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Some Remarks to the Paper "The Goose that Laid the Golden Egg"

Neka zapažanja o članku "Guska koja nese zlatno jaje"

SAŽETAK

Članak [1] potaknuo me da povežem transcedentalne LAMÉove krivulje sa svojim opažnjima o realnim ptičjim jajima [2]. Dijelovi dviju različitih krivulja spajaju se u točkama na osi y. Dobiveni jajoliki oblik izgleda lijepo, ali nas oko vara. U točki spajanja krivulje imaju zajedničku tangentu, ali su polumjeri zakrivljenosti 0 i ∞. Te nezamjetne pojave sažete su u dva teorema.

Ključne riječi: jajolike krivulje, ptičja jaja, zlatni rez

Some Remarks to the Paper "The Goose that Laid the Golden Egg"

ABSTRACT

The paper [1] induced me to combine the transcendental curves of LAMÉ with my observations on real bird-eggs [2]. Two segments of different curves are connected at junctions points located on the y-axis. The obtained egg-like shape looks nice, but the eye is cheating. It turns out that the curvature of these curves at the junction point has some surprising properties. These imperceptible phenomena are summarized in two theorems.

Key words: bird-egs, egg-shaped curves, golden ratio

MSC 2000: 53B99, 92B05

Introduction

It is natural to determine the form of bird-eggs in order to distinguish and describe the eggs of different bird species. It is an old problem for nature scientists named oologists (zoologists studying bird eggs).

Geometers interpret all bird-eggs as bodies with rotational symmetry. Their longitudinal cross section is an egg-curve [2]. As dimensions we usually use the length l, the width denoted by $2b$, and the axial sections denoted by p and $q = l - p$ $(p \geqslant q)$ [3]. These can exactly be measured. To distinguish bird-eggs the oologists use the quotients p/q and $l/2b$, which only by few bird species can be equal to the golden ratio.

In the following we apply the real measured data on Fibonacci-related functions [1] which give egglike shapes.

This type of egglike curve has the advantage that it can easily be modified in order to meet the measurements given by oologists.

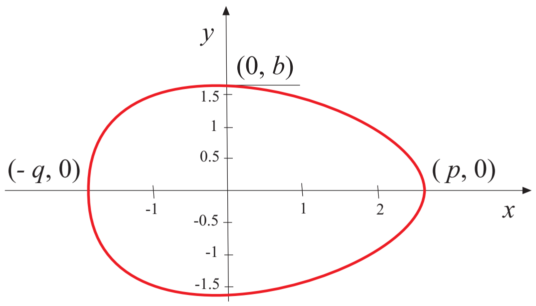


Fig.1 The notation of measured data

1 Basic formulas

In [1] the author used the golden ratio

$$\theta = \frac{\sqrt{5}+1}{2} = 1.618033989$$

for obtaining different shapes of biological forms. Two parts of curves determined by equations

$$\mathbf{a})\ x^{\theta} + y^{\theta} = 1; \quad \mathbf{b})\ x^{\theta^2} + y^{\theta^2} = 1; \quad \mathbf{c})\ x^{\theta} + \left(\frac{y}{\theta}\right)^{\theta} = 1$$

were connected. The junction points of the two segments were located on the y-axis.

Among the presented examples there were also egg-curves, but only of the eggs which have the characteristics $p/q = \theta$, i.e. $l/2b = (1+\theta)/2$.

Now this method will be generalised. For this purpose it is necessary to modify the above equations into

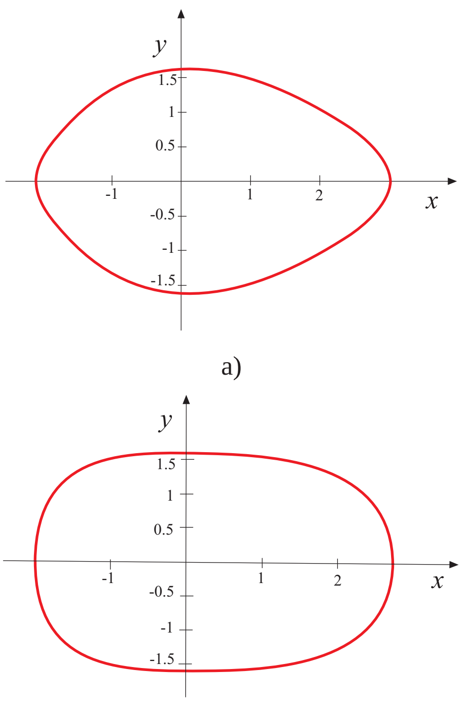
$$\mathbf{b1})\quad \left(\frac{|y|}{b}\right)^{\theta^2} + \left(\frac{|x|}{q}\right)^{\theta^2} = 1,$$

$$\mathbf{c1})\quad \left(\frac{|y|}{b}\right)^{\theta} + \left(\frac{|x|}{p}\right)^{\theta} = 1.$$

(The curves of this kind are known as transcendent curves of LAMÉ.)

If x is running from $-q$ to 0, the plot of $\mathbf{b}1)$ presents the blunt end (see Fig.1, left hand side). For $0 \leqslant x \leqslant p$ the plot of $\mathbf{c}1)$ gives the pointed end of the egg-curve. The x-axis is the axis of symmetry. With the data of the peewit egg (*vanellus vanellus*) $b = 16.45mm$, $p = 26.3mm$, $q = 19mm$ we get the "usual" egg-curve given in Fig.1.

If also the left segment obeys formula $\mathbf{c}1)$, but p is replaced by q and x by $-x$, we get the shape of bird-eggs with two pointed ends in Fig.2a (see "zweispitzige Eier" in [3]). The two segments are corresponding under a perspective affine transformation.



a)

b)

Fig.2 The egg-curve with two pointed ends and two blunt ends

When only the formula $\mathbf{b}1)$ is used, we get the bird-eggs of the form in Fig.2b ("wurstförmige Eier" in [3]).

Both possibilities can also be used for $p = q$. Contrary to the cases presented in [2], we obtain plots which are not ellipses or circles. Since also the exponent θ can vary, these composed egg-curves offer more possibilities than all the others before.

2 About the curvature in the end points

2.1 The right part

If we observe the formulas of the curves composed in Fig.1 we can also make some conclusions about the variation of the curvature. Let us first observe the right part of the curve:

$$|y| = b\left(1 - \left(\frac{x}{p}\right)^{\theta}\right)^{\frac{1}{\theta}}. \tag{1}$$

The first and second derivatives of $y \geqslant 0$ are

$$y' = -\frac{bx^{\theta-1}}{\left(1-\left(\frac{x}{p}\right)^{\theta}\right)^{1-\frac{1}{\theta}} \cdot p^{\theta}} \tag{2}$$

and

$$y'' = \frac{-(\theta-1)b}{p^{\theta}} \cdot \left[\frac{x^{2\theta-2}}{\left(1-\left(\frac{x}{p}\right)^{\theta}\right)^{2-\frac{1}{\theta}} \cdot p} + \frac{1}{\left(1-\left(\frac{x}{p}\right)^{\theta}\right)^{1-\frac{1}{\theta}} \cdot x^{1-\frac{1}{\theta}}}\right]. \tag{3}$$

The first derivative vanishes if $x = 0$. This gives the maximum point.

The denominator of y'' is equal zero if $x = 0$ or $x = p$. It means that the second derivative is undetermined at $x = 0$, but $\lim\limits_{x\to 0} y'' = -\infty$. In the point $(p, 0)$ we have the same situation.

2.2 The left part

The part determined by $\mathbf{b}1)$ for $-q \leqslant x \leqslant 0$, $y \geqslant 0$ reads:

$$y_1 = b\left(1 - \left(\frac{-x}{q}\right)^{\theta^2}\right)^{\frac{1}{\theta^2}}. \tag{4}$$

The derivatives are

$$y_1' = \frac{b(-x)^{\theta^2-1}}{\left(1-\left(\frac{-x}{q}\right)^{\theta^2}\right)^{1-\frac{1}{\theta^2}} q^{\theta^2}} \tag{5}$$

and

$$y_1'' = -\frac{\theta b}{q^{\theta^2}}\left[\frac{(-x)^{2\theta}}{\left(1-\left(\frac{-x}{q}\right)^{\theta^2}\right)^{\theta} q^{\theta^2}} + \frac{(-x)^{\theta^2-2}}{\left(1-\left(\frac{-x}{q}\right)^{\theta^2}\right)^{1-\frac{1}{\theta^2}}}\right]. \tag{6}$$

y_1' vanishes if $x=0$, and it is undetermined for $x=-q$. Exactly the same holds for y_1'' but

$$\lim_{x\to -q+0} y_1' = \infty \quad \text{and} \quad \lim_{x\to -q+0} y_1'' = -\infty.$$

2.3 The curvature radius

The formula for the curvature radius (since $y" < 0$ for all $x \in [-q,p]$) is

$$r = -\frac{|1+y'|^{3/2}}{y''} \tag{7}$$

and, on the right hand part of the egg-curve, gives $r=0$ if $x=0$ or $x=p$. It means that at the observed points the curvature $c=1/r$ of the right part tends forwards infinity.

Substitution of y_1' and y_1'' into the definition (7) shows that the curvature radius of the left part at the points $x=0$ and $x=-q$ tends to infinity.

We summarise this surprising facts in

Theorem 1 *In the Fig.1 at the point $(0,b)$ the two curves have the same tangent, but the curvature radii as well as the curvatures are extremely different.*

Theorem 2 *The composed egg-curve given in Fig.1 has at the blunt end $(-q,0)$ the curvature radius $r=\infty$, at the pointed end $(p,0)$ is $r=0$.*

REMARK:

In Fig.1 the curvature radius at the left side appear the minimal value $r \sim 1.58$ in $x \sim -1.6$, and the maximal value of the right side $r \sim 4.2$ in $x \sim 1.1$.

The investigation of the other Fibonacci-related functions listed in [1] is advisable using data appearing in the nature.

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