$-A\,\delta w_A \;+\; P\,\delta w_P \;+\; M\,\delta\varphi_M \;=\; 0$$

$$\delta\varphi_M \;=\; \frac{\delta w_A}{\ell}$$

$$\delta w_P \;=\; (\ell - a)\,\delta\varphi_M \;=\; \frac{\ell - a}{\ell}\,\delta w_A$$

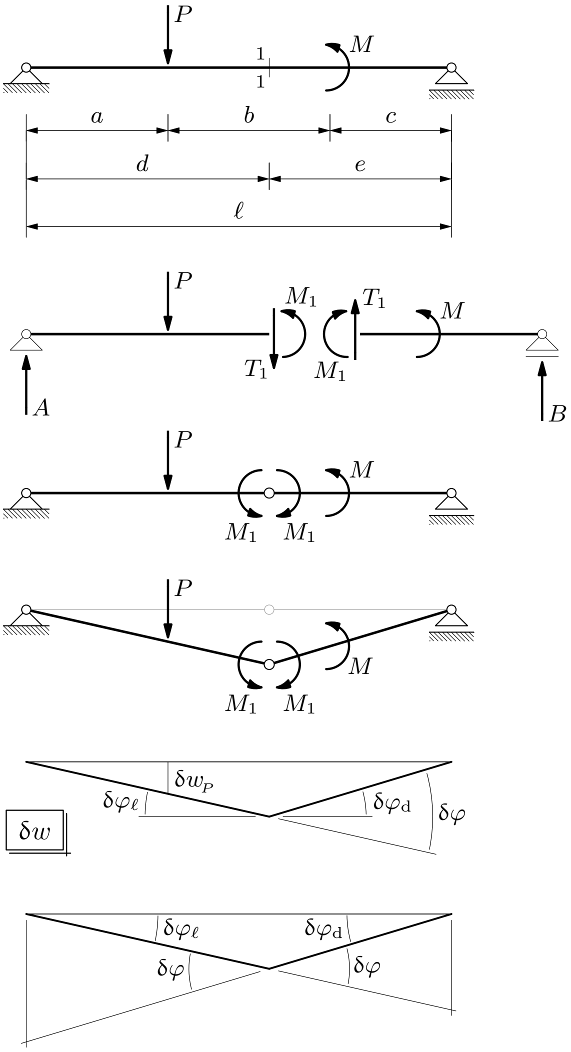
$$-A\,\delta w_A \;+\; P\,\frac{\ell - a}{\ell}\,\delta w_A \;+\; M\,\frac{\delta w_A}{\ell} \;=\; 0$$

$$\left(-A \;+\; P\,\frac{\ell - a}{\ell} \;+\; M\,\frac{1}{\ell}\right)\delta w_A \;=\; 0 \qquad \forall\,\delta w_A$$

$$-A \;+\; P\,\frac{\ell - a}{\ell} \;+\; M\,\frac{1}{\ell} \;=\; 0$$

$$A \;=\; \frac{\ell - a}{\ell}\,P \;+\; \frac{1}{\ell}\,M$$

izračunavanje vrijednostî unutarnjih sila:



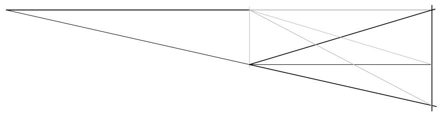
vrijednost momenta savijanja u presjeku 1−1:

rad na virtualnim pomacima izazvanima relativnim zaokretom $\delta\varphi$ u presjeku 1−1

$$P\,\delta w_P \;-\; M_1\,\delta\varphi_\ell \;-\; M_1\,\delta\varphi_\mathrm{d} \;+\; M\,\delta\varphi_\mathrm{d} \;=\; 0$$

$$P\,\delta w_P \;-\; M_1\,(\delta\varphi_\ell \;+\; \delta\varphi_\mathrm{d}) \;+\; M\,\delta\varphi_\mathrm{d} \;=\; 0$$

$$P\,\delta w_P \;-\; M_1\,\delta\varphi \;+\; M\,\delta\varphi_\mathrm{d} \;=\; 0$$



$$\delta\varphi_\ell\,\ell \;=\; \delta\varphi\, e \quad\Rightarrow\quad \delta\varphi_\ell \;=\; \frac{e}{\ell}\,\delta\varphi$$

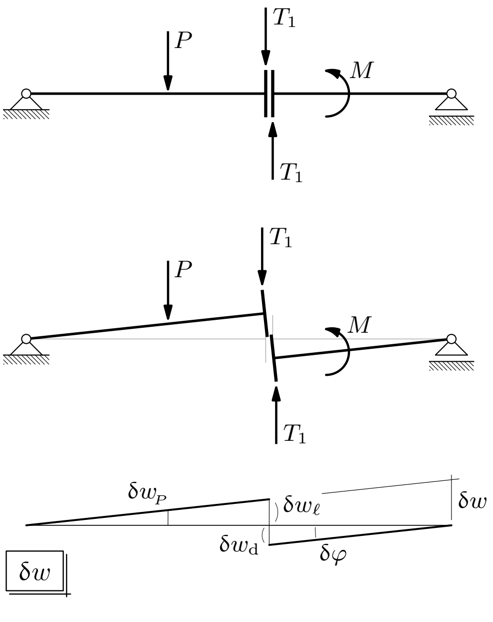
$$\delta\varphi_\mathrm{d}\,\ell \;=\; \delta\varphi\, d \quad\Rightarrow\quad \delta\varphi_\mathrm{d} \;=\; \frac{d}{\ell}\,\delta\varphi$$

$$\delta w_P \;=\; a\,\delta\varphi_\ell \;=\; \frac{a\,e}{\ell}\,\delta\varphi$$

$$\left(P\,\frac{a\,e}{\ell} \;-\; M_1 \;+\; M\,\frac{d}{\ell}\right)\delta\varphi \;=\; 0 \qquad \forall\,\delta\varphi$$

$$P\,\frac{a\,e}{\ell} \;-\; M_1 \;+\; M\,\frac{d}{\ell} \;=\; 0$$

$$M_1 \;=\; \frac{a\,e}{\ell}\,P \;+\; \frac{d}{\ell}\,M$$



vrijednost poprečne sila u presjeku 1−1:

rad na virtualnim pomacima izazvanima relativnim pomakom δw u presjeku 1−1

$$-P\,\delta w_P \;-\; T_1\,\delta w_\ell \;-\; T_1\,\delta w_\mathrm{d} \;+\; M\,\delta\varphi \;=\; 0$$

$$-P\,\delta w_P \;-\; T_1\,(\delta w_\ell \;+\; \delta w_\mathrm{d}) \;+\; M\,\delta\varphi \;=\; 0$$

$$-P\,\delta w_P \;-\; T_1\,\delta w \;+\; M\,\delta\varphi \;=\; 0$$

$$\delta\varphi \;=\; \frac{\delta w}{\ell}$$

$$\delta w_P \;=\; a\,\delta\varphi \;=\; \frac{a}{\ell}\,\delta w$$

$$\left(-P\,\frac{a}{\ell} \;-\; T_1 \;+\; M\,\frac{1}{\ell}\right)\delta w \;=\; 0 \qquad \forall\,\delta w$$

$$-P\,\frac{a}{\ell} \;-\; T_1 \;+\; M\,\frac{1}{\ell} \;=\; 0$$

$$T_1 \;=\; -\frac{a}{\ell}\,P \;+\; \frac{1}{\ell}\,M$$