GS 2. — 28. kolovoza 2024.

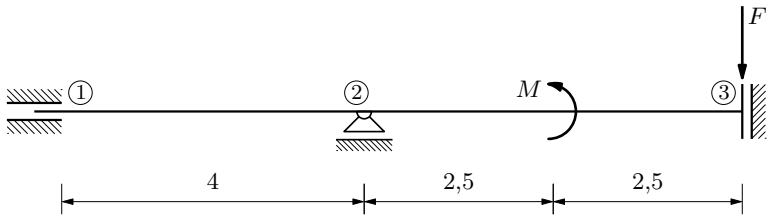
Zadatak 2.a.

Pomoću utjecajne linije nacrtane inženjerskom metodom pomakā izračunajte vrijednost reaktivnoga momenta u desnome ležaju!

$EI = \text{const.}$

$F = 125$ kN

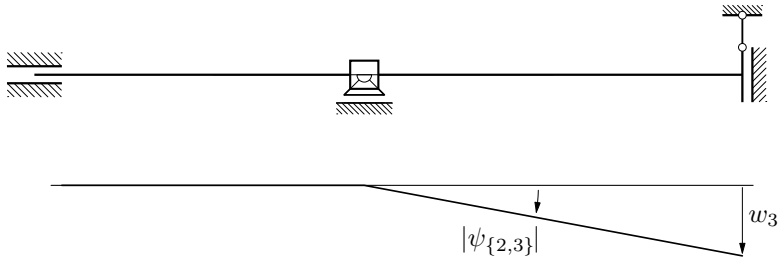
$M = 75$ kNm



$$k_{1,2} = \frac{EI}{4}, \qquad k_{2,3} = \frac{EI}{5}$$

nepoznanice za inženjersku metodu pomakā: φ_2 & w_3

uz statičku kondenzaciju pomaka $\vec{w}_3$: φ_2



$$\psi_{\{2,3\}} = -\frac{w_3}{\ell_{\{2,3\}}} = -\frac{w_3}{5}$$

dvije nepoznanice — $\boldsymbol{\varphi_2}$ & $\boldsymbol{w_3}$:

izrazi za vrijednosti momenata na krajevima štapova:

$$M_{1,2} = 2\,k_{\{1,2\}}\,\varphi_2 = \frac{EI}{2}\,\varphi_2$$

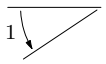
$$M_{2,1} = 4\,k_{\{1,2\}}\,\varphi_2 = EI\,\varphi_2$$

$$M_{2,3} = 4\,k_{\{2,3\}}\,\varphi_2 - 6\,k_{\{2,3\}}\,\psi_{\{2,3\}} + \overline{M}_{2,3} = \frac{4\,EI}{5}\,\varphi_2 + \frac{6\,EI}{25}\,w_3 + \overline{M}_{2,3}$$

$$M_{3,2} = 2\,k_{\{2,3\}}\,\varphi_2 - 6\,k_{\{2,3\}}\,\psi_{\{2,3\}} + \overline{M}_{3,2} = \frac{2\,EI}{5}\,\varphi_2 + \frac{6\,EI}{25}\,w_3 + \overline{M}_{3,2}$$

vrijednosti momenata upetosti za jedinični zaokret kraja 3 štapa $\{2,3\}$:

smisao vrtnje reaktivnoga momenta: 

jedinični kut: 

$$\overline{M}_{3,2} = 4\,k_{\{1,2\}} \cdot 1 = \frac{4\,EI}{5} \qquad \& \qquad \overline{M}_{2,3} = 2\,k_{\{2,3\}} \cdot 1 = \frac{2\,EI}{5}$$

jednadžba ravnoteže momenata u čvoru 2:

$$-M_{2,1} - M_{2,3} = 0 \qquad \Rightarrow \qquad M_{2,1} + M_{2,3} = 0$$

$$EI\,\varphi_2 + \frac{4\,EI}{5}\,\varphi_2 + \frac{6\,EI}{25}\,w_3 + \frac{2\,EI}{5} = 0$$

$$\frac{9\,EI}{5}\,\varphi_2 + \frac{6\,EI}{25}\,w_3 = -\frac{2\,EI}{5} \qquad \Rightarrow \qquad \frac{9}{5}\,\varphi_2 + \frac{6}{25}\,w_3 = -\frac{2}{5} \qquad (☠)$$

jednadžba virtualnih radova:

$$\big(M_{2,3} + M_{3,2}\big)\,\delta\psi_{\{2,3\}} = 0 \quad \forall\,\delta\psi_{\{2,3\}} \qquad \Rightarrow \qquad M_{2,3} + M_{3,2} = 0$$

$$\frac{6\,EI}{5}\,\varphi_2 + \frac{12\,EI}{25}\,w_3 = -\frac{6\,EI}{5} \qquad \Rightarrow \qquad \frac{6}{5}\,\varphi_2 + \frac{12}{25}\,w_3 = -\frac{6}{5} \qquad (🏴‍☠️)$$

Hmm, matrica sustava jednadžbi (☠) & (🏴‍☠️) nije simetrična?!

S jednadžbom virtualnoga rada može se malo petljati… Može se reći da smo prerano upotrijebili $\forall$ da uklonimo δ. Izrazimo li $\delta\psi_{\{2,3\}}$ kao funkciju δw_3 (na isti način kao što smo $\psi_{\{2,3\}}$ izrazili kao funkciju w_3), dobit ćemo

$$\big(M_{2,3} + M_{3,2}\big)\left(-\frac{\delta w_3}{5}\right) = 0 \quad \forall\,\delta w_3 \qquad \Rightarrow \qquad -\frac{1}{5}\big(M_{2,3} + M_{3,2}\big) = 0$$

$$-\frac{6\,EI}{5}\,\varphi_2 - \frac{12\,EI}{25}\,w_3 = \frac{6\,EI}{5} \quad \Big/ \quad \times(-1)$$

$$\frac{6\,EI}{5}\,\varphi_2 + \frac{12\,EI}{25}\,w_3 = -\frac{6\,EI}{5} \qquad \Rightarrow \qquad \frac{6}{25}\,\varphi_2 + \frac{12}{125}\,w_3 = -\frac{6}{25} \qquad (🏴‍☠️)$$

Matrica sustava jednadžbi (☠) & (🏴‍☠️) jest simetrična.

Sustavi (☠) & (🏴‍☠️) i (☠) & (🏴‍☠️) su ekvivalentni — imaju isto rješenje. Pomnožimo li neku jednadžbu sustava nekim brojem (različitim od nule), rješenje sustava se neće promijeniti — množenje (ili dijeljenje) jednadžbe brojem jedna je od elementarnih operacija Gaußova eliminacijskog postupka za rješavanje sustava linearnih jednadžbi.

rješenje sustava (☠) & (🏴‍☠️) (i, dakako, (☠) & (🏴‍☠️)):

$$\varphi_2 = \frac{1}{6} \qquad \& \qquad w_3 = -\frac{35}{12}$$

vrijednosti momenata na krajevima štapova:

$$M_{1,2} = \frac{EI}{2}\cdot\frac{1}{6} = \frac{EI}{12}$$

$$M_{2,1} = EI\cdot\frac{1}{6} = \frac{EI}{6}$$

$$M_{2,3} \;=\; \frac{4\,EI}{5}\cdot\frac{1}{6} \;-\; \frac{6\,EI}{25}\cdot\frac{35}{12} \;+\; \frac{2\,EI}{5} \;=\; -\frac{EI}{6}$$

$$M_{3,2} \;=\; \frac{2\,EI}{5}\cdot\frac{1}{6} \;-\; \frac{6\,EI}{25}\cdot\frac{35}{12} \;+\; \frac{4\,EI}{5} \;=\; \frac{EI}{6}$$

ili: (samo) **jedna nepoznanica — $\boldsymbol{\varphi_2}$**:

izrazi za vrijednosti momenata na krajevima štapova:

$$M_{1,2} \;=\; 2\,k_{\{1,2\}}\,\varphi_2 \;=\; \frac{EI}{2}\,\varphi_2$$

$$M_{2,1} \;=\; 4\,k_{\{1,2\}}\,\varphi_2 \;=\; EI\,\varphi_2$$

$$M^{\mathrm{c}}_{2,3} \;=\; k_{\{2,3\}}\,\varphi_2 \;+\overline{M}^{\mathrm{c}}_{2,3} \;=\; \frac{EI}{5}\,\varphi_2 \;+\overline{M}^{\mathrm{c}}_{2,3}$$

$$M^{\mathrm{c}}_{3,2} \;=\; -k_{\{2,3\}}\,\varphi_2 \;+\overline{M}^{\mathrm{c}}_{3,2} \;=\; -\frac{EI}{5}\,\varphi_2 \;+\overline{M}^{\mathrm{c}}_{3,2}$$

vrijednosti momenata upetosti za jedinični zaokret kraja 3 štapa $\{2,3\}$:

$$\overline{M}^{\mathrm{c}}_{3,2} \;=\; k_{\{1,2\}}\cdot 1 \;=\; \frac{EI}{5} \qquad \&\qquad \overline{M}^{\mathrm{c}}_{2,3} \;=\; -k_{\{2,3\}}\cdot 1 \;=\; -\frac{EI}{5}$$

jednadžba ravnoteže momenata u čvoru 2:

$$-M_{2,1} \;-\; M^{\mathrm{c}}_{2,3} \;=\; 0 \qquad \Rightarrow \qquad M_{2,1} \;+\; M^{\mathrm{c}}_{2,3} \;=\; 0$$

$$EI\,\varphi_2 \;+\; \frac{EI}{5}\,\varphi_2 \;-\; \frac{EI}{5} \;=\; 0 \qquad \Rightarrow \qquad \frac{6}{5}\,\varphi_2 \;=\; \frac{1}{5}$$

. . . i njezino rješenje:

$$\varphi_2 \;=\; \frac{1}{6}$$

vrijednosti momenata na krajevima štapova:

$$M_{1,2} \;=\; \frac{EI}{2}\cdot\frac{1}{6} \;=\; \frac{EI}{12}$$

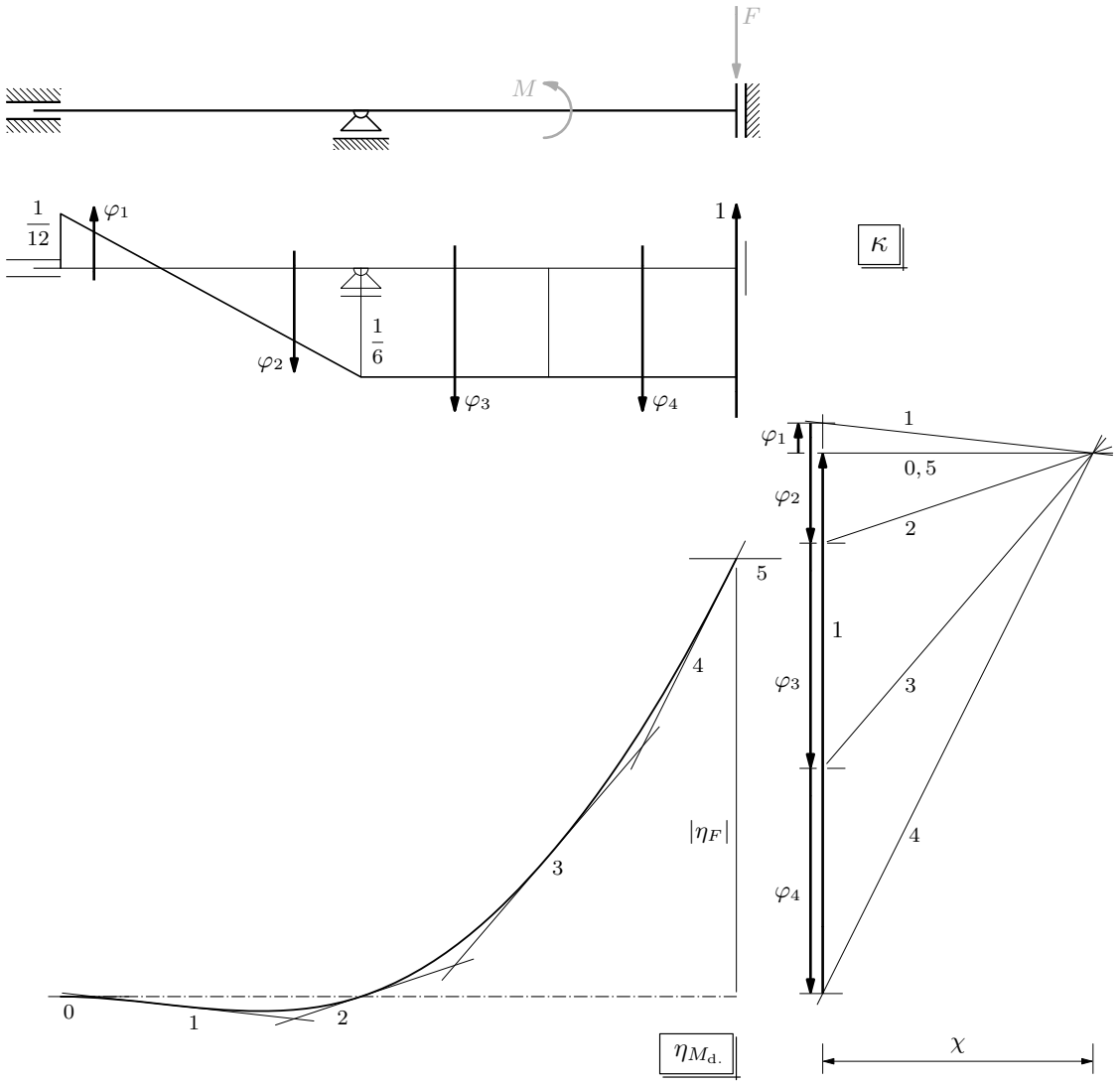
$$M_{2,1} \;=\; EI\cdot\frac{1}{6} \;=\; \frac{EI}{6}$$

$$M_{2,3} \;=\; M^{\mathrm{c}}_{2,3} \;=\; \frac{EI}{5}\cdot\frac{1}{6} \;-\; \frac{EI}{5} \;=\; -\frac{EI}{6}$$

$$M_{3,2} \;=\; M^{\mathrm{c}}_{2,3} \;=\; -\frac{EI}{5}\cdot\frac{1}{6} \;+\; \frac{EI}{5} \;=\; \frac{EI}{6}$$

utjecajna linija:

mjerilo duljina: $\quad 1\,\mathrm{cm} \;::\; 1\,\mathrm{m}$



$$\varphi_1 \;=\; \frac{1}{2}\cdot\frac{1}{12}\cdot\left(\frac{1}{3}\cdot 4\right) \;=\; \frac{1}{18}\,, \qquad\qquad \varphi_2 \;=\; \frac{1}{2}\cdot\frac{1}{6}\cdot\left(\frac{2}{3}\cdot 4\right) \;=\; \frac{2}{9}$$

$$\varphi_3 \;=\; \varphi_4 \;=\; \frac{1}{6}\cdot\left(\frac{1}{2}\cdot 5\right) \;=\; \frac{5}{12}$$

provjera: $\quad -\dfrac{1}{18} \;+\; \dfrac{2}{9} \;+\; \dfrac{5}{12} \;+\; \dfrac{5}{12} \;-\; 1 \;=\; 0$ $\qquad$ [zašto?]

mjerilo kutova: $\quad 1\text{ cm} \;::\; \dfrac{5}{36}$

$$\widetilde{\varphi}_1 \;=\; \frac{2}{5} \;=\; 0{,}4\,\text{cm}, \qquad\qquad \widetilde{\varphi}_2 \;=\; \frac{8}{5} \;=\; 1{,}6\,\text{cm},$$

$$\widetilde{\varphi}_3 \;=\; \widetilde{\varphi}_4 \;=\; 3\,\text{cm}, \qquad\qquad \tilde{1} \;=\; \frac{36}{5} \;=\; 7{,}2\,\text{cm},$$

$$\chi \;=\; \frac{1}{2} \qquad \Rightarrow \qquad \widetilde{\chi} \;=\; 3{,}6\,\text{cm}$$

vrijednost momenta u desnome lažaju:

očitano: $|\widetilde{\eta}_F| \,=\, 58 \text{ i } {}^1\!/_3 \text{ mm} \,\simeq\, 5{,}83 \text{ cm}$

$$\chi \,=\, \frac{1}{2} \qquad \Rightarrow \qquad n \,=\, 2$$

$$|\eta_F| \,=\, \frac{m}{n}\,\tilde{\eta}_F \,=\, \frac{1}{2}\cdot 5{,}83 \,=\, 2{,}915, \qquad\qquad \eta_F \,=\, -2{,}915$$

nagib tangente na $\eta_{M_{\mathrm{d.}}}$ u hvatištu momenta:

$$\operatorname{tg}\alpha_M \,=\, \varphi_1 \,-\, \varphi_2 \,-\, \varphi_3 \,=\, \frac{1}{18} \,-\, \frac{2}{9} \,-\, \frac{5}{12} \,=\, -\frac{7}{12} \,=\, -0{,}583$$

$$M_{\mathrm{d.}} \,=\, F\,\eta_F \,+\, M\,(-\operatorname{tg}\alpha_M) \,=\, 125\cdot(-2{,}915) \,+\, 75\cdot\big(-(-0{,}583)\big) \,=\, -320{,}65\,\mathrm{kNm}$$

(smisao vrtnje je suprotan od pretpostavljenoga)

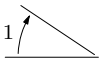
Zadatak 2.b.

Pomoću utjecajne linije nacrtane relaksacijskim postupkom izračunajte vrijednost momenta savijanja iznad srednjega ležaja!

nepoznanice za inženjersku metodu pomakā: kao u zadatku 2.b.

relaksacija bez statičke kondenzacije:

jedinični kut zaokreta neposredno lijevo od srednjega ležaja (zaokret kraja 2 štapa $\{1,2\}$):

smisao vrtnje momenta:  jedinični kut: 

vrijednosti momenata upetosti:

$$\overline{M}_{1,2} \;=\; 2\,k_{\{1,2\}}\cdot(-1) \;=\; -\frac{EI}{2} \qquad \& \qquad \overline{M}_{2,1} \;=\; 4\,k_{\{1,2\}}\cdot(-1) \;=\; -EI$$

razdjelni koeficijenti:

$$k_2 \;=\; 4\,k_{1,2} \,+\, 4\,k_{2,3} \;=\; 4\,\frac{EI}{4} \,+\, 4\,\frac{EI}{5} \;=\; \frac{9}{5}\,EI$$

$$\mu_{2,1} \;=\; \frac{4\,k_{1,2}}{k_2} \;=\; \frac{4\,\frac{EI}{4}}{\frac{9\,EI}{5}} \;=\; \frac{5}{9}$$

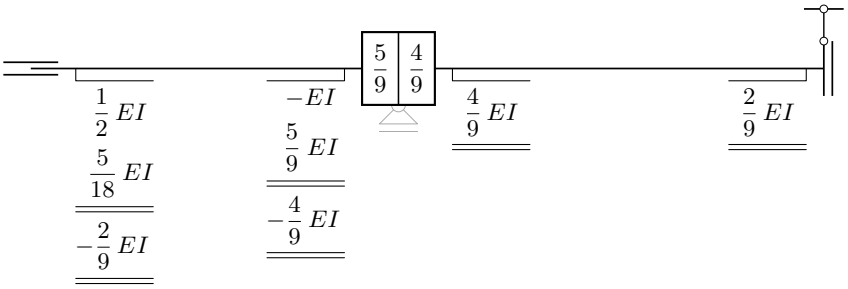
$$\mu_{2,3} \;=\; \frac{4\,k_{2,3}}{k_2} \;=\; \frac{4\,\frac{EI}{5}}{\frac{9\,EI}{5}} \;=\; \frac{4}{9}$$

$$\mu_{2,1} \,+\, \mu_{2,3} \;=\; \frac{5}{9} \,+\, \frac{4}{9} \;=\; 1$$

prijenosni koeficijenti:

$$2 \;\to\; 1 \;:\; \frac{1}{2}, \qquad\qquad 2 \;\to\; 3 \;:\; \frac{1}{2}$$

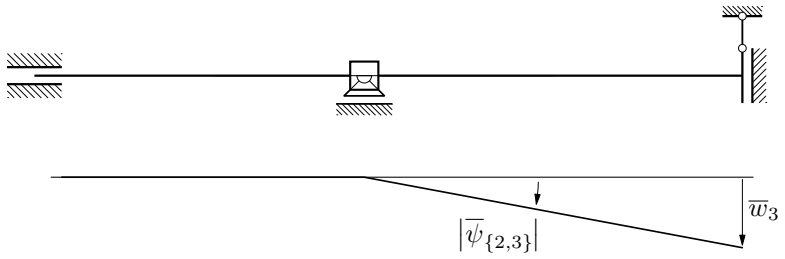
relaksacija bez iteracije:



reakcija u dodanom spoju:

$$R_1 \;=\; T_{3,2} \;=\; \frac{1}{\ell_{\{2,3\}}}\left(M_{2,3} \,+\, M_{3,2}\right) \;=\; \frac{1}{5}\left(\frac{4}{9}\,EI \,+\, \frac{2}{9}\,EI\right) \;=\; \frac{2}{15}\,EI$$

prisilni pomak:

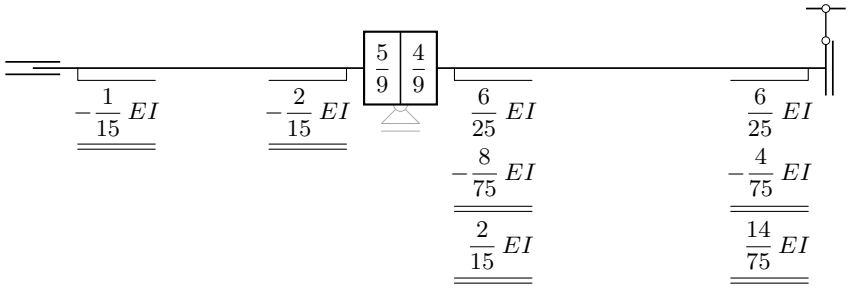


$$\overline{w}_3 \;=\; 1, \qquad \overline{\psi}_{\{2,3\}} \;=\; -\frac{\overline{w}_3}{\ell_{\{2,3\}}} \;=\; -\frac{1}{5}$$

vrijednosti momenata upetosti:

$$\overline{M}_{2,3} \;=\; \overline{M}_{3,2} \;=\; -6\,k_{\{2,3\}}\,\overline{\psi}_{\{2,3\}} \;=\; -6\,\frac{EI}{5}\left(-\frac{1}{5}\right) \;=\; \frac{6}{25}\,EI$$

relaksacija bez iteracije:



reakcija u dodanom spoju:

$$R_2 \;=\; T_{3,2}^{(2)} \;=\; \frac{1}{5}\left(\frac{2}{15}\,EI \;+\; \frac{14}{75}\,EI\right) \;=\; \frac{8}{125}\,EI$$

$$R_1 \;+\; \varrho\,R_2 \;=\; 0 \qquad \Rightarrow \qquad \varrho \;=\; -\frac{R_1}{R_2} \;=\; -\frac{\frac{2}{15}\,EI}{\frac{8}{125}\,EI} \;=\; -\frac{25}{12}$$

konačne vrijednosti momenata:

$$M_{1,2} \;=\; M_{1,2}^{(1.\,\mathrm{Cross})} \;+\; \varrho\,M_{1,2}^{(2.\,\mathrm{Cross})} \;=\; -\frac{2}{9}\,EI \;-\; \frac{25}{12}\left(-\frac{1}{15}\,EI\right) \;=\; -\frac{1}{12}\,EI$$

$$M_{2,1} \;=\; -\frac{4}{9}\,EI \;-\; \frac{25}{12}\left(-\frac{2}{15}\,EI\right) \;=\; -\frac{1}{6}\,EI$$

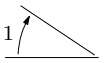
$$M_{2,3} \;=\; \frac{4}{9} \;-\; \frac{25}{12}\cdot\frac{2}{15}\,EI \;=\; \frac{1}{6}\,EI$$

$$M_{3,2} \;=\; \frac{2}{9} \;-\; \frac{25}{12}\cdot\frac{14}{75}\,EI \;=\; -\frac{1}{6}\,EI$$

ili: jedinični kut zaokreta neposredno desno od srednjega ležaja (zaokret kraja 2 štapa $\{2,3\}$):
domaća zabava!

ili: **relaksacija uz statičku kondenzaciju**:

jedinični kut zaokreta neposredno lijevo od srednjega ležaja (zaokret kraja 2 štapa $\{1,2\}$):

smisao vrtnje momenta:  jedinični kut: 

vrijednosti momenata upetosti:

$$\overline{M}_{1,2} \;=\; 2\,k_{\{1,2\}}\cdot(-1) \;=\; -\frac{EI}{2} \qquad \& \qquad \overline{M}_{2,1} \;=\; 4\,k_{\{1,2\}}\cdot(-1) \;=\; -EI$$

razdjelni koeficijenti:

$$k_2 \;=\; 4\,k_{1,2} \,+\, k_{2,3} \;=\; 4\,\frac{EI}{4} \,+\, \frac{EI}{5} \;=\; \frac{6}{5}\,EI$$

$$\mu_{2,1} \;=\; \frac{4\,k_{1,2}}{k_2} \;=\; \frac{\frac{4\,EI}{4}}{\frac{6\,EI}{5}} \;=\; \frac{5}{6}$$

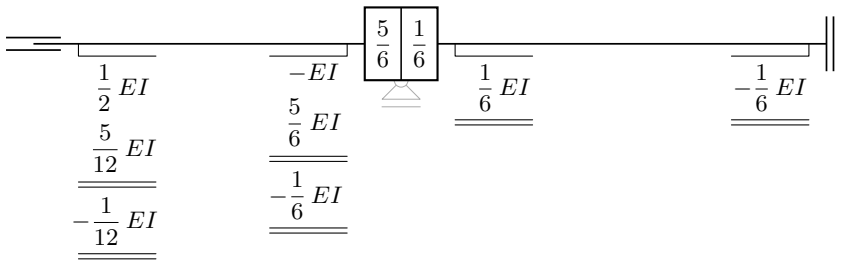
$$\mu_{2,3} \;=\; \frac{k_{2,3}}{k_2} \;=\; \frac{\frac{EI}{5}}{\frac{6\,EI}{5}} \;=\; \frac{1}{6}$$

$$\mu_{2,1} \,+\, \mu_{2,3} \;=\; \frac{5}{6} \,+\, \frac{1}{6} \;=\; 1$$

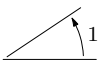
prijenosni koeficijenti:

$$2 \;\to\; 1 \;:\; \frac{1}{2}, \qquad\qquad 2 \;\to\; 3 \;:\; -1$$

relaksacija bez iteracije:



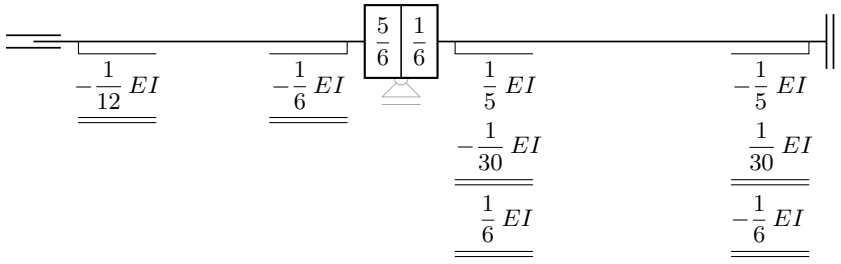
ili: jedinični kut zaokreta neposredno desno od srednjega ležaja (zaokret kraja 2 štapa $\{2,3\}$):

smisao vrtnje momenta:  jedinični kut: 

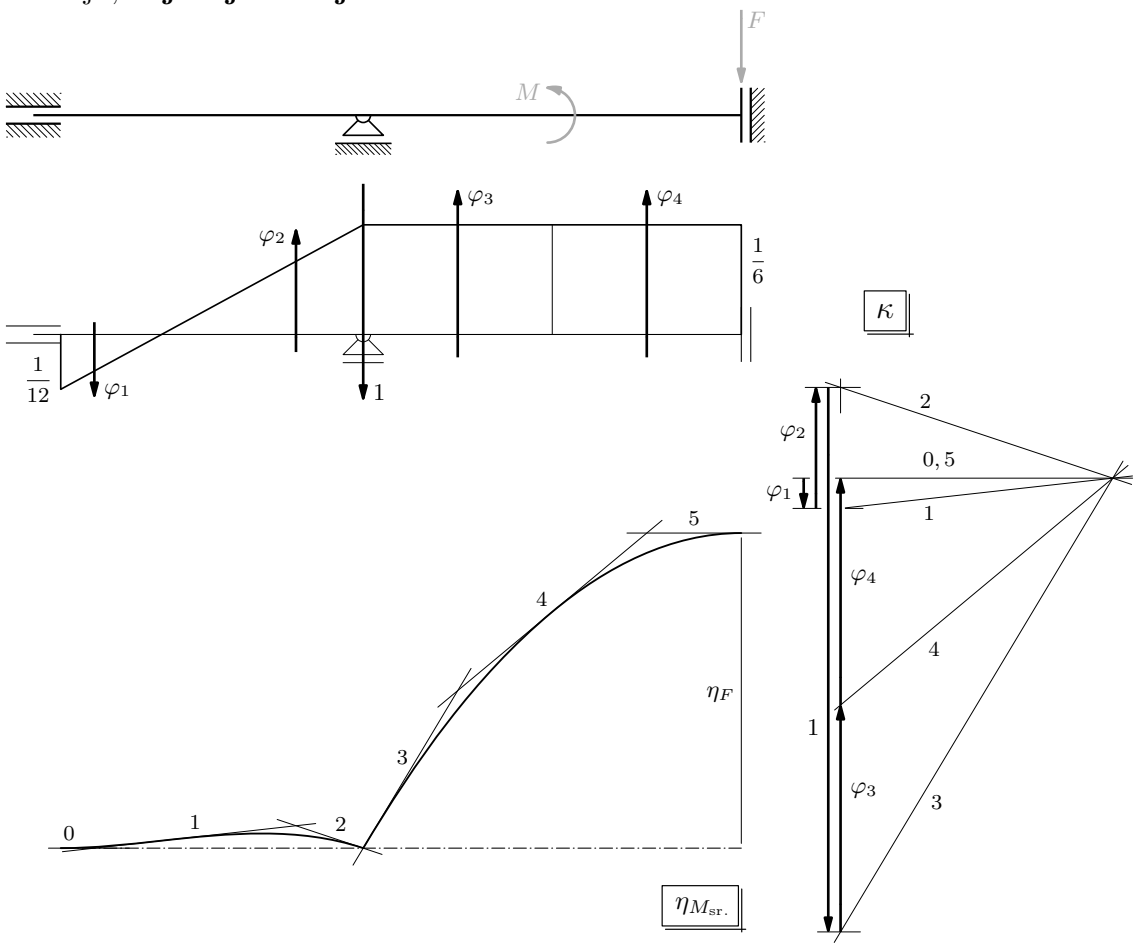
vrijednosti momenata upetosti:

$$\overline{M}_{2,3} \;=\; k_{\{2,3\}}\cdot 1 \;=\; \frac{EI}{5} \qquad \& \qquad \overline{M}_{3,2} \;=\; -k_{\{2,3\}}\cdot 1 \;=\; -\frac{EI}{5}$$

relaksacija bez iteracije:



i, na kraju, **utjecajna linija**. . . :



mjerilo duljina: 1 cm :: 1 m

$$\varphi_1 = \frac{1}{2}\cdot\frac{1}{12}\cdot\left(\frac{1}{3}\cdot 4\right) = \frac{1}{18}$$

$$\varphi_2 = \frac{1}{2}\cdot\frac{1}{6}\cdot\left(\frac{2}{3}\cdot 4\right) = \frac{2}{9}$$

$$\varphi_3 = \varphi_4 = \frac{1}{6}\cdot\left(\frac{1}{2}\cdot 5\right) = \frac{5}{12}$$

provjera: $\frac{1}{18} - \frac{2}{9} + 1 - \frac{5}{12} - \frac{5}{12} = 0$ [zašto?]

mjerilo kutova: $1\text{ cm} :: \frac{5}{36}$

$\widetilde{\varphi}_1 = \frac{2}{5} = 0{,}4\,\text{cm}$, $\widetilde{\varphi}_2 = \frac{8}{5} = 1{,}6\,\text{cm}$,

$\widetilde{\varphi}_3 = \widetilde{\varphi}_4 = 3\,\text{cm}$, $\tilde{1} = \frac{36}{5} = 7{,}2\,\text{cm}$

$\chi = \frac{1}{2} \quad \Rightarrow \quad \widetilde{\chi} = 3{,}6\,\text{cm}$

... **i primjena**:

očitano: $|\widetilde{\eta}_F| = 41 \text{ i } {}^2/_3 \text{ mm} \simeq 4{,}17 \text{ cm}$

$\chi = \frac{1}{2} \quad \Rightarrow \quad n = 2$

$|\eta_F| = \frac{m}{n}\,\tilde{\eta}_F = \frac{1}{2} \cdot 4{,}17 = 2{,}085, \qquad \eta_F = -2{,}085$

nagib tangente na $\eta_{M_{\text{d.}}}$ u hvatištu momenta:

$\operatorname{tg}\alpha_M = -\varphi_4 = -\frac{5}{12} = -0{,}417$

$M_{\text{s.}} = F\,\eta_F + M\,(-\operatorname{tg}\alpha_M) = 125 \cdot (-2{,}085) + 75 \cdot \big(-(-0{,}417)\big) = -229{,}35\,\text{kNm}$

(smisao vrtnje je suprotan od pretpostavljenoga)