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BLAŽENKA DIVJAK

Notes on Taxicab Geometry

Bilješke o taxicab geometriji

SAŽETAK

Taxicab geometrija jedna je od neeuklidskih geometrija. Tu je geometriju još prije 100 godina razmatrao Hermann Minkowski (Gesammelte Abhandlungen), a od tada doživljavala je periode marginalizacije i gotovo zaborava, ali i periode velikog zanimanja i široke primjene. Danas se može pronaći čitav spektar upotrebe i primjene taxicab geometrije. Za to postoji više razloga.

Prvo, taxicab geometrija bliska je euklidskoj geometriji i lagana za razumijevanje. Može se promatrati kao metrički sustav u kojem točke korespondiraju križanjima ulica u zamišljenom gradu gdje ulice idu samo horizontalno i vertikalno i nema jednosmjernih ulica (odatle i naziv taxicab geometrija). Kao takva pogodna je za izučavanje na dodiplomskom studiju u obliku eseja, seminarskih radova i diplomskih radova (kao što je opisano u [7] i [9]).

Drugo, taxicab geometrija zanimljiva je za izučavanje i sa stanovišta teorijske geometrije. Moguće ju je opisati upotrebom sintetičkog pristupa (koji je uveo David Hilbert), ali i metričkim pristupom (za koji je zaslužan George David Birkhoff). Oba ova pristupa pojašnjena su i upotrebljena za uvođenje taxicab geometrije u [6]. Postoji i treći pristup u geometriji preko apstraktne algebre i teorije grupa koji su uveli Felix Klein i Arthur Cayley koji tvrdi da se geometrija treba proučavati preko djelovanja grupe gibanja na zadani skup. Nadalje, dokazat ćemo neke poučke o elipsi u taxicab geometriji.

Treće, praktična je vrijednost taxicab geometrije njezina široka primjenjivost na stvarne (urbane) probleme transporta, planiranja gradova itd. O tim primjena govori se npr. u [10].

Ključne riječi: neeuklidska geometrija, taxicab geometrija

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Notes on Taxicab Geometry

ABSTRACT

The taxicab geometry is one of non-Euclidean geometries. Hermann Minkowski (Gesammelte Abhandlungen) introduced this geometry more than 100 years ago. In this hundred years there were the periods of marginalization and the periods of great interest and wide application of this geometry. Today there is a whole specter of application and implementation of the taxicab geometry. There are several reasons for this.

First, the taxicab geometry is similar to Euclidean geometry and easy to understand. It can be observed as such a metric system where the points correspond to the intersections of the streets in the imagined city, streets run only horizontally and vertically. There are no one-way streets. From the previous description the name taxicab geometry arises. The taxicab geometry is appropriate to discuss out during the undergraduate study in the form of essays, seminar works and diploma thesis as it is described in [7] and [9].

Second, the taxicab geometry is interesting for theoretical geometry study, too. It can be analyzed by synthetic approach (introduced by David Hilbert), or by metric approach (described by George David Birkhoff). Mentioned approaches are described and discussed in [6]. There is the third approach in geometry using abstract groups and group theory. This approach was introduced by Felix Klein and Arthur Cayley. They claimed that geometry had to be studied through acting the group of motions on the given set. Further, some propositions about ellipses in the taxicab geometry are proved.

Third, the practical value of the taxicab geometry is its wide application in (urban) transportation problems, city planning and so on. This application has been described in [10].

Key words: non-Euclidean geometry, taxicab geometry

1 Introduction to taxicab geometry

Let's imagine an ideal city in which the streets run only horizontally and vertically and city blocks have the same size. The intersections of streets correspond to the points in space. Now we would like to take a taxi and drive as fast as possible from the point $A(x_1,y_1)$ to the point $B(x_2,y_2)$. To achieve this goal, it would be reasonable to measure the distance between points A and B. We can use Euclidean ("school") way to compute the distance. Then distance between A and B is

$$d(A,B)=\sqrt{(x_2-x_1)^2+(y_2-y_1)^2}. \tag{1}$$

But this number show us the distance between A and B as a bird flies and it is useless for us if we take a taxi. So we use another measure so called **taxicab distance** which counts the number of blocks we would have to travel to get from A to B. Obviously for taxicab distance between A and B the following formula holds

$$d_T(A,B)=|x_1-x_2|+|y_1-y_2|. \tag{2}$$

According to the above description we could define the taxicab geometry as the metric system $(\mathbb{Z}^2,d_T)$ where points are represented by the ordered pairs of whole numbers. This definition is appropriate for student's use in a way it will be described in section 3.

For pure mathematical research the taxicab geometry is defined on $\mathbb{R}^2$ and it will be introduced like this in the next section.

As it often happens, the approach which is not interesting for pure mathematical research is more useful.

2 Approaches to taxicab geometry

There are three fundamental approaches to geometry:

1. synthetic approach,
2. metric approach,
3. group approach.

2.1 Metric approach

In the **metric approach** the concepts of distance and angle measure are added to an incidence geometry. American mathematician G. D. Birkhoff is founder of this approach (1932).

Let we explain metric approach briefly (it is done in details in [1]).

Definition 1 *An **abstract geometry** $\mathcal{A}$ consists of a set S (set of points) and a collection L of non-empty subsets of S (collection of lines) such that:*

1. *$\forall A,B\in S\Longrightarrow\exists l\in L$ with $A\in l$ and $B\in l$.*
2. *Every line has at least two points.*

Proposition 2 *The **Cartesian plane** is a model $\mathcal{C}=\{\mathbb{R}^2,L_e\}$, where L_e is a collection of vertical $(x=a)$ and non-vertical $(y=ax+b)$ lines of $\mathbb{R}^2$. $\mathcal{C}$ is an abstract geometry.*

Definition 3 *An abstract geometry $\{\mathcal{S},\mathcal{L}\}$ is an **incidence geometry** if*

1. *Every two distinct points in $\mathcal{S}$ lie on a unique line.*
2. *There exits three points which do not lie on one line.*

Proposition 4 *The Cartesian plane is an incidence geometry.*

Definition 5 *A **distance function** on a set $\mathcal{S}$ is a function $d:\mathcal{S}\times\mathcal{S}\to\mathbb{R}$ such that for all $A,B\in\mathcal{S}$*

1. *$d(A,B)\geq 0$;*
2. *$d(A,B)=0\Leftrightarrow A=B$;*
3. *$d(A,B)=d(B,A)$.*

Proposition 6 *The **taxicab distance** $d:\mathbb{R}^2\times\mathbb{R}^2\to\mathbb{R}$ given by*

$$d_T(A,B)=|x_1-x_2|+|y_1-y_2| \tag{3}$$

where $A=(x_1,y_2)$ and $B=(x_1,y_2)$, is a distance function on $\mathbb{R}^2$.

Definition 7 *Let l be a line in an incidence geometry $\{\mathcal{S},\mathcal{L}\}$. Assume that there is a distance function d on $\mathcal{S}$. A function $f:l\to\mathbb{R}$ is a **ruler** (or **coordinate system**) for l if*

1. f is a bijection;
2. $\forall A,B\in l$

$$|f(A)-f(B)|=d(A,B) \tag{4}$$

(**Ruler equation**).

Definition 8 *An incidence geometry $\{\mathcal{S},\mathcal{L}\}$ together with a distance function d satisfies the ruler postulate if every line $l\in\mathcal{S}$ has a ruler. In this case we say $\mathcal{M}=\{\mathcal{S},\mathcal{L},d\}$ is a **metric geometry**.*

Proposition 9 *The Cartesian plane with the taxicab distance is a metric geometry.*

Proof. Ruler for vertical line $x = 0$ is $f((a,y)) = y$.

Ruler for nonvertical line $y = ax + b$ is $f((x,y)) = (1+|a|)(x)$. ■

Definition 10 *The model* $T = \{\mathbb{R}^2, L_e, d_T\}$ *is called the* ***taxicab plane****.*

By using the distance function we define betweenness and betweenness allow us to define elementary figures such as segments, rays, angles and triangles.

Definition 11 *B is* ***between*** *A and C if A,B and C are distinct collinear points in the metric geometry* $\{S, L, d\}$ *and if*

$$d(A,B) + d(B,C) = d(A,C). \tag{5}$$

Remark 12 *In the taxicab plane we can find three points which are not collinear but which satisfy the equation* [5]*. For example if we take* $A(-1,-1), B(0,0), C(2,1)$ *the equation* [5] *is satisfied but these three point are not collinear. This shows why the definition of between requires collinear points.*

Definition 13 *Let* $\{S, L, d\}$ *be a metric geometry and let* $S_1 \subset S$. S_1 *is said to be convex if for every two points P, Q* $\in S_1$*, the segment* $\overline{PQ}$ *is a subset of* S_1*.*

Definition 14 *We say that a metric geometry* $\{S, L, d\}$ *satisfies the plane separation axiom (PSA) if for every* $l \in L$*, there are two sets* H_1 *and* H_2 *of* S *such that*

1. S-l=$H_1 \cup H_2$;
2. H_1 *and* H_2 *are disjoint and each is convex;*
3. *If* $A \in H_1$ *and* $B \in H_2$ *then* $\overline{AB} \cap l \neq \emptyset$.

Definition 15 *A* ***Pasch geometry*** *is a metric geometry which satisfies PSA.*

Proposition 16 *The Taxicab geometry is a Pasch geometry.*

Definition 17 *Let* r_0 *be a fixed positive number. In a Pasch geometry, an angle measure (or protractor) based on* r_0 *is a function m from the set* $\mathcal{A}$ *of all angles to the set of real numbers such that*

1. *If* $\angle ABC \in \mathcal{A}$ *then* $0 < m(\angle ABC) < r_0$;
2. *If* $\overrightarrow{AB}$ *lies in the edge of the half plane* H_1 *and if* θ *is a real number with* $0 < \theta < r_0$*, then there is a unique ray* $\overrightarrow{BC}$ *with* $C \in H_1$ *and* $m(\angle ABC) = \theta$;
3. *If* $D \in int(\angle ABC)$ *then*

$$m(\angle ABD) + m(\angle DBC) = m(\angle ABC). \tag{6}$$

Definition 18 *A* ***protractor geometry*** $\{S, L, d, m\}$ *is a Pasch geometry* $\{S, L, d\}$ *together with an angle measure m.*

Proposition 19 *Assume that* m_e *is an angle measure for Euclidean metric geometry. Then* m_e *is an angle measure for the Taxicab plane* $\{\mathbb{R}^2, L_e, d_T\}$*. So,* $\{\mathbb{R}^2, L_e, d_T, m_e\}$ *is a protractor geometry.*

There is another concept of the angle measure in Taxicab plane as it is introduced in [2]. We will briefly describe it here.

Definition 20 *A* ***t-radian*** *is an angle whose vertex is the center of a unit (taxicab) circle and which intercepts an arc of length 1. The taxicab measure of a taxicab angle* θ *is the number of t-radians subtended by the angle on the unit taxicab circle about the vertex.*

It follows that a taxicab unit circle has 8 t-radians and so we can say that in taxicab geometry $\pi = 4$. This taxicab angle measure has a nice application to parallax (the apparent shift of an object due to the motion of the observer). The parallax is used for estimating the distance to a nearby object. In the article is discovered that the taxicab method yields the same formula commonly used in the Euclidean case with the exception that the taxicab formula is exact.

Definition 21 *A* ***neutral geometry*** *(or absolute geometry) is a protractor geometry which satisfies Side-Angle-Side Axiom (SAS).*

The Taxicab plane is not a neutral geometry.

2.2 Synthetic approach

In **synthetic** (or axiomatic) **approach** we have to decide what are the important properties of the geometry and then define these properties axiomatically. This approach was first used in Euclid's famous Elements but was introduced completely by D. Hilbert in 1899, who put several mathematical areas on axiomatic ground.

We have to choose axioms first and in this some criterions should be obeyed (correspondence to an intuitive picture, richness of the theory and consistency). In the synthetic approach, congruence axioms replace distance and angular measure.

2.3 Group approach

The **group approach** was introduced by F. Klein and A. Cayley and involved transformation group of a geometry. Then geometry studies those properties which are invariant under the group of motions.

The taxicab metric in $\mathbb{R}^2$ is an interesting example of non-Euclidean metric and it is a special case of the more general Minkowski metric where the defining unit circle is a certain symmetric closed convex curve. It has been shown in [6] that the respective isometries in taxicab space form a subgroup of the whole space.

3 Taxicab in classroom

Why to introduce taxicab in classroom?

Let me cite some experts.

1. B. Grünbaum, J. Mycielski: ”Undergraduate courses on the foundations of geometry need examples different from the traditional Euclidean plane in order to give students experience in dealing with axiomatic systems.” ([7])

2. M. Artigue: ”...much of our knowledge remains strongly contextual, that is, dependent on the situations from which it arises.”

   ”...learning in mathematics is not a continuous process, (it is necessity sometimes to make) breaks with earlier knowledge and models of thought.” ([8])

The main problem in introducing geometrical systems different from Euclidean on technical faculties is a program frame. In that case solution is mathematical essay or diploma work, but then with special emphasis on applications aspects of taxicab geometry (see [2] and[4]).

On mathematical faculties it is recommended for students to investigate some theoretical aspects, such as circles (as it is done in [5]) or other curves of second order in taxicab geometry (see [3]). It will be easier, and for this maybe more motivating, to take ”student's version” of taxicab geometry (points are from $\mathbb{Z}^2$).

In addition some results of such investigation on ellipse will be mentioned.

Definition 22 *Let $F_1(-c,0), F_2(c,0)$ be two point in $(\mathbb{Z}^2, d_T)$ (focuses). The ellipse E is the set of points $T(x,y)$ from $\mathbb{Z}^2$ such that*

$$|d_T(F_1,T) + d_T(F_1,T)| = a, a \in \mathbb{Z}. \tag{7}$$

Proposition 23 *The number of points on an ellipse E is $2a$.*

Definition 24 *For an ellipse E the halfaxis b is*

$$b = \max\{|y_2 - y_1|, \text{ for all } A(x_1,y_2), B(x_2,y_2) \in E\}. \tag{8}$$

Proposition 25 *For an ellipse E the following holds*

$$a = b + c. \tag{9}$$

4 Applications of the taxicab geometry

At the end, let us say a few facts about applications of the taxicab geometry in a real world problems.

In the real world the shortest distance between two places for trains, planes and even for plains, is not as a bird flies (Euclidean distance). For example, trains want to take the shortest route possible from one place to another. The lattice points with respect to taxicab geometry would be the depots or turns of the track. We want to consider another variable to travel along taxicab distance, so called taxicab time. But sometimes it is not good enough, because the shortest route is selected thus **minimizing taxicab distance** and minimizing costs. The minimal taxicab distance, time or costs can be determined by the same method described in [9].

Let us explain another natural problem for taxicab geometry. A taxi company has a number of stations across the city. In order to dispatch the calls, the company want to know for any point in the city, which station is the closest, so that the taxi can get there as fast as possible. Obviously, our task is to find the locus of points which are equidistant to their two closest taxi stations. These boundaries will tessellate the plane in regions. In Euclidean geometry, the corresponding graph is called **Vornoi diagram** and is widely used in computational geometry and pattern recognition (see [10]). This task can be a interesting diploma work on mathematical or technical oriented faculty (a Java program can be demanded).

A city planning makes also attractive use of taxicab geometry. From the historical point of view there are two types of cities: organically grown cities and planned cities (see Fig. 1 and Fig. 2). Organically grown cities develop slowly, tend to plan around preserving their natural beauty and have unstructured design with curved shapes and sporadic placements of streets, buildings and landscape (early European cities).

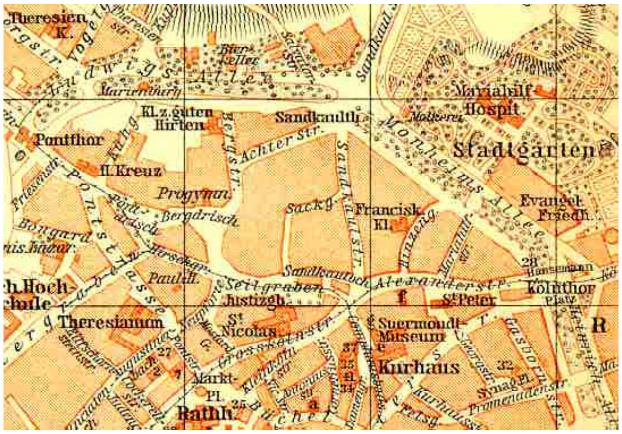

Figure 1: Aachen - the example of a “spider” city.

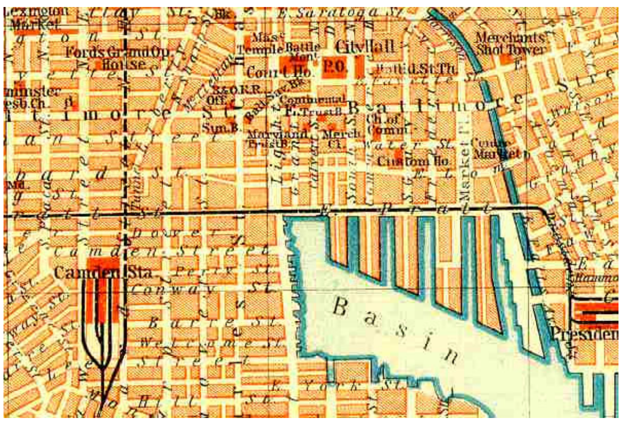

Figure 2: Baltimore - the example of a planned city.

The growth and plans of planned cities are on ”high levels” coordinated with a population growth (predicted mathematically on the basis of cohort survival method). The patterns in the planned cities are more vertical and horizontal (and more suitable for application of taxicab geometry). In the 1960s the central focus of city planning moved from artistic beauty of the city towards location and efficiency.

But even for the curved streets we can set up models such as the shortest rout problem or Vornoi diagrams. The curved streets can be represent as a straight edge form with the distance used as edge weighs and the street intersections as the nodes. Besides a distance we have to consider traffic volume, speed limits, traffic signals and so on.

Additionally, the street patterns tells a lot about the city’s history.

We can conclude by noticing that parts of city planning involve the use of geometric models and mathematical models in general. Taxicab geometry is used in many ways for planning city and the traversed routes between and through cities. ”Taxicab geometry is a great model for the artificially grid planned world that man has created” (cite [4]).

References

[1] Millman R. S.,Parker G.D.; *Geometry: A Metric Approach with Models*, Springer-Verlag, New York (1991).

[2] Thompson K., Dray T.; Taxicab Angles and Trigonometry, *Pi Mu Epsilon Journal* (2000) *(*to apear)

[3] Prevost F.J.; The conic sections in taxicab geometry: Some investigations for high school students, *The Mathematics Theachers, Reston,* Apr (1998).

[4] http://www2.gvsu.edu/~vanbelkj/Project.html

[5] Tian S., So S., Chen G.; Concerning circles in taxicab geometry, *International Journal of Mathematical Education in Science and Technology*, Sep/Oct (1997).

[6] Greenberg M.;*Euclidean and Non-Euclidean Geometry*, Freeman, San Francisco (1980).

[7] Grünbaum B., Mycielski J.; Some models of Plane Geometries, *AMM*, Vol 97, No9, Nov (1990) pp.839-847.

[8] Artigue M.; The Teaching and Learning of Mathematics at the University Level: Crucial Questions for Contemporary Reasearh in Education, *Notices of the AMS*, Dec (1999).

[9] Hesse R., Gene W.; *Applied Management Science A Quick & Dirty*

Approach, Science Research Associates, Inc. (1980), pp. 96-99.

[10] Graham R., Yao F.; A Whirlwind of Computational Geometry, *The American Mathematical Monthly*, Vol. 97, No. 8, Oct (1990), pp. 687-702.

Dr. sc. Blaženka Divjak
University of Zagreb
Faculty of Organization and Informatics
e-mail: bdivjak@foi.hr