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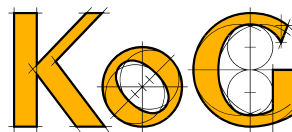
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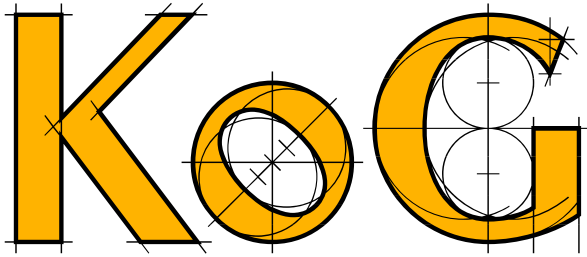
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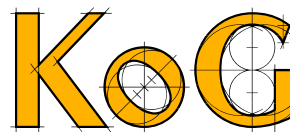
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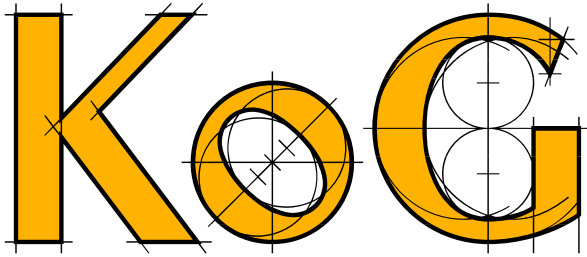
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## U povodu izlaska 5. broja KoG-a

Moglo bi se reći da kasnimo. Broj za 2000. godinu trebao je kao i oni koji su mu prethodili izaći još u veljači. S druge bi se strane moglo reći da on izlazi i nešto ranije. Naime, veći dio članaka primljen je i prihvaćen ove godine. Zbog cijelog niza razloga, uglavnom vezanih uz organizaciju rada uredništva, lipanj se čini pogodnijim i za izdavanje sljedećih brojeva. Nadamo se da će se Ministarstvo znanosti i tehnologije Republike Hrvatske, uz čiju se financijsku podršku časopis izdaje, suglasiti s našom odlukom da termin izlaska časopisa KoG ubuduće bude lipanj.

Novost je i digitalni tisak, koji je za gotovo istu cijenu omogućio izdavanje časopisa u boji. Vjerujemo da će ta tehnologija, svakako primjerena časopisu s područja geometrije i grafike, u narednim godinama znatno pridonijeti njegovoj kvaliteti u svakom pogledu.

U prvom broju KoG-a prof. Vlasta Ščurić, tadašnja predsjednica Hrvatskog društva za konstruktivnu geometriju i kompjutorsku grafiku (HDKGIKG), na kraju uvodnog teksta *Osnivanje Društva i poziv na suradnju*, kaže: "KoG je znanstveno-stručno informativni časopis našega Društva. U rukama držite njegov prvi broj, koji je rezultat volonterskog rada i entuzijazma sadašnjeg članstva. Koristim priliku da izrazim zahvalnost recenzentima i svima koji su znanjem pridonijeli njegovu sadržaju. Planiram da časopis izlazi jednom godišnje. Prihvaćamo sugestije i radove te pozivamo na različite oblike stručne suradnje sve one koji se bave geometrijom i grafikom."

Danas, nakon pet godina, na našim je čitateljima i suradnicima da prosude koliko smo i kako bili na tragu onoga što smo planirali, jesmo li u tome imali uspjeha i jesmo li dovoljno otvoreni za traženje novog u našem području. S naše bi strane valjalo sada konkretizirati kome upućujemo svoju zahvalnost, istaknuti one čija se imena ne nalaze u impresumu niti na stranicama KoG-a, a bez čije bi podrške i znanja njegovo izdavanje graničilo s nemogućim. Dakle, zahvaljujemo svim članovima geometrijskog seminara Matematičkog odjela PMF-a u Zagrebu, kojeg je voditelj prof. Mirko Polonijo, posebno prof. Vladimiru Volencu, na čije smo se dugogodišnje iskustvo s uređivanjem znanstvenih i stručnih časopisa uvijek mogli osloniti, prof. Hellmuthu Stachelu (Institut za geometriju Tehničkog sveučilišta iz Beča), prof. Emilu Molnáru (Institut za matematiku Tehničkog sveučilišta iz Budimpešte) te prof. Gunteru Weissu (Institut za geometriju Tehničkog sveučilišta u Dresdenu).

**Izdavački savjet**

## The journal KoG issue No 5.

It could be said we are behind the schedule. The issue for the year 2000 should have been published, as it was in previous years, in February. Yet it could be said it was published ahead of time. Namely, we received and accepted most articles this year. Besides, for a number of other organisational reasons related to the work of the editorial board, June proved to be more convenient for publishing. Therefore we hope that the Ministry of Science and Technology of the Republic of Croatia, which provides funds for the publishing of the journal, will support our decision to change the publishing date of the journal KoG.

Digital printing has been introduced as a technical novelty in this issue and has allowed us to print the journal in colour almost without increasing the costs. We believe that this technology, certainly appropriate for a journal on geometry and graphics, will contribute a great deal to raising the quality of the journal in the years to come.

In the first issue of KoG, professor Vlasta Ščurić, the chairperson of the Croatian Society for Constructive Geometry and Computer Graphics at the time, at the end of her editorial said: "KoG is a scientific professional and informative journal of our Society. The first issue that you are holding in your hands is the result of voluntary work and great enthusiasm of our current members. Let me express my gratitude to reviewers and all contributors who enriched its content by their knowledge. According to our plan the journal will be issued annually. We are open to your suggestions, we welcome your articles and we invite all those engaged in the field of geometry and graphics to various forms of professional cooperation." Today, five years later, let our readers and collaborators judge whether our work has been along the lines of the goal set five years ago, whether we have been successful and open enough in the search of the new in the realm of geometry and graphics. It is upon us to express our gratitude to the people, whose names are not on the editorial or other pages of KoG, but without whose unfailing helpfulness and knowledge the publishing of this journal would be almost impossible. Hence, we would like to thank all the members of the Geometrical Seminar of the Department of Mathematics - University of Zagreb, presided by professor Mirko Polonijo, we thank particularly professor Vladimir Volenec for his willingness to share his professional knowledge based on his long experience in editing scientific and professional journals, professor Hellmuth Stachel (Institute of Geometry, Vienna University of Technology), professor Emil Molnar (Institute of mathematics, Technical University of Budapest) and professor Gunter Weiss (Institute of Geometry, Technical University of Dresden).

**Editorial Board**

Review

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BLAŽENKA DIVJAK

# Notes on Taxicab Geometry

## Bilješke o taxicab geometriji

### SAŽETAK

Taxicab geometrija jedna je od neeuklidskih geometrija. Tu je geometriju još prije 100 godina razmatrao Hermann Minkowski (Gesammelte Abhandlungen), a od tada doživljavala je periode marginalizacije i gotovo zaborava, ali i periode velikog zanimanja i široke primjene. Danas se može pronaći čitav spektar upotrebe i primjene taxicab geometrije. Za to postoji više razloga.

Prvo, taxicab geometrija bliska je euklidskoj geometriji i lagana za razumijevanje. Može se promatrati kao metrički sustav u kojem točke korespondiraju križanjima ulica u zamišljenom gradu gdje ulice idu samo horizontalno i vertikalno i nema jednosmjernih ulica (odatle i naziv taxicab geometrija). Kao takva pogodna je za izučavanje na dodiplomskom studiju u obliku eseja, seminarskih radova i diplomskih radova (kao što je opisano u [7] i [9]).

Drugo, taxicab geometrija zanimljiva je za izučavanje i sa stanovišta teorijske geometrije. Moguće ju je opisati upotrebom sintetičkog pristupa (koji je uveo David Hilbert), ali i metričkim pristupom (za koji je zaslužan George David Birkhoff). Oba ova pristupa pojašnjena su i upotrebljena za uvođenje taxicab geometrije u [6]. Postoji i treći pristup u geometriji preko apstraktne algebre i teorije grupa koji su uveli Felix Klein i Arthur Cayley koji tvrdi da se geometrija treba proučavati preko djelovanja grupe gibanja na zadani skup. Nadalje, dokazat ćemo neke poučke o elipsi u taxicab geometriji.

Treće, praktična je vrijednost taxicab geometrije njezina široka primjenjivost na stvarne (urbane) probleme transporta, planiranja gradova itd. O tim primjena govori se npr. u [10].

**Gljučne riječi:** neeuklidska geometrija, taxicab geometrija

**MSC 2000:** 53A35, 51K99, 97D30

## Notes on Taxicab Geometry

### ABSTRACT

The taxicab geometry is one of non-Euclidean geometries. Hermann Minkowski (Gesammelte Abhandlungen) introduced this geometry more than 100 years ago. In this hundred years there were the periods of marginalization and the periods of great interest and wide application of this geometry. Today there is a whole specter of application and implementation of the taxicab geometry. There are several reasons for this.

First, the taxicab geometry is similar to Euclidean geometry and easy to understand. It can be observed as such a metric system where the points correspond to the intersections of the streets in the imagined city, streets run only horizontally and vertically. There are no one-way streets. From the previous description the name taxicab geometry arises. The taxicab geometry is appropriate to discuss out during the undergraduate study in the form of essays, seminar works and diploma thesis as it is described in [7] and [9].

Second, the taxicab geometry is interesting for theoretical geometry study, too. It can be analyzed by synthetic approach (introduced by David Hilbert), or by metric approach (described by George David Birkhoff). Mentioned approaches are described and discussed in [6]. There is the third approach in geometry using abstract groups and group theory. This approach was introduced by Felix Klein and Arthur Cayley. They claimed that geometry had to be studied through acting the group of motions on the given set. Further, some propositions about ellipses in the taxicab geometry are proved.

Third, the practical value of the taxicab geometry is its wide application in (urban) transportation problems, city planning and so on. This application has been described in [10].

**Key words:** non-Euclidean geometry, taxicab geometry

## 1 Introduction to taxicab geometry

Let's imagine an ideal city in which the streets run only horizontally and vertically and city blocks have the same size. The intersections of streets correspond to the points

in space. Now we would like to take a taxi and drive as fast as possible from the point  $A(x_1, y_1)$  to the point  $B(x_2, y_2)$ . To achieve this goal, it would be reasonable to measure the distance between points  $A$  and  $B$ . We can use Euclidean ("school") way to compute the distance. Then distance be-

tween  $A$  and  $B$  is

$$d(A, B) = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}. \quad (1)$$

But this number show us the distance between  $A$  and  $B$  as a bird flies and it is useless for us if we take a taxi. So we use another measure so called **taxicab distance** which counts the number of blocks we would have to travel to get from  $A$  to  $B$ . Obviously for taxicab distance between  $A$  and  $B$  the following formula holds

$$d_T(A, B) = |x_1 - x_2| + |y_1 - y_2|. \quad (2)$$

According to the above description we could define the taxicab geometry as the metric system  $(\mathbb{Z}^2, d_T)$  where points are represented by the ordered pairs of whole numbers. This definition is appropriate for student's use in a way it will be described in section 3.

For pure mathematical research the taxicab geometry is defined on  $\mathbb{R}^2$  and it will be introduced like this in the next section.

As it often happens, the approach which is not interesting for pure mathematical research is more useful.

## 2 Approaches to taxicab geometry

There are three fundamental approaches to geometry:

1. synthetic approach,
2. metric approach,
3. group approach.

### 2.1 Metric approach

In the **metric approach** the concepts of distance and angle measure are added to an incidence geometry. American mathematician G. D. Birkhoff is founder of this approach (1932).

Let we explain metric approach briefly (it is done in details in [1]).

**Definition 1** An **abstract geometry**  $A$  consists of a set  $S$  (set of points) and a collection  $L$  of non-empty subsets of  $S$  (collection of lines) such that:

1.  $\forall A, B \in S \implies \exists l \in L$  with  $A \in l$  and  $B \in l$ .
2. Every line has at least two points.

**Proposition 2** The **Cartesian plane** is a model  $C = \{\mathbb{R}^2, L_e\}$ , where  $L_e$  is a collection of vertical ( $x = a$ ) and non-vertical ( $y = ax + b$ ) lines of  $\mathbb{R}^2$ .  $C$  is an abstract geometry.

**Definition 3** An abstract geometry  $\{S, L\}$  is an **incidence geometry** if

1. Every two distinct points in  $S$  lie on a unique line.
2. There exists three points which do not lie on one line.

**Proposition 4** The Cartesian plane is an incidence geometry.

**Definition 5** A **distance function** on a set  $S$  is a function  $d: S \times S \rightarrow \mathbb{R}$  such that for all  $A, B \in S$

1.  $d(A, B) \geq 0$ ;
2.  $d(A, B) = 0 \Leftrightarrow A = B$ ;
3.  $d(A, B) = d(B, A)$ .

**Proposition 6** The **taxicab distance**  $d: \mathbb{R}^2 \times \mathbb{R}^2 \rightarrow \mathbb{R}$  given by

$$d_T(A, B) = |x_1 - x_2| + |y_1 - y_2| \quad (3)$$

where  $A = (x_1, y_1)$  and  $B = (x_2, y_2)$ , is a distance function on  $\mathbb{R}^2$ .

**Definition 7** Let  $l$  be a line in an incidence geometry  $\{S, L\}$ . Assume that there is a distance function  $d$  on  $S$ . A function  $f: l \rightarrow \mathbb{R}$  is a **ruler** (or **coordinate system**) for  $l$  if

1.  $f$  is a bijection;
2.  $\forall A, B \in l$

$$|f(A) - f(B)| = d(A, B) \quad (4)$$

(**Ruler equation**).

**Definition 8** An incidence geometry  $\{S, L\}$  together with a distance function  $d$  satisfies the ruler postulate if every line  $l \in S$  has a ruler. In this case we say  $M = \{S, L, d\}$  is a **metric geometry**.

**Proposition 9** The Cartesian plane with the taxicab distance is a metric geometry.



**Proof.** Ruler for vertical line  $x = 0$  is  $f((a, y)) = y$ .

Ruler for nonvertical line  $y = ax + b$  is  $f((x, y)) = (1 + |a|)(x)$ . ■

**Definition 10** The model  $T = \{\mathbb{R}^2, L_e, d_T\}$  is called the *taxicab plane*.

By using the distance function we define betweenness and betweenness allow us to define elementary figures such as segments, rays, angles and triangles.

**Definition 11**  $B$  is *between*  $A$  and  $C$  if  $A, B$  and  $C$  are distinct collinear points in the metric geometry  $\{S, L, d\}$  and if

$$d(A, B) + d(B, C) = d(A, C). \quad (5)$$

**Remark 12** In the taxicab plane we can find three points which are not collinear but which satisfy the equation [5]. For example if we take  $A(-1, -1), B(0, 0), C(2, 1)$  the equation [5] is satisfied but these three point are not collinear. This shows why the definition of between requires collinear points.

**Definition 13** Let  $\{S, L, d\}$  be a metric geometry and let  $S_1 \subset S$ .  $S_1$  is said to be *convex* if for every two points  $P, Q \in S_1$ , the segment  $PQ$  is a subset of  $S_1$ .

**Definition 14** We say that a metric geometry  $\{S, L, d\}$  satisfies the *plane separation axiom (PSA)* if for every  $l \in L$ , there are two sets  $H_1$  and  $H_2$  of  $S$  such that

1.  $S - l = H_1 \cup H_2$ ;
2.  $H_1$  and  $H_2$  are disjoint and each is convex;
3. If  $A \in H_1$  and  $B \in H_2$  then  $\overline{AB} \cap l \neq \emptyset$ .

**Definition 15** A *Pasch geometry* is a metric geometry which satisfies PSA.

**Proposition 16** The Taxicab geometry is a Pasch geometry.

**Definition 17** Let  $r_0$  be a fixed positive number. In a Pasch geometry, an *angle measure (or protractor)* based on  $r_0$  is a function  $m$  from the set  $A$  of all angles to the set of real numbers such that

1. If  $\angle ABC \in A$  then  $0 < m(\angle ABC) < r_0$ ;
2. If  $\overrightarrow{AB}$  lies in the edge of the half plane  $H_1$  and if  $\theta$  is a real number with  $0 < \theta < r_0$ , then there is a unique ray  $\overrightarrow{BC}$  with  $C \in H_1$  and  $m(\angle ABC) = \theta$ ;

3. If  $D \in \text{int}(\angle ABC)$  then

$$m(\angle ABD) + m(\angle DBC) = m(\angle ABC). \quad (6)$$

**Definition 18** A *protractor geometry*  $\{S, L, d, m\}$  is a Pasch geometry  $\{S, L, d\}$  together with an angle measure  $m$ .

**Proposition 19** Assume that  $m_e$  is an angle measure for Euclidean metric geometry. Then  $m_e$  is an angle measure for the Taxicab plane  $\{\mathbb{R}^2, L_e, d_T\}$ . So,  $\{\mathbb{R}^2, L_e, d_T, m_e\}$  is a protractor geometry.

There is another concept of the angle measure in Taxicab plane as it is introduced in [2]. We will briefly describe it here.

**Definition 20** A *t-radian* is an angle whose vertex is the center of a unit (taxicab) circle and which intercepts an arc of length 1. The taxicab measure of a taxicab angle  $\theta$  is the number of t-radians subtended by the angle on the unit taxicab circle about the vertex.

It follows that a taxicab unit circle has 8 t-radians and so we can say that in taxicab geometry  $\pi = 4$ . This taxicab angle measure has a nice application to parallax (the apparent shift of an object due to the motion of the observer). The parallax is used for estimating the distance to a nearby object. In the article is discovered that the taxicab method yields the same formula commonly used in the Euclidean case with the exception that the taxicab formula is exact.

**Definition 21** A *neutral geometry (or absolute geometry)* is a protractor geometry which satisfies Side-Angle-Side Axiom (SAS).

The Taxicab plane is not a neutral geometry.

## 2.2 Synthetic approach

In **synthetic (or axiomatic) approach** we have to decide what are the important properties of the geometry and then define these properties axiomatically. This approach was first used in Euclid's famous Elements but was introduced completely by D. Hilbert in 1899, who put several mathematical areas on axiomatic ground.

We have to choose axioms first and in this some criterions should be obeyed (correspondence to an intuitive picture, richness of the theory and consistency). In the synthetic approach, congruence axioms replace distance and angular measure.

### 2.3 Group approach

The **group approach** was introduced by F. Klein and A. Cayley and involved transformation group of a geometry. Then geometry studies those properties which are invariant under the group of motions.

The taxicab metric in  $\mathbb{R}^2$  is an interesting example of non-Euclidean metric and it is a special case of the more general Minkowski metric where the defining unit circle is a certain symmetric closed convex curve. It has been shown in [6] that the respective isometries in taxicab space form a subgroup of the whole space.

## 3 Taxicab in classroom

Why to introduce taxicab in classroom?

Let me cite some experts.

1. B. Grünbaum, J. Mycielski: "Undergraduate courses on the foundations of geometry need examples different from the traditional Euclidean plane in order to give students experience in dealing with axiomatic systems." ([7])
2. M. Artigue: "...much of our knowledge remains strongly contextual, that is, dependent on the situations from which it arises."  
"...learning in mathematics is not a continuous process, (it is necessity sometimes to make) breaks with earlier knowledge and models of thought." ([8])

The main problem in introducing geometrical systems different from Euclidean on technical faculties is a program frame. In that case solution is mathematical essay or diploma work, but then with special emphasis on applications aspects of taxicab geometry (see [2] and [4]).

On mathematical faculties it is recommended for students to investigate some theoretical aspects, such as circles (as it is done in [5]) or other curves of second order in taxicab geometry (see [3]). It will be easier, and for this maybe more motivating, to take "student's version" of taxicab geometry (points are from  $\mathbb{Z}^2$ ).

In addition some results of such investigation on ellipse will be mentioned.

**Definition 22** Let  $F_1(-c, 0), F_2(c, 0)$  be two point in  $(\mathbb{Z}^2, d_T)$  (foci). The ellipse  $E$  is the set of points  $T(x, y)$  from  $\mathbb{Z}^2$  such that

$$|d_T(F_1, T) + d_T(F_2, T)| = a, a \in \mathbb{Z}. \quad (7)$$

**Proposition 23** The number of points on an ellipse  $E$  is  $2a$ .

**Definition 24** For an ellipse  $E$  the halfaxis  $b$  is

$$b = \max \{|y_2 - y_1|, \text{ for all } A(x_1, y_1), B(x_2, y_2) \in E\}. \quad (8)$$

**Proposition 25** For an ellipse  $E$  the following holds

$$a = b + c. \quad (9)$$

## 4 Applications of the taxicab geometry

At the end, let us say a few facts about applications of the taxicab geometry in a real world problems.

In the real world the shortest distance between two places for trains, planes and even for plains, is not as a bird flies (Euclidean distance). For example, trains want to take the shortest route possible from one place to another. The lattice points with respect to taxicab geometry would be the depots or turns of the track. We want to consider another variable to travel along taxicab distance, so called taxicab time. But sometimes it is not good enough, because the shortest route is selected thus **minimizing taxicab distance** and minimizing costs. The minimal taxicab distance, time or costs can be determined by the same method described in [9].

Let us explain another natural problem for taxicab geometry. A taxi company has a number of stations across the city. In order to dispatch the calls, the company want to know for any point in the city, which station is the closest, so that the taxi can get there as fast as possible. Obviously, our task is to find the locus of points which are equidistant to their two closest taxi stations. These boundaries will tessellate the plane in regions. In Euclidean geometry, the corresponding graph is called **Vornoi diagram** and is widely used in computational geometry and pattern recognition (see [10]). This task can be an interesting diploma work on mathematical or technical oriented faculty (a Java program can be demanded).

A city planning makes also attractive use of taxicab geometry. From the historical point of view there are two types of cities: organically grown cities and planned cities (see Fig. 1 and Fig. 2). Organically grown cities develop slowly, tend to plan around preserving their natural beauty and have unstructured design with curved shapes and sporadic placements of streets, buildings and landscape (early European cities).

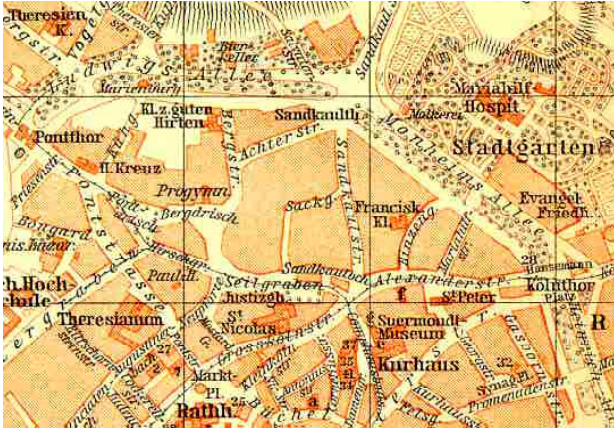


Figure 1: Aachen - the example of a "spider" city.

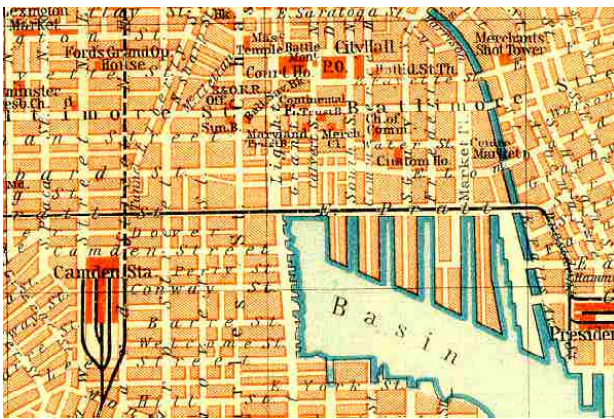


Figure 2: Baltimore - the example of a planned city.

The growth and plans of planned cities are on "high levels" coordinated with a population growth (predicted mathematically on the basis of cohort survival method). The patterns in the planned cities are more vertical and horizontal (and more suitable for application of taxicab geometry). In the 1960s the central focus of city planning moved from artistic beauty of the city towards location and efficiency.

But even for the curved streets we can set up models such as the shortest rout problem or Voronoi diagrams. The curved streets can be represent as a straight edge form with the distance used as edge weighs and the street intersections as the nodes. Besides a distance we have to consider traffic volume, speed limits, traffic signals and so on.

Additionally, the street patterns tells a lot about the city's history.

We can conclude by noticing that parts of city planning involve the use of geometric models and mathematical models in general. Taxicab geometry is used in many ways for

planning city and the traversed routes between and through cities. "Taxicab geometry is a great model for the artificially grid planned world that man has created" (cite [4]).

## References

- [1] Millman R. S., Parker G.D.; *Geometry: A Metric Approach with Models*, Springer-Verlag, New York (1991).
- [2] Thompson K., Dray T.; Taxicab Angles and Trigonometry, *Pi Mu Epsilon Journal* (2000) (to appear)
- [3] Prevost F.J.; The conic sections in taxicab geometry: Some investigations for high school students, *The Mathematics Teachers*, Reston, Apr (1998).
- [4] <http://www2.gvsu.edu/~vanbelkj/Project.html>
- [5] Tian S., So S., Chen G.; Concerning circles in taxicab geometry, *International Journal of Mathematical Education in Science and Technology*, Sep/Oct (1997).
- [6] Greenberg M.; *Euclidean and Non-Euclidean Geometry*, Freeman, San Francisco (1980).
- [7] Grünbaum B., Mycielski J.; Some models of Plane Geometries, *AMM*, Vol 97, No9, Nov (1990) pp.839-847.
- [8] Artigue M.; The Teaching and Learning of Mathematics at the University Level: Crucial Questions for Contemporary Research in Education, *Notices of the AMS*, Dec (1999).
- [9] Hesse R., Gene W.; *Applied Management Science A Quick & Dirty Approach*, Science Research Associates, Inc. (1980), pp. 96-99.
- [10] Graham R., Yao F.; A Whirlwind of Computational Geometry, *The American Mathematical Monthly*, Vol. 97, No. 8, Oct (1990), pp. 687-702.

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# The Consequences of Descartes's Method for Factorization of 4th Degree Polynomial

## Posljedice Descartesove metode za faktorizaciju polinoma 4. stupnja

### SAŽETAK

U članku je dan detaljan opis Descartesove metode za faktorizaciju polinoma četvrtog stupnja (nad poljem  $\mathbf{R}$ ) koji je dan u sljedećem reduciranom obliku

$$P_4(x) \equiv x^4 + a_2x^2 + a_1x + a_0 \equiv (x^2 + Ax + B)(x^2 + Cx + D).$$

Nakon što je riješen sustav od četiri jednačbe sa četiri nepoznanice, koji slijedi iz gornjeg identiteta, dobiva se sljedeća kubna rezolventa  $P_3(t) \equiv t^3 + 2a_2t^2 + (a_2^2 - 4a_0)t - a_1^2$ , gdje je  $t = A^2$ . Formulirana su i dokazana dva teorema. U prvom se otkriva korespondencija između tipova korijena od  $P_3(t)$  i od  $P_4(x)$  dok se u drugom daje karakterizacija tih tipova korijena od  $P_3(t)$ .

**Cljučne riječi:** Descartesova metoda, faktorizacija, kubna rezolventa, tipovi korijena, karakterizacija tipova korijena, ravninske krivulje četvrtog reda

## The Consequences of Descartes's Method for Factorization of 4th Degree Polynomial

### ABSTRACT

In this article we give in details description of Descartes's method for factorization of the fourth degree polynomial (over the field  $\mathbf{R}$ ) in the following reduced form

$$P_4(x) \equiv x^4 + a_2x^2 + a_1x + a_0 \equiv (x^2 + Ax + B)(x^2 + Cx + D).$$

When we seek the solution for  $A$  we get the following cubic resolvent  $P_3(t) \equiv t^3 + 2a_2t^2 + (a_2^2 - 4a_0)t - a_1^2$ , where  $t = A^2$ . At the end, we formulate and prove two theorems. In the first one, we find the correspondences between the types of the roots of  $P_3(t)$  and  $P_4(x)$  while in the second one, we give the characterizations of types of roots for  $P_3(t)$ .

**Key words:** Descartes's method, factorization, cubic resolvent, types of roots, characterizations of types of roots, plane quartic curves

**MSC 2000:** 14H45

If we can find the roots of the equation

$$P_4(x) \equiv a_4x^4 + a_3x^3 + a_2x^2 + a_1x + a_0 = 0$$

$$a_i \in \mathbf{R}, i = 1, 2, 3, 4; a_4 \neq 0, \quad (1)$$

then we can also solve the problem of factorization of the polynomial  $P_4(x)$  over the field  $\mathbf{R}$ . The history of the mathematics (see [1]) knows two basic methods for finding the roots of (1) the Ferrari's and Euler's method. However, the inverse reasoning is also true, if we know the factorization of  $P_4(x)$  over  $\mathbf{R}$  then we can also find all roots easily by solving two quadratic equations.

In the Descartes's method (see [5]) for factorization of  $P_4(x)$  we can first suppose without losing generality that  $a_4 = 1$  and that  $a_3 = 0$ . It is known that if  $a_3 \neq 0$  then by substituting  $x = y - a_3/4$  we get  $a_3 = 0$ . Now we shall describe the method for factorization of the polynomial

$$P_4(x) \equiv x^4 + a_2x^2 + a_1x + a_0 \equiv (x^2 + Ax + B)(x^2 + Cx + D), \quad (2)$$

over  $\mathbf{R}$ .

From (2) we get the following system of nonlinear equations

$$\begin{aligned} A + C &= 0 \\ B + D + AC &= a_2 \\ AD + BC &= a_1 \\ BD &= a_0. \end{aligned} \quad (3)$$

When we substitute  $C = -A$  in the second and the third equation we get

$$\begin{aligned} B + D &= A^2 + a_2 \\ -B + D &= a_1/A \\ BD &= a_0. \end{aligned} \quad (4)$$

Let us suppose that  $A \neq 0$ , then from the first and the second equation we get easily

$$\begin{aligned} B &= \frac{A^2 + a_2}{2} - \frac{a_1}{2A} \\ D &= \frac{A^2 + a_2}{2} + \frac{a_1}{2A}. \end{aligned} \quad (5)$$

Finally, from (5) and from the last equation in (4) it follows

$$A^6 + 2a_2A^4 + (a_2^2 - 4a_0)A^2 - a_1^2 = 0. \quad (6)$$

After substituting  $t = A^2$  in (6), we get the following cubic equation (resolvent)

$$t^3 + 2a_2t^2 + (a_2^2 - 4a_0)t - a_1^2 = 0. \quad (7)$$

Denote by  $P_3(t)$  the left side of (7). Since  $\lim_{t \rightarrow +\infty} P_3(t) = +\infty$  and  $P_3(0) = -a_1^2$ , when  $a_1 \neq 0$ , it follows that there is a positive root of (7). Hence, there is a real root of (6) that is different from zero, and we can calculate  $B$  and  $D$  using the formulas (5). There is only the case  $a_1 = 0$  left to be examined. Then we get the following system from (3)

$$\begin{aligned} A + C &= 0 \\ B + D + AC &= a_2 \\ AD + BC &= 0 \\ BD &= a_0. \end{aligned} \quad (8)$$

Analogously we get from (8)

$$\begin{aligned} B + D &= A^2 + a_2 \\ A(D - B) &= 0 \\ BD &= a_0. \end{aligned} \quad (9)$$

From the second equation in (9) we get  $A = 0$  or  $D = B$ . If  $A = 0$  we get from (9)

$$\begin{aligned} B + D &= a_2 \\ BD &= a_0. \end{aligned} \quad (10)$$

From (10) we get

$$B = \frac{a_2}{2} + \sqrt{\frac{a_2^2}{4} - a_0} \quad (11)$$

$$D = \frac{a_2}{2} - \sqrt{\frac{a_2^2}{4} - a_0},$$

and if  $a_2^2 - 4a_0 \geq 0$  we have the complete solution of (8). But when  $a_2^2 - 4a_0 < 0$ , then evidently  $a_0 > 0$  and we must apply the second case  $D = B$ , which yields

$$\begin{aligned} D = B &= \sqrt{a_0} \\ A &= \sqrt{2\sqrt{a_0} - a_2}. \end{aligned} \quad (12)$$

Thus, we always have a nonnegative root of the equation (6) and a complete solution of the system (3) in real numbers. There is only the equation (7) left to be examined. Using the known substitution (see [4])  $t = z - 2a_2/3$  we get from (7)

$$z^3 + pz + q = 0, \quad (13)$$

and the connection between  $p, q, a_0, a_1, a_2$  is

$$\begin{aligned} p &= -4a_0 - \frac{1}{3}a_2^2 \\ q &= \frac{8}{3}a_0a_2 - a_1^2 - \frac{2}{27}a_2^3. \end{aligned} \quad (14)$$

Let  $a_1 \neq 0$  and

$$D_1 = \frac{q^2}{4} + \frac{p^3}{27}. \quad (15)$$

It is known (see [3]) if  $D_1 > 0$  then

$$z = \sqrt[3]{-\frac{q}{2} + \sqrt{D_1}} + \sqrt[3]{-\frac{q}{2} - \sqrt{D_1}} \quad (16)$$

is only a real root of (13), so that

$$t = \sqrt[3]{-\frac{q}{2} + \sqrt{D_1}} + \sqrt[3]{-\frac{q}{2} - \sqrt{D_1}} - \frac{2a_2}{3} \quad (17)$$

is surely a positive root of (7). If  $D_1 = 0$

$$z = \begin{cases} 2\sqrt[3]{-\frac{q}{2}} & \text{if } q < 0 \\ \sqrt[3]{\frac{q}{2}} & \text{if } q > 0 \\ 0 & \text{if } q = 0 \end{cases} \quad (18)$$

is maximal nonnegative root of (13) (see [4]), so that

$$t = z - \frac{2a_2}{3} \quad (19)$$

is surely a positive root of (7). If  $D_1 < 0$ , it can be shown, that if

$$\varphi = \arccos \left[ -\frac{q}{2} \left( -\frac{3}{p} \right)^{\frac{2}{3}} \right], \quad (20)$$

then maximal real root of (13) is (see [4])

$$z = 2\sqrt{-\frac{p}{3}} \cos \frac{\varphi}{3}. \quad (21)$$

Hence,

$$t = 2\sqrt{-\frac{p}{3}} \cos \frac{\varphi}{3} - \frac{2a_2}{3} \quad (22)$$

is a positive root of (7). In all these cases

$$A = \sqrt{t}, \quad (23)$$

is a positive root of (6) and the other values of unknowns of the system (3) are given by (5).

Now we shall formulate and prove the theorem that gives the correspondences between the types of roots of  $P_4(x)$  and the types of roots of its cubic resolvent  $P_3(t)$  and a theorem that gives the characterizations for the types of roots of  $P_3(t)$ . We have three main possibilities for the types of roots of  $P_3(t)$  (see Figure 1.).

In the **first case**,  $P_3(t)$  has only one real nonnegative root and two conjugate complex roots or one real nonnegative root and one real negative double root.

In the **second case**,  $P_3(t)$  has one real nonnegative root and two different real nonpositive roots.

In the **third case**,  $P_3(t)$  has three real nonnegative roots (the cases of double and triple roots are included in this case).

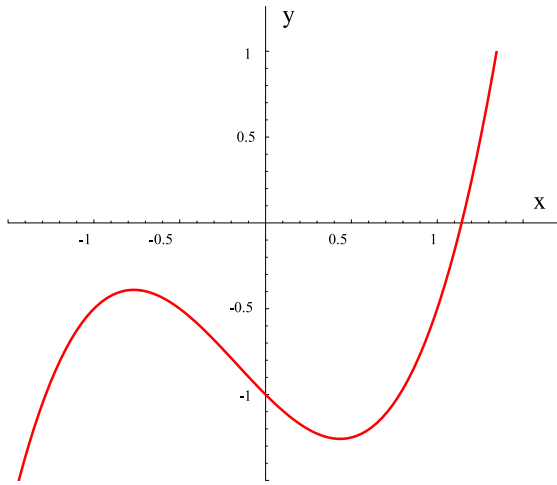


Fig. 1a: 1st case

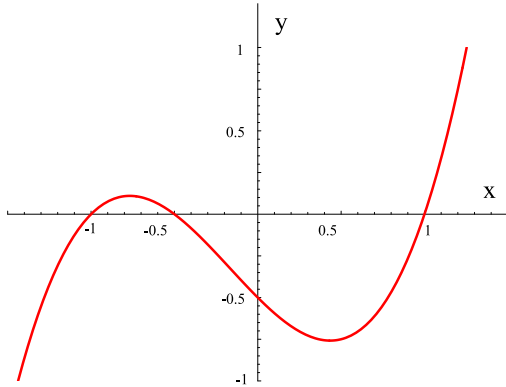


Fig. 1b: 2nd case

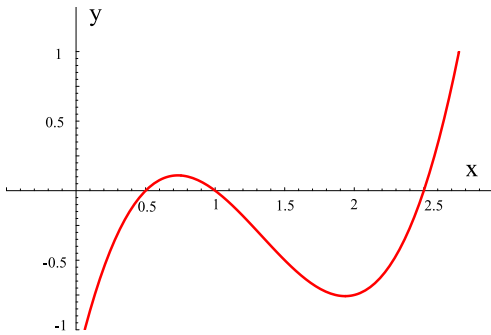


Fig. 1c: 3rd case

### Theorem 1.

1st case  $\iff P_4(x)$  has two real and two complex roots

2nd case  $\iff P_4(x)$  has only complex roots

3rd case  $\iff P_4(x)$  has only real roots

**Proof.** Let  $P_4(x)$  have two real and two complex roots

$$\begin{aligned}
 P_4(x) &= (x - x_1)(x - x_2)(x - a - bi)(x - a + bi) = \\
 &= [x^2 - (x_1 + x_2)x + x_1x_2][x^2 - 2ax + a^2 + b^2] = \\
 &= [x^2 - (x_1 + a + bi)x + (a + bi)x_1] \\
 &\quad [x^2 - (x_2 + a - bi)x + (a - bi)x_2] = \\
 &= [x^2 - (x_1 + a - bi)x + (a - bi)x_1] \\
 &\quad [x^2 - (x_2 + a + bi)x + (a + bi)x_2].
 \end{aligned} \tag{24}$$

Let  $P_4(x)$  have only complex roots

$$\begin{aligned}
 P_4(x) &= (x - a - bi)(x - a + bi)(x - c - di)(x - c + di) = \\
 &= [x^2 - 2ax + a^2 + b^2][x^2 - 2cx + c^2 + d^2] = \\
 &= [x^2 - (b + d)ix + ac - bd + (ad + bc)i] \\
 &\quad [x^2 + (b + d)ix + ac - bd - (ad + bc)i] = \\
 &= [x^2 - (b - d)ix + ac + bd + (bc - ad)i] \\
 &\quad [x^2 + (b - d)ix + ac + bd + (ad - bc)i].
 \end{aligned} \tag{25}$$

Let  $P_4(x)$  have only real roots

$$\begin{aligned}
 P_4(x) &= (x - x_1)(x - x_2)(x - x_3)(x - x_4) = \\
 &= [x^2 - (x_1 + x_2)x + x_1x_2][x^2 - (x_3 + x_4)x + x_3x_4] = \\
 &= [x^2 - (x_1 + x_3)x + x_1x_3][x^2 - (x_2 + x_4)x + x_2x_4] = \\
 &= [x^2 - (x_1 + x_4)x + x_1x_4][x^2 - (x_2 + x_3)x + x_2x_3].
 \end{aligned} \tag{26}$$

First, we shall prove the "only if" direction. Let  $P_4(x)$  have two real and two complex roots. Then we have the first row in the factorizations (24) and the remaining three rows we get by considering all possibilities of factorizations with two quadratic polynomials having a unit as a leading coefficient (regardless of it having real coefficients or complex coefficients). These possibilities are closely connected with the roots of the Descartes's cubic resolvent  $P_3(t)$ , because  $t = A^2$ , where  $A$  is a coefficient of  $x$  in one of these two quadratic polynomials (no matter which one, because they differ only in the sign). As the sum of these two coefficients of  $x$  is the same in all those possibilities and de facto represents the coefficient of  $x^3$  (which is zero), we get an important relation

$$x_1 + x_2 + 2a = 0. \tag{27}$$

Now, we shall consider two cases  $x_1 = x_2$  and  $x_1 \neq x_2$ . In the first case, we obtain  $x_1 + a = x_2 + a = 0$ , from (27) and finally, from this one and from (24), it follows

$$(A^2)_1 = 4a^2 \geq 0; (A^2)_{2,3} = -b^2 < 0. \tag{28}$$

Thus  $-b^2$  is a double root of  $P_3(t)$  in the first case. In the second case from (27) we obtain

$$x_2 + a = -(x_1 + a). \quad (29)$$

From (29) we conclude that  $x_1 + a \neq 0$  and  $x_2 + a \neq 0$ , because on the contrary, (29) leads to  $x_1 + a = x_2 + a = 0$  or equivalently to  $x_1 = x_2$ , which is a contradiction. Thus, in the second case we obtain from (24)

$$(A^2)_{2,3} = (x_1 + a)^2 - b^2 \pm 2b(x_1 + a)i. \quad (30)$$

That means  $(A^2)_{2,3}$  is a pair of conjugate complex numbers (because  $b \neq 0$  and  $x_1 + a \neq 0$ ).

Now, let  $P_4(x)$  have two pairs of conjugate complex numbers. It means that in (25)  $b$  and  $d$  are different from zero, which implies that

$$(b + d)^2 \neq (b - d)^2. \quad (31)$$

From (25) and (31) we get easily

$$(A^2)_2 = -(b + d)^2 \neq (A^2)_3 = -(b - d)^2. \quad (32)$$

Thus,  $(A^2)_{2,3}$  are different nonpositive real numbers, and  $(A^2)_1 = 4a^2$  is evidently a nonnegative real number.

Finally, let  $P_4(x)$  have only real roots. From (26) it is easy to see that  $(A^2)_{1,2,3} \geq 0$ .

We shall furthermore prove the "if" direction. If we want to prove that the first case implies  $P_4(x)$  having two real roots and a pair of conjugate complex roots we suppose the opposite, that the first case holds and for example  $P_4(x)$  has two pairs of conjugate complex roots. We have proved before that if  $P_4(x)$  has two pairs of conjugate complex roots, then it implies the second case. As the first and the second case are mutually exclusive cases, we come to a contradiction. The same type of a proof is valid if we suppose that the first case holds and  $P_4(x)$  has four real roots. Hence, the exclusive property of the cases is the main tool in all remaining proofs.

Q.E.D.

### Theorem 2.

1st case  $\iff$

$$D_1 > 0$$

$$\text{or } (D_1 = 0 \text{ and } (a_2^2 - 4a_0 < 0 \text{ or } (a_2^2 - 4a_0 > 0 \text{ and } a_2 > 0)))$$

$$\text{or } (D_1 = 0 \text{ and } a_2^2 - 4a_0 = 0 \text{ and } a_2 > 0 \text{ and } a_1 \neq 0)$$

2nd case  $\iff$

$$(D_1 < 0 \text{ and } (a_2^2 - 4a_0 < 0 \text{ or } (a_2^2 - 4a_0 \geq 0 \text{ and } a_2 > 0)))$$

$$\text{or } (a_1 = 0 \text{ and } a_2^2 - 4a_0 = 0 \text{ and } a_2 > 0)$$

3rd case  $\iff D_1 \leq 0$  and  $a_2^2 - 4a_0 \geq 0$  and  $a_2 \leq 0$ .

(33)

### Proof.

$$P_3(t) = t^3 + 2a_2t^2 + (a_2^2 - 4a_0)t - a_1^2$$

$$P_3'(t) = 3t^2 + 4a_2t + a_2^2 - 4a_0 \quad (34)$$

$$P_3''(t) = 6t + 4a_2.$$

Since in the third case all roots are real (double and triple roots are included in that case) it is equivalent to  $D_1 \leq 0$  (see [3]), but all roots are not only real, but all roots are nonnegative, which is equivalent to  $D_1 \leq 0$  and both roots of  $P_3'(t)$  are nonnegative. This last statement is equivalent to  $D_1 \leq 0$  and  $P_3'(0) = a_2^2 - 4a_0 \geq 0$  and  $P_3''(0) = 4a_2 \leq 0$ .

Since in the second case all roots are real and different (except in one special case which will be soon considered), which is equivalent to  $D_1 < 0$ , but as two roots are nonpositive and one nonnegative we conclude that either  $P_3'(t)$  has one root negative and one root positive or both roots are nonpositive and different. The first case is equivalent to  $D_1 < 0$  and  $P_3'(0) = a_2^2 - 4a_0 < 0$ . The second case is equivalent to  $D_1 < 0$  and  $P_3'(0) = a_2^2 - 4a_0 \geq 0$  and  $P_3''(0) = 4a_2 > 0$ . It remains only to consider the special case of the second case. In that special case one root of  $P_3(t)$  is negative and two other roots are equal to zero. It is equivalent to  $P_3(0) = -a_1^2 = 0$  (thus  $a_1 = 0$ ) and one root of  $P_3'(t)$  is negative while the other is zero. It is equivalent to  $a_1 = 0$  and  $P_3'(0) = a_2^2 - 4a_0 = 0$  and  $P_3''(0) = 4a_2 > 0$ .

Since in the first case we have two quite different possibilities, we shall first consider the first possibility in which only one root of  $P_3(t)$  is real and nonnegative. That first possibility is equivalent to  $D_1 > 0$  (see [3]). In the second possibility we have one double negative real root and one nonnegative real root. It is equivalent to  $D_1 = 0$  (see [3]) and at least one root of  $P_3'(t)$  is negative. That is equivalent to  $D_1 = 0$  and one root of  $P_3'(t)$  is negative while the other is positive or both roots of  $P_3'(t)$  are nonpositive and different. That means in the first case  $D_1 = 0$  and  $P_3'(0) = a_2^2 - 4a_0 < 0$  or in the second case  $D_1 = 0$  and  $P_3'(0) = a_2^2 - 4a_0 \geq 0$  and  $P_3''(0) = 4a_2 > 0$ . But we need to separate this second case additionally in two cases to make a distinction between it and a special case of the second case (see remark 1. and remark 2.).

Q.E.D.

**Remark 1.** It is easy to see that conditions  $a_1 = 0$  and  $a_2^2 - 4a_0 = 0$  imply  $D_1 = 0$ .

**Remark 2.** To make a distinction between the following two possibilities (see Figure 2.) we introduce these conditions in order to characterise the first case and the second case.

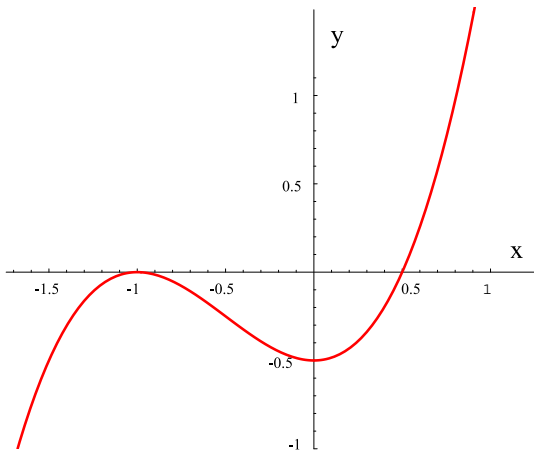


Fig. 2a: 1st case  $D_1 = 0, a_1 \neq 0, P'_3(0) = a_2^2 - 4a_0 = 0,$   
 $P''_3(0) = 4a_2 > 0.$

**Remark 3.** Everywhere in (33) the symbol "or" is used only in the exclusive sense. Although the characterization of the first case and of the second case is quite complicated, their main parts are not so complicated (the main parts are those in which possibility  $D_1 = 0$  is excluded). Hence, the main part of the first case is  $D_1 > 0$  and the main part of the second case is  $D_1 < 0$  and ( $a_2^2 - 4a_0 < 0$  or ( $a_2^2 - 4a_0 \geq 0$  and  $a_2 > 0$ )). These main parts, especially in the second case, are of considerable importance in the theory of plane quartic curves.

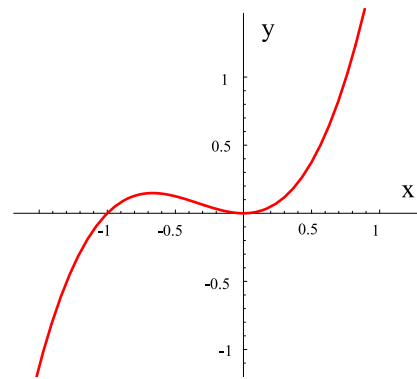


Fig. 2b: 2nd case  $a_1 = 0, P'_3(0) = a_2^2 - 4a_0 = 0,$   
 $P''_3(0) = 4a_2 > 0.$

## References

- [1] W. W. ROUSE BALL, *A short account of the history of mathematics*, Dover publications, New York, 1960.
- [2] A. B. BASSET, *An elementary treatise on cubic and quartic curves*, Deighton Bell and Co., Cambridge, 1901.
- [3] B. PAVKOVIĆ, D. VELJAN, *Elementarna matematika I*, Tehnička knjiga, Zagreb, 1992.
- [4] J. PLEMELJ, *Algebra in teorija števil*, Slovenska akademija znanosti in umetnosti, Ljubljana, 1962.
- [5] V. SEDMAK, *Uvod u algebru*, Sveučilište u Zagrebu, Zagreb, 1961.

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# Triangles from the Feuerbach Triangle

## Trokuti iz Feuerbachovog trokuta

### SAŽETAK

U članku dokazujemo četiri neobična teorema o Feuerbachovom trokutu  $A_d B_d C_d$  zadanog trokuta  $ABC$  kome su vrhovi u točkama  $A_d$ ,  $B_d$ , i  $C_d$  gdje pripisane kružnice dotiču izvana kružnicu devet točaka. Ti rezultati odnose se na problem određivanja trokuta  $XYZ$  za koje će segmenti  $A_d X$ ,  $B_d Y$ , i  $C_d Z$  biti stranice trokuta. Pronađeno je pet trokuta  $XYZ$  (uključujući i degenerirani trokut u točki  $D$  gdje upisana kružnica iznutra dodiruje kružnicu devet točaka) pridruženih trokutu  $ABC$  takvih da segmenti  $A_d X$ ,  $B_d Y$ , i  $C_d Z$  nikada nisu stranice bilo kakvog trokuta. Na pozitivnoj strani, otkrivena su tri trokuta  $XYZ$  takva da su segmenti  $A_d X$ ,  $B_d Y$ , i  $C_d Z$  uvijek stranice nekog trokuta. Dokazi se provode čistom algebarskom metodom upotrebom analitičke geometrije ravnine. Također se pokazuje kako se ti i njima slični rezultati mogu otkriti pomoću računalnog programa Geometer's Sketchpad (Geometrova Crtanka).

**Gljučne riječi:** trokut, upisana kružnica, pripisane kružnice, kružnica devet točaka, Feuerbachov trokut, centralne točke trokuta, Feuerbachova točka, Geometer's Sketchpad

## Triangles from the Feuerbach Triangle

### ABSTRACT

We prove four unusual theorems about the Feuerbach triangle  $A_d B_d C_d$  of the given triangle  $ABC$  whose vertices are points  $A_d$ ,  $B_d$ , and  $C_d$  in which the excircles touch from outside the nine-point circle. These results concern the problem to determine for which triangles  $XYZ$  will the segments  $A_d X$ ,  $B_d Y$ , and  $C_d Z$  be sides of a triangle. We shall find five triangles  $XYZ$  (including the degenerate triangle at the point  $D$  in which the incircle touches from inside the nine-point circle) associated to a triangle  $ABC$  such that  $A_d X$ ,  $B_d Y$ , and  $C_d Z$  are never sides of a triangle. On the positive side, we discover three triangles  $XYZ$  such that the segments  $A_d X$ ,  $B_d Y$ , and  $C_d Z$  are always sides of a triangle. We give an algebraic method of proof for these results based on simple analytic geometry in the plane. We also show how one can discover these and other related results using the Geometer's Sketchpad.

**Key words:** triangle, incircle, excircles, nine-point circle, Feuerbach triangle, central points, Feuerbach point, Geometer's Sketchpad

**MSC 2000:** Primary 51N20, 51M04, Secondary 14A25, 14Q05

## 1 Introduction

Recall the construction of a triangle  $ABC$  whose sides are three given segments  $a$ ,  $b$ , and  $c$  (see Figure 1). First pick a point  $B$  in the plane and select a point  $C$  on a circle with centre at  $B$  and radius  $a$ . Then draw circles with centres at  $B$  and  $C$  and radii  $c$  and  $b$ . Intersections of these two circles determine two possibilities for the third vertex  $A$ . Hence, there is only one solution when we require that going from  $A$  to  $B$  and then to  $C$  is in the counterclockwise direction.

The condition for the existence of solutions is that the inequalities  $a < b + c$ ,  $b < c + a$ , and  $c < a + b$  hold. Since  $a$ ,  $b$ , and  $c$  are positive, this condition is equivalent with the requirement that  $T[a] > 0$ , where  $[a]$  is a short notation for the triple  $(a, b, c)$  and  $T[a]$  is the product  $(a + b + c)(b + c - a)(c + a - b)(a + b - c)$  which ex-

pands to  $2(b^2 c^2 + c^2 a^2 + a^2 b^2) - (a^4 + b^4 + c^4)$ . When it is positive,  $T[a]$  is equal to 16 times the square of the area  $S$  of  $ABC$ .

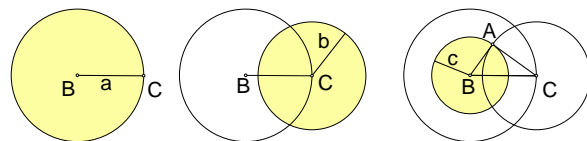


Fig. 1 Construction of a triangle from three segments.

The function  $T$  can be therefore utilised to decide when three segments are sides of a triangle. These three segments could be given in myriad of ways. One of the most natural is to take them as segments  $PX$ ,  $QY$ , and  $RZ$  joining corresponding vertices of triangles  $PQR$  and  $XYZ$  (see Fig-

ure 2(a)) or as segments  $PW$ ,  $QW$ , and  $RW$  joining vertices of a triangle  $PQR$  with a point  $W$  (see Figure 2(b)). We write  $XYZ \in \Omega(PQR)$  and  $W \in \Omega(PQR)$  when  $T[PX] > 0$  and  $T[PW] > 0$ , respectively.

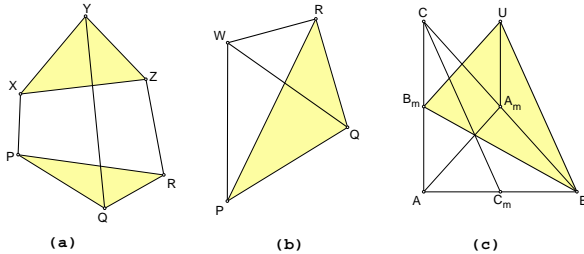


Fig. 2 (a) Triangles from segments joining vertices with vertices.  
 (b) Triangles from segments joining vertices with a point.  
 (c) The triangle from the medians.

For example, if  $A_m$ ,  $B_m$ , and  $C_m$  are midpoints of sides  $BC$ ,  $CA$ , and  $AB$  of the base triangle  $ABC$ , then the statement that (the complementary triangle)  $A_m B_m C_m$  is in  $\Omega(ABC)$  is equivalent to the well-known fact that medians  $AA_m$ ,  $BB_m$ , and  $CC_m$  are sides of a triangle.

The simplest proof of  $A_m B_m C_m \in \Omega(ABC)$  is based on the Figure 2(c) from [3, p. 282]. The segments  $A_m U$  and  $B_m U$  are parallel to  $AB_m$  and  $AA_m$  so that the triangle  $BB_m U$  has medians as sides.

Another entirely algebraic proof that is equally simple if we do it with a computer requires first to find lengths of medians  $AA_m$ ,  $BB_m$ , and  $CC_m$  and then to show that  $T[AA_m] > 0$ . Since  $2AA_m$  is equal  $\sqrt{2b^2 + 2c^2 - a^2}$ , and  $2BB_m$  and  $2CC_m$  are  $\sqrt{2c^2 + 2a^2 - b^2}$  and  $\sqrt{2a^2 + 2b^2 - c^2}$ , we easily find  $T[AA_m] = \frac{9}{16} S^2 > 0$ .

The present article takes in problems shown in Figures 2(a) and 2(b) for the triangle  $PQR$  the Feuerbach triangle  $A_d B_d C_d$  of a given triangle  $ABC$  and searches for triangles and points in  $\Omega(A_d B_d C_d)$  or its complement among various triangles and points naturally associated to  $ABC$ . If you wonder what is so special about the Feuerbach triangle, keep in mind that some of the most beautiful theorems in triangle geometry have been proved about it in the last 250 years and that our results below show surprising role of this triangle even in such a basic question as when three segments are sides of a triangle. The surprise comes from the black and white nature of our results: from segments joining vertices of  $A_d B_d C_d$  with vertices of some triangles we always get a triangle while there are triangles when we never get a triangle in this way. An interesting recent article about the Feuerbach triangle is [6].

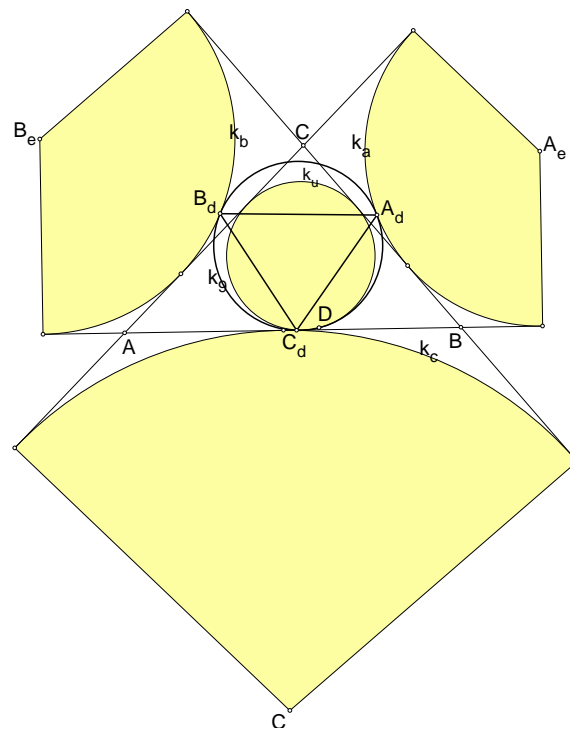


Fig. 3 The nine-point circle is touched from outside by the excircles at vertices of the Feuerbach triangle and from inside by the incircle at the Feuerbach point of  $ABC$ .

The vertices  $A_d$ ,  $B_d$ , and  $C_d$  of the Feuerbach triangle are points in which the excircles  $k_a$ ,  $k_b$ , and  $k_c$  touch from outside the circumcircle  $k_9$  of the complementary triangle  $A_m B_m C_m$ . The circle  $k_9$  is also known as the nine-point circle because it goes through the midpoints  $A_m$ ,  $B_m$ , and  $C_m$  of sides, the feet  $A_o$ ,  $B_o$ , and  $C_o$  of altitudes, and the midpoints  $A_f$ ,  $B_f$ , and  $C_f$  of segments joining the orthocentre  $H$  (concurrency point of altitudes) with vertices  $A$ ,  $B$ , and  $C$ .

The above statement about excircles touching the nine-point circle from outside is just a part of the famous Feuerbach theorem from 1834 which also established that the incircle  $k_u$  makes a touch with the nine-point circle from inside at the so called Feuerbach point  $D$  of  $ABC$  (see [3] and Figure 3 above).

## 2 Statement of Theorems

In order to describe our main results we need the triangle  $A_p B_p C_p$  at points of contact of the incircle with sides of  $ABC$  and the triangle  $A_n B_n C_n$  at points of intersection of internal angle bisectors with sides of  $ABC$  (i. e., the intouch triangle (cevian triangle of the Gergonne point) and the incentral triangle (cevian triangle of the incentre)).

**Theorem 1** *In every triangle  $ABC$  it is impossible to construct a triangle from segments  $A_dD$ ,  $B_dD$ , and  $C_dD$  joining the vertices of its Feuerbach triangle with its Feuerbach point.*

In our notation, this theorem simply claims that  $D \notin \Omega(ABC)$  holds for every triangle  $ABC$ . The next theorem is a similar statement for four triangles.

Let  $A_g$ ,  $B_g$  and  $C_g$  denote points diametrically opposite on the nine-point circle to the points  $A_f$ ,  $B_f$  and  $C_f$ .

**Theorem 2** *In every triangle  $ABC$  it is impossible to construct a triangle from segments  $A_dX$ ,  $B_dY$ , and  $C_dZ$  when  $XYZ$  is  $A_xB_xC_x$  for  $x \in \{g, m, n, p\}$ .*

The next two theorems are different because they describe situations when segments to vertices of the Feuerbach triangle are always sides of a triangle.

**Theorem 3** *If the triangle  $ABC$  is not isosceles, then the segments  $A_dA_o$ ,  $B_dB_o$ , and  $C_dC_o$  joining the vertices of its Feuerbach triangle with the vertices of its orthic triangle  $A_oB_oC_o$  are sides of a triangle.*

Let  $A_q$ ,  $B_q$  and  $C_q$  denote points diametrically opposite on the nine-point circle to the midpoints  $A_m$ ,  $B_m$  and  $C_m$  of sides.

**Theorem 4** *In every triangle  $ABC$ , the segments joining the vertices of its Feuerbach triangle  $A_dB_dC_d$  with the vertices of either its Euler triangle  $A_fB_fC_f$  or the antipodal  $A_qB_qC_q$  of its complementary triangle  $A_mB_mC_m$  are sides of an acute triangle.*

### 3 Theorems and the Geometer's Sketchpad

In this section we shall explain how one can discover and check our theorems using the computer software the Geometer's Sketchpad. This program allows one to explore properties of geometric objects and constructions in a dynamical fashion because it remembers relationships and readjusts all calculations as you move objects around (on the screen).

Let us first describe how to make a script `test.gss` which will test if segments  $PX$ ,  $QY$ , and  $RZ$  from Figure 2(a) are sides of a triangle. For this one must draw this figure, measure lengths of these segments, and calculate  $T[PX]$ . Hide everything except the vertices and the calculation, select with Shift key all visible objects (one can do this also

with Ctrl+a), and then use Work menu to make the script. In applying script `test.gss` one must select six points in correct order. Its action will give a value of the function  $T$  for segments joining corresponding vertices of two triangles. As we move points around we must look whether this value is positive. Then the segments are sides of a triangle. On the other hand, when this value is zero or negative, then the segments are not sides of a triangle. The same test applies to the situation of Figure 2(b). Simply select the point  $W$  three times.

The next task is to make scripts for all points and triangles which appear in our theorems. Of course, some of them, like `h.gss` for the orthocentre of a triangle or `tr.m.gss` for the complementary triangle of a triangle, are straightforward. Others, like `d.gss` for the Feuerbach point of a triangle and `tr.d.gss` for the Feuerbach triangle of a triangle, are a bit tricky. It would be wrong to use Figure 3 and the Feuerbach theorem because the Geometer's Sketchpad has difficulties in finding intersections of two circles and for it touching point is always a pair of points.

A way out from these difficulties is to construct these points from their trilinear coordinates i.e., from any triple of real numbers proportional to their distances from sidelines of the base triangle. We shall illustrate this by describing the script `d.gss` for the Feuerbach point  $D$ . It is well-known (see references [4] and [2] and section 6 below) that  $D$  has trilinears  $t_a = 1 - \cos(B - C)$ ,  $t_b = 1 - \cos(C - A)$ , and  $t_c = 1 - \cos(A - B)$  and that the actual distance of  $D$  from  $BC$  is  $d_a = 2St_a / (t_aBC + t_bCA + t_cAB)$  while  $d_b$  and  $d_c$  have similar expressions.

Draw a triangle  $ABC$  and rotate points  $C$  and  $A$  around points  $B$  and  $C$  for 90 degrees in the counterclockwise direction to get points  $C'$  and  $A'$ . Calculate sides  $BC$ ,  $CA$ , and  $AB$ , angles  $A$ ,  $B$ , and  $C$ , and distances  $d_a$  and  $d_b$ . Then dilate points  $C'$  and  $A'$  with respect to centres  $B$  and  $C$  for marked ratios  $\lambda d_a/BC$  and  $\lambda d_b/CA$  to get points  $C''$  and  $A''$ , where  $\lambda$  is the product of signs of angles  $A$ ,  $B$ , and  $C$ . Finally,  $D$  will be the intersection of parallels through  $C''$  and  $A''$  to  $BC$  and  $CA$  (see Figure 4). Hide everything except  $A$ ,  $B$ ,  $C$ , and  $D$ , select with Shift these four points, and use the Work menu to make the script. Its action produces the Feuerbach point of a triangle whose vertices have been selected.

For the Feuerbach triangle we must know that trilinears of  $A_d$  are  $-\sin^2(\frac{B-C}{2})$ ,  $\cos^2(\frac{C-A}{2})$ , and  $\cos^2(\frac{A-B}{2})$ , and that trilinears of  $B_d$  and  $C_d$  are their cyclic permutations. In section 5 below we shall show how one can compute these trilinears.

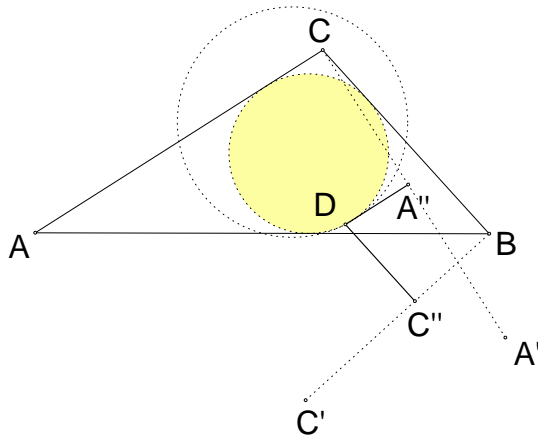


Fig. 4 Sketch explaining the script `d.gss` for the Feuerbach point of a triangle.

How does one check the claim  $D \notin \Omega(ABC)$  from the Theorem 1? Well, draw  $ABC$ , use `tr_d.gss` and `d.gss` to get its Feuerbach triangle  $A_d B_d C_d$  and its Feuerbach point  $D$ , and then apply `test.gss` to compute the triangle test for the triple  $[A_d D]$ . As we move the point  $C$  around the value of the test is never positive so that these segments are never sides of a triangle. The arguments for other our claims are analogous.

Of course, computer software like the Geometer's Sketchpad can only help us to discover theorems and quickly eliminate false conjectures but they can not give us (for now) mathematically sound proofs. For one thing, the Geometer's Sketchpad has limited precision so that no point is determined precisely. Therefore, how can we expect to prove something about the Feuerbach triangle when we don't even know the position of its vertices?

This is a nice example showing the need for rigour in mathematical proofs and for work we are going to do below in proving our theorems. Our idea is to use analytic geometry in the plane and position the base triangle in the coordinate system so that most calculations are rather simple especially when done with software using symbolic computation (like Maple, Mathematica, and Derive) which is nowadays quite common.

#### 4 Placement of $ABC$

We shall position the triangle  $ABC$  in the following fashion with respect to the rectangular coordinate system in order to simplify our calculations. The vertex  $A$  is the origin with coordinates  $(0, 0)$ , the vertex  $B$  is on the  $x$ -axis and has coordinates  $(rh, 0)$ , and the vertex  $C$  has coordinates  $(gqr/k, 2fgr/k)$ , where  $h = f + g$ ,  $k = fg - 1$ ,  $p = f^2 + 1$ ,  $q = f^2 - 1$ ,  $s = g^2 + 1$ ,  $t = g^2 - 1$ ,  $u = f^4 + 1$ ,

$v = g^4 + 1$ , and  $w = f - g$ . The three parameters  $r$ ,  $f$ , and  $g$  are the inradius and the cotangents of half of angles at vertices  $A$  and  $B$ . Without loss of generality, we can assume that both  $f$  and  $g$  are larger than 1 (i. e., that angles  $A$  and  $B$  are acute).

Nice features of this placement are that most important points related to the triangle  $ABC$  (including all central points from Table 1 in [4]), have rational functions in  $f$ ,  $g$ , and  $r$  as coordinates and that we can easily switch from  $f$ ,  $g$ , and  $r$  to side lengths  $a$ ,  $b$ , and  $c$  and back with substitutions

$$a = \frac{rfs}{k}, \quad b = \frac{rgp}{k}, \quad c = rh,$$

$$f = \frac{(b+c)^2 - a^2}{\sqrt{T([a])}}, \quad g = \frac{(a+c)^2 - b^2}{\sqrt{T([a])}}, \quad r = \frac{\sqrt{T([a])}}{2(a+b+c)}.$$

Moreover, since we use the Cartesian coordinate system, computation of distances of points and all other formulas and techniques of analytic geometry are available and well-known to widest audience. A price to pay for these conveniences is that symmetry has been lost.

The third advantage of the above position of the base triangle is that we can easily find coordinates of a point with given trilinears. More precisely, if a point  $P$  with coordinates  $x$  and  $y$  has projections  $P_a$ ,  $P_b$ , and  $P_c$  onto the side lines  $BC$ ,  $CA$ , and  $AB$  and  $\lambda = PP_a/PP_b$  and  $\mu = PP_b/PP_c$ , then

$$x = \frac{gh(p\mu + q)r}{fs\lambda\mu + gp\mu + hk}, \quad y = \frac{2fghr}{fs\lambda\mu + gp\mu + hk}.$$

This formulas will greatly simplify our exposition because there will be no need to give explicitly coordinates of points but only its trilinear coordinates. For example, the centre  $A_e$  of the  $A$ -excircle  $k_a$  obviously has trilinears  $-1 : 1 : 1$ . Then we use the above formulas with  $\lambda = -1$  and  $\mu = 1$  to get the coordinates  $(rfgh/k, rgh/k)$  of  $A_e$  in our coordinate system.

#### 5 Computation of coordinates of points

In this section we shall explain how to compute coordinates of all points from statements of our theorems.

Let  $I$  be the incenter of  $ABC$ . Then the inner angle bisectors  $AI$  and  $BI$  and the external angle bisector at the vertex  $B$  have equations

$$\mathbf{e}_1 : x - fy = 0, \quad \mathbf{e}_2 : x + gy - hr = 0,$$

$$\mathbf{e}_3 : gx - y - ghr = 0.$$

The solution of equations  $\mathbf{e}_1$  and  $\mathbf{e}_2$  will give us coordinates  $(fr, r)$  of the incenter  $I$  while the solution of  $\mathbf{e}_1$  and  $\mathbf{e}_3$  determines the coordinates  $A_e(fghr/k, ghr/k)$  of the center

of the  $A$ -excircle. Hence, the equations of the incircle and the  $A$ -excircle are

$$e_4: (x - fr)^2 + (y - r)^2 = r^2,$$

$$e_5: \left(x - \frac{fghr}{k}\right)^2 + \left(y - \frac{ghr}{k}\right)^2 = \left(\frac{ghr}{k}\right)^2.$$

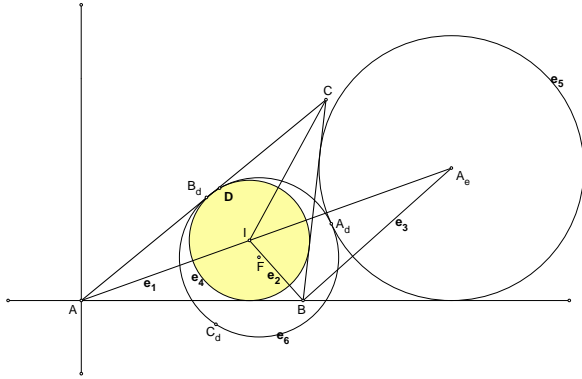


Fig. 5 Lines and circles used for the computation of coordinates.

On the other hand, since the midpoints of sides have coordinates

$$A_m \left( \frac{r(fs + 2gq)}{2k}, \frac{fgr}{k} \right),$$

$$B_m \left( \frac{gqr}{2k}, \frac{fgr}{k} \right), \quad C_m \left( \frac{hr}{2}, 0 \right),$$

it is easy to compute coordinates

$$F \left( \frac{r(ft + 3gq)}{4k}, \frac{r((k+2)^2 - w^2)}{8k} \right)$$

of the center of the nine-point circle and its radius  $\frac{prs}{8k}$ . It follows that the nine-point circle has the equation

$$e_6: \left(x - \frac{r(ft + 3gq)}{4k}\right)^2 + \left(y - \frac{r((k+2)^2 - w^2)}{8k}\right)^2 = \left(\frac{prs}{8k}\right)^2.$$

The equations  $e_5$  and  $e_6$  have only one solution which determines coordinates  $\left(\frac{u_a}{km_a}, \frac{v_a}{km_a}\right)$  of the touching  $A_d$  of the  $A$ -excircle and the nine-point circle, where  $u_a = ghr(fs + 6gq + 4fg^2)$ ,  $v_a = 2ghr(k+2)^2$ , and  $m_a = ps + 8(k+s)$ . In an analogous way one can compute coordinates  $\left(\frac{u_b}{km_b}, \frac{v_b}{km_b}\right)$  and  $\left(\frac{u_c}{m_c}, \frac{v_c}{m_c}\right)$  of the

other two vertices  $B_d$  and  $C_d$  of the Feuerbach triangle, where  $u_b = hr(qs + 6fgq - 4f^2)$ ,  $v_b = 2fhr(k+2)^2$ ,  $m_b = ps + 8(k+q)$ ,  $u_c = gr(3qs + 6fgq + 4g^2 - 4q)$ ,  $v_c = -2fgrw^2$ , and  $m_c = ps + 8fgk$ .

Similarly, the equations  $e_4$  and  $e_6$  have only one solution which determines coordinates  $\left(\frac{u_0}{m_0}, \frac{v_0}{m_0}\right)$ , of the touching point  $D$  of the nine-point circle and the incircle, where  $u_0 = r(fqs - 6gq + 4f)$ ,  $v_0 = 2rw^2$ , and  $m_0 = ps - 8k$ .

We can now compute the trilinear coordinates of  $D$  as follows. The third trilinear coordinate of  $D$  is proportional to  $v_0/m_0$  (the ordinate of  $D$  is its distance from  $AB$ ). On the other hand,

$$\begin{aligned} 1 - \cos(A - B) &= 2 \sin^2 \left( \frac{A - B}{2} \right) \\ &= \frac{2(B^* - A^*)^2}{(B^* - A^*)^2 + (1 + B^*A^*)^2} = \frac{2w^2}{ps}, \end{aligned}$$

where  $A^* = \cot(A/2) = f$  and  $B^* = \cot(B/2) = g$ . But,  $ps/m_0$  represented in terms of  $a$ ,  $b$ , and  $c$  is easily seen to be symmetric, so that the third trilinear of  $D$  is indeed  $1 - \cos(A - B)$ . The other two are  $1 - \cos(B - C)$  and  $1 - \cos(C - A)$ . We can verify this using the transfer formulas from trilinears to our coordinates.

## 6 Proof of $D \notin \Omega(A_dB_dC_d)$

With the standard formula  $d(P, Q) = \sqrt{(p-q)^2 + (x-y)^2}$  for the Euclidean distance between points  $P(p, x)$  and  $Q(q, y)$ , we find

$$A_dD = \frac{prs|i_a|}{k\sqrt{m_0m_a}}, \quad B_dD = \frac{prs|i_b|}{k\sqrt{m_0m_b}}, \quad C_dD = \frac{prs|w|}{\sqrt{m_0m_c}},$$

for distances from points  $A_d$ ,  $B_d$ , and  $C_d$  to the point  $D$ , where  $i_a = ft - 2g$  and  $i_b = gq - 2f$ . Hence,  $T[A_dD] = -32p^4r^4s^4w^2i_a^2i_b^2m_1/(k^4m_a^2m_b^2m_c^2)$ , where  $m_1$  is the following polynomial

$$\begin{aligned} (k+2)(2k+1)h^6 + k(4k^3 + 43k^2 + 68k + 32)h^4 \\ + k^2(2k^2 + 11k + 8)(k^2 + 14k + 16)h^2 + k^7. \end{aligned}$$

Since the expression  $m_1$  is clearly positive (recall that  $h > 2$  and  $k > 0$ ), we conclude that  $T[A_dD]$  is never positive so that the segments  $A_dD$ ,  $B_dD$ , and  $C_dD$  can not be sides of a triangle regardless of the shape of the triangle  $ABC$ .

### 7 Proof of $A_m B_m C_m \notin \Omega(A_d B_d C_d)$

In the same way as above we can compute distances of points  $A_d$ ,  $B_d$ , and  $C_d$  from the midpoints of sides  $A_m$ ,  $B_m$ , and  $C_m$ .

$$A_d A_m = \frac{|i_a| m_2}{k \sqrt{m_a}}, \quad B_d B_m = \frac{|i_b| m_2}{k \sqrt{m_b}},$$

$$C_d C_m = \frac{|w| m_2}{\sqrt{m_c}}, \quad m_2 = \frac{r \sqrt{p s}}{2}$$

and find  $T[A_d A_m] = -2 p^2 r^4 s^2 w^2 i_a^2 i_b^2 m_3 / (k^4 m_a^2 m_b^2 m_c^2)$ , where  $m_3$  is the polynomial

$$k(2k-3)h^6 + (k-2)(4k^3 + 19k^2 - 56k + 36)h^4 + (k-2)^2(2k^2 + 3k - 6)(k^2 + 10k - 8)h^2 + (k-2)^7.$$

Since  $m_3$  becomes a polynomial with all coefficients positive after the substitutions  $f = 1 + f'$  and  $g = 1 + g'$  (recall that  $f > 1$  and  $g > 1$  and thus  $f' > 0$  and  $g' > 0$ ), we conclude that  $T[A_d A_m]$  is never positive so that the segments  $A_d A_m$ ,  $B_d B_m$ , and  $C_d C_m$  also can not be sides of a triangle for any triangle  $ABC$ .

### 8 Remarks on proofs of remaining cases

The proofs of the remaining six cases are almost identical to the case with midpoints of sides. The only difference is that polynomials corresponding to the polynomial  $m_3$  become far more complicated and difficult to write down. This is even more so for polynomials that we obtain after the above substitutions because they have hundreds (up to 570) terms. It is now clear that our method of proof is almost impossible without use of computers. Also, in order to check our claims in the rest of the paper, the reader should make a try with some package for symbolic computation (like Maple, Mathematica, or Derive). We shall only give some expressions that can serve as pointers to all those who will attempt such a work-out. Therefore, this paper is an example of a new type of articles in mathematics which can be fully appreciated only by those readers that are willing to read it interactively. The standards for exposition of such papers is only emerging so that our presentation might appear unusual or inadequate to some readers.

The author does not rule out the possibility that our results have much simpler proofs with traditional geometric methods. Hence, our paper and its approach might challenge readers to think of such old-fashioned proofs for Theorems 1 - 4.

### 9 Proof of Theorem 3

Points  $A_o$ ,  $B_o$ , and  $C_o$  are projections of the vertices  $A$ ,  $B$ , and  $C$  onto the opposite sides  $BC$ ,  $CA$ , and  $AB$ . Hence, we get

$$A_o \left( \frac{4g^2 hr}{s^2}, \frac{2ghrt}{s^2} \right),$$

$$B_o \left( \frac{hq^2 r}{p^2}, \frac{2fhqr}{p^2} \right), \quad C_o \left( \frac{gqr}{k}, 0 \right).$$

The distances of points  $A_d$ ,  $B_d$ , and  $C_d$  from the points  $A_o$ ,  $B_o$ , and  $C_o$  are

$$A_d A_o = \frac{ghr|ft - 2g|\sqrt{p}}{k\sqrt{sm_a}}, \quad B_d B_o = \frac{fhr|gq - 2f|\sqrt{s}}{k\sqrt{pm_b}},$$

$$C_d C_o = \frac{fgr|w|\sqrt{ps}}{k\sqrt{m_c}},$$

Then  $T[A_d A_o] = r^4 w^2 i_a^2 i_b^2 m_4 / (k^4 p^2 s^2 m_a^2 m_b^2 m_c^2)$ , where  $m_4 = \sum_{i=0}^7 k_i k^{\lambda_i} h^{2i}$  with  $\lambda_i = 12, 8, 4, 2, 0, 0, 0, 0$  for  $i = 0, 1, \dots, 7$  and  $k_i$  is a (product of) polynomial(s) in the variable  $k$  represented as sequences  $(a_0, \dots, a_n)$  of their integer coefficients as follows:

- $k_0$  (4(1, 1)<sup>3</sup>)
- $k_1$  (512, 1024, 600, 136, 35)
- $k_2$  2(1, 1)(8192, 32768, 52224, 44032, 23422, 9332, 2631, 331)
- $k_3$  (32768, 229376, 666624, 1056768, 1002704, 583760, 203133, 38174, 2845)
- $k_4$  (16384, 147456, 563200, 1200128, 1564604, 1277204, 636048, 176284, 20820)
- $k_5$  (512, 6144, 22360, 37624, 33077, 15050, 2845)
- $k_6$  (452, 1916, 3150, 2364, 662)
- $k_7$  (5, 7)(7, 5).

For example, polynomials  $k_0$  and  $k_6$  are  $4(k+1)^3$  and  $662k^4 + 2364k^3 + 3150k^2 + 1916k + 452$ .

The above is the first example of our method of writing down in compact form rather lengthy polynomials like  $m_4$ . We simply write in parenthesis their coefficients in the increasing order starting with the trailing coefficient and since ours are polynomials in variables  $h$  and  $k$  we give polynomials of  $k$  as coefficients of powers of  $h$ .

The polynomial  $m_4$  has all coefficients positive so that the triangle test  $T[A_d A_o]$  is always positive unless  $ABC$  is

isosceles when it is zero. It follows that segments  $A_dA_o$ ,  $B_dB_o$ , and  $C_dC_o$  will be sides of a triangle for any triangle  $ABC$  which is not isosceles.

## 10 Proof of $A_fB_fC_f \in \Omega(A_dB_dC_d)$

Points  $A_f$ ,  $B_f$ , and  $C_f$  are midpoints of the segments  $AH$ ,  $BH$ , and  $CH$ , where  $H$  is the orthocenter of  $ABC$ . Since  $H$  has coordinates  $(gqr/k, qrt/2k)$ , we get

$$A_f \left( \frac{gqr}{2k}, \frac{grt}{4k} \right), B_f \left( \frac{r(2gq+ft)}{2k}, \frac{qrt}{4k} \right),$$

$$C_f \left( \frac{gqr}{k}, \frac{r((k+2)^2 - w^2)}{4k} \right).$$

Hence, the distances  $A_dA_f$ ,  $B_dB_f$ , and  $C_dC_f$  are  $m_5m_6/\sqrt{m_a}$ ,  $m_5m_7/\sqrt{m_b}$ , and  $m_5m_8/\sqrt{m_c}$ , where  $m_5 = r\sqrt{ps}/(4k)$ ,  $m_6 = qs + 4fg + 4g^2$ ,  $m_7 = pt + 4fg + 4f^2$ , and  $m_8 = qt + 4f^2g^2$ .

We obtain  $T[A_dA_f] = p^2r^4s^2m_9/(256k^4m_a^2m_b^2m_c^2)$ , where  $m_9 = \sum_{i=0}^8 k_i k^{\lambda_i} h^{2i}$  with  $\lambda_i = 11, 6, 4, 2, 1, 0, 0, 0, 0$  for  $i = 0, \dots, 8$  and

$k_0$	$(4, 3)^2(128, 256, 152, 27)$
$k_1$	$8(32768, 172032, 377344, 454400, 332544, 154112, 44938, 7632, 585)$
$k_2$	$4(131072, 966656, 2772992, 4162560, 3627008, 1897856, 589452, 100840, 7341)$
$k_3$	$8(32768, 319488, 1248256, 2487808, 2785280, 1809408, 670562, 130568, 10415)$
$k_4$	$2(32768, 434176, 1930240, 3859968, 3956992, 2140648, 572816, 58393)$
$k_5$	$(4096, 120832, 706560, 1445888, 1323504, 550912, 83320)$
$k_6$	$(6144, 66048, 142608, 112992, 29364)$
$k_7$	$(2736, 7488, 4680)$
$k_8$	$(243)$ .

The polynomial  $m_9$  has all coefficients positive so that the triangle test  $T[A_dA_f]$  is always positive. It follows that segments  $A_dA_f$ ,  $B_dB_f$ , and  $C_dC_f$  will be sides of a triangle for any triangle  $ABC$ .

In order to show that the triangle with sides  $A_dA_f$ ,  $B_dB_f$ , and  $C_dC_f$  is acute recall that the triangle  $ABC$  is acute, right, or obtuse if and only if the product  $U[a] = (b^2 + c^2 - a^2)(a^2 - b^2 + c^2)(a^2 + b^2 - c^2)$  is positive, zero, or negative (see [1]). Here, this product is  $U[A_dA_f] = p^4r^6s^4m_{10}m_{11}m_{12}/(4096k^6m_a^3m_b^3m_c^3)$ , where the polynomials  $m_{10}$ ,  $m_{11}$ , and  $m_{12}$  become polynomials in  $f'$  and  $g'$  with all coefficients positive after the substitution  $f = 1 + f'$  and  $g = 1 + g'$ .

## 11 Other proofs and extensions

We leave proofs of  $A_qB_qC_q \in \Omega(A_dB_dC_d)$  and  $A_xB_xC_x \notin \Omega(A_dB_dC_d)$  for  $x = p, n, g$  to the reader because they are almost identical to the above proofs. The point here is not that our method is elegant or simple (in the traditional sense), but that the same method applies to all cases.

Another method of proof of our theorems is to express everything in terms of the side lengths  $a$ ,  $b$ , and  $c$ . For the Theorem 1 and the part of Theorem 2 for  $A_mB_mC_m$ , the triangle test is easily seen to be always negative. For other cases the procedure is to write the numerator of the triangle test in terms of the three basic symmetric polynomials in variables  $a$ ,  $b$ , and  $c$  and then use the fact [5, p. 7] that they are roots of the polynomial  $x^3 - 2\sigma x^2 + (\sigma^2 + r(r+4R))x - 4rR\sigma$ , where  $\sigma$  is the semi-perimeter,  $r$  is the inradius, and  $R$  is the circumradius. In this way we obtain a polynomial in  $\sigma$  with coefficients polynomials in  $r$  and  $R$ . Now using the Euler inequality  $R \geq 2r$  and the fundamental inequalities between  $\sigma$ ,  $r$ , and  $R$  (see [5, Chapter I]) in each case we can argue that the triangle test function is either always positive or is never positive. However, without some help from computers this approach is also difficult.

We close with the following claims which are possible projects from geometry of triangles. Our method with polynomials applies here too.

When  $ABC$  is not equilateral, then the centre of the Kiepert hyperbola [2] has the same property as  $D$  in Theorem 1.

When  $ABC$  is not quite special, then the centre of the Jarabek hyperbola, which goes through the vertices  $A$ ,  $B$ ,  $C$ , the orthocentre  $H$ , and the circumcentre  $O$  has the same property as  $D$  in Theorem 1.

When  $ABC$  is not isosceles, then the segments  $DA_p$ ,  $DB_p$ , and  $DC_p$  are always sides of an obtuse triangle.

When  $ABC$  does not have angles of either  $\pi/3$  or  $2\pi/3$  radians, then the segments  $DA_f$ ,  $DB_f$ , and  $DC_f$  are always sides of a triangle.

The segments  $XA_d$ ,  $YB_d$ , and  $ZC_d$  are never sides of a triangle, where  $X$ ,  $Y$ ,  $Z$  are second intersections of  $DA_d$ ,  $DB_d$ , and  $DC_d$  with the incircle. The same is true for segments  $XD$ ,  $YD$ , and  $ZD$ .

An interesting project is to decide which central points  $X$  of the triangle  $ABC$  have the property that the segments  $XA_d$ ,  $XB_d$ , and  $XC_d$  are always (never) sides of a triangle.

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## References

- [1] C. Ciamberlini, *Rassegna*, **5** (1925), 241 – 245.
- [2] R. H. Eddy and R. Fritsch, *The conics of Ludwig Kiepert: A comprehensive lesson in the geometry of the triangle*, *Mathematics Magazine*, **67** (1994), 188–205.
- [3] Roger A. Johnson, *Advanced Euclidean Geometry*, Dover Publ., Washington, 1964.
- [4] Clark Kimberling, *Central points and central lines in the plane of a triangle*, *Mathematics Magazine*, **67** (1994), 163–187.
- [5] D. S. Mitrinović, J. E. Pečarić, and V. Volenec, *Recent Advances in Geometric Inequalities*, Kluwer Academic Publishers, Dordrecht, 1989.
- [6] C. F. Parry, *Steiner – Lehmus and the Feuerbach triangles*, *Math. Gaz.*, **80** (1996), 275–285.



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MILJENKO LAPAINE

# Pramen konika zadan pomoću dviju dvostrukih točaka

**Pramen konika zadan pomoću dviju dvostrukih točaka**

**SAŽETAK**

Prikazan je algoritam za određivanje koeficijenata u jednadžbi pramena konika kad je pramen zadan s četiri točke od kojih su dvije dvostruke. Da bi se moglo raditi s neizmjerljivo dalekim točkama uvedene su homogene koordinate. Postupak je ilustriran s nekoliko primjera.

**Cljučne riječi:** pramen konika, homogene koordinate, računalna grafika

**Conic Section Pencil Given by two Double Points**

**ABSTRACT**

The algorithm is presented for the determination of coefficients in the equation of conic section pencils when the pencil is given by four base points two of which are double points. In order to work with the points in infinity, the homogeneous coordinates are introduced. The approach is illustrated by several examples.

**Key words:** conic section pencil, homogeneous coordinates, computer graphics

**MSC 2000:** 51N20, 51N15

## 1. Uvod

Pramen konika općenito je određen s četiri realne i različite točke  $A, B, C$  i  $D$ . Pramen konika tipa VI (Ščurić 1996) karakteriziran je svojstvom da se temeljne točke  $A$  i  $B$  te  $C$  i  $D$  podudaraju. Tim dvostrukim točkama prolaze zajedničke tangente svih konika pramena.

Grafički prikaz pramena konika može se učinkovito ostvariti primjenom računala i priključenog crtala. Matematička osnova razvijenog softvera opisana je u radu (Lapaine 1997). U jednom prethodnom radu (Lapaine 1999) opisan je algoritam za računanje koeficijenata u jednadžbi pramena konika tipa IV, gdje je pramen zadan s tri točke (od kojih je jedna dvostruka) i zajedničkom tangentom svih konika pramena. Na analogan način, u ovome radu daje se algoritam za računanje koeficijenata u jednadžbi pramena tipa VI, gdje je pramen zadan s dvije dvostruke točke. Postupak se temelji na primjeni homogenih koordinata kako bi se i neizmjerljivo daleke točke ravnine mogle analitički obuhvatiti posve analogno kao i točke u konačnosti.

## 2. Pramen konika

Neka su

$$F(x, y) = a_1x^2 + 2b_1xy + c_1y^2 + 2d_1x + 2e_1y + f_1 = 0, \quad (1)$$

$$G(x, y) = a_2x^2 + 2b_2xy + c_2y^2 + 2d_2x + 2e_2y + f_2 = 0, \quad (2)$$

jednadžbe dviju konika. Za proizvoljni  $\mu \in \mathbf{R}$  sastavimo izraz

$$H(x, y) = F(x, y) + \mu G(x, y). \quad (3)$$

Polinom  $H$  je oblika

$$H(x, y) = ax^2 + 2bxy + cy^2 + 2dx + 2ey + f, \quad (4)$$

gdje smo označili

$$a = a_1 + \mu a_2, \dots, \quad f = f_1 + \mu f_2. \quad (5)$$

Za svaki pojedini  $\mu \in \mathbf{R}$ , izraz

$$H(x, y) = F(x, y) + \mu G(x, y) = 0 \quad (6)$$

je jednačba konike ako je barem jedan od brojeva  $a$ ,  $b$  i  $c$  različit od nule. Ako je  $a = b = c = 0$  tada se radi o specijalnim, ali jednostavnim slučajevima (Lapaine 1997).

Za zadane realne brojeve  $a_1, b_1, \dots, f_1, a_2, b_2, \dots, f_2$ , i  $\mu \in \mathbf{R}$  skup svih konika obuhvaćenih jednačbom (6) naziva se pramenom konika. Konike pomoću kojih je pramen definiran i kojima odgovaraju jednačbe

$$F(x, y) = 0 \quad \text{i} \quad G(x, y) = 0$$

nazivaju se osnovnim konikama pramena.

Za svaki čvrsti  $\mu \in \mathbf{R}$  jednačba (4) predstavlja jednu krivulju iz pramena ili u specijalnom slučaju prazan skup. Jedan način određivanja tipa konike s mogućnošću grafičkog prikazivanja pomoću računala objašnjen je u radu (Lapaine i Jovičić 1996).

### 3. Pramen konika zadan pomoću dviju dvostrukih točaka

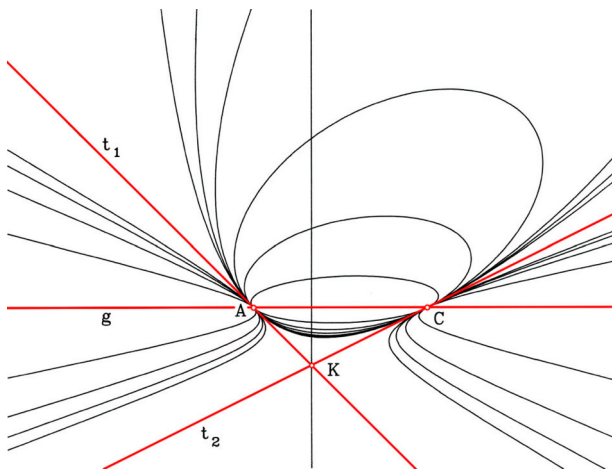
Ako su zadane četiri točke  $P_i(x_i, y_i)$   $i = 1, 2, 3, 4$  u ravnini, od kojih ni koje tri nisu kolinearne, te ako je

$$g_{ik} = a_{ik}x + b_{ik}y + c_{ik} = 0, \quad i \neq k, \quad (7)$$

jednačba pravca  $P_iP_k$ , tada je jednačbom

$$g_{12}g_{34} + \mu g_{13}g_{24} = 0 \quad (8)$$

predočen pramen konika kojem su točke  $P_i$  temeljne. Sve konike pramena prolaze točkama  $P_i$  (Cesarec 1957).



Slika 1. Pramen konika tipa VI

Neka su sada zadane dvije točke  $A$  i  $C$  i neka je  $t_1$  bilo koji pravac koji prolazi točkom  $A$ , ali ne sadrži točku  $C$  i neka

je  $t_2$  bilo koji pravac koji prolazi točkom  $C$ , a ne prolazi točkom  $A$ . Označimo s  $g$  pravac kroz točke  $A$  i  $C$  (vidi sliku 1). Ako su

$$t_1 = 0, \quad t_2 = 0 \quad \text{i} \quad g = 0 \quad (9)$$

jednačbe navedenih pravaca, tada je

$$g^2 + \mu t_1 t_2 = 0 \quad (10)$$

jednačba pramena konika kojem su  $A$  i  $C$  temeljne točke, a  $t_1$  i  $t_2$  zajedničke tangente svih konika pramena. Zaista, točka  $A$  je temeljna točka pramena jer leži na pravcima  $g$  i  $t_1$ , a točka  $C$  je temeljna jer je sjecište pravaca  $g$  i  $t_2$ . Za svaki zadani  $\mu$  jednačba (10) predstavlja jednu koniku pramena. Za točku  $(x, y)$  koja je zajednička proizvoljnoj konici pramena i pravcu  $t_1(x, y) = 0$  mora biti

$$g(x, y) = 0, \quad (11)$$

što vrijedi samo za točku koja je istovremeno na pravcima  $t_1$  i  $g$ . Dakle, radi se o točki  $A$ . Odatle zaključujemo da je pravac  $t_1$  zajednička tangenta svih konika pramena jer sa svakom konikom ima samo jednu zajedničku točku. Ta zajednička točka  $A$  naziva se dvostrukom temeljnom točkom pramena.

Analogno se pokazuje da je pravac  $t_2$  zajednička tangenta svih konika pramena i točka  $C$  dvostruka temeljna točka pramena. Može se također reći i da je pramen zadan s četiri točke od kojih su dvije i dvije pale zajedno.

### 4. Jednačba pramena konika zadanog pomoću dviju dvostrukih točaka

Neka su s pomoću homogenih koordinata (vidi npr. Lapaine 1999) zadane točke

$$A = (x_{0A}, x_{1A}, x_{2A}) \quad (12)$$

$$C = (x_{0C}, x_{1C}, x_{2C}).$$

Može se vidjeti da se jednačba pravca  $g$  koji prolazi točkama  $A$  i  $C$  može napisati u obliku

$$g_x x + g_y y + g_z = 0, \quad (13)$$

gdje smo označili

$$\begin{aligned} g_x &= \begin{vmatrix} x_{2C} & x_{2A} \\ x_{0C} & x_{0A} \end{vmatrix}, \\ g_y &= - \begin{vmatrix} x_{1C} & x_{1A} \\ x_{0C} & x_{0A} \end{vmatrix}, \\ g_z &= \begin{vmatrix} x_{1C} & x_{1A} \\ x_{2C} & x_{2A} \end{vmatrix}. \end{aligned} \quad (14)$$

Pravac  $t_1$  prolazi točkom  $A$ , a njegov smjer neka određuje jedna pomoćna točka  $T_1$  s homogenim koordinatama

$$T_1 = (x_{0T_1}, x_{1T_1}, x_{2T_1}). \quad (15)$$

Jednadžba pravca  $t_1$  koji prolazi točkama  $A$  i  $T$ , glasi tada

$$t_{1x}x + t_{1y}y + t_{1z}z = 0, \quad (16)$$

gdje smo označili

$$\begin{aligned} t_{1x} &= \begin{vmatrix} x_{2T_1} & x_{2A} \\ x_{0T_1} & x_{0A} \end{vmatrix}, \\ t_{1y} &= - \begin{vmatrix} x_{1T_1} & x_{1A} \\ x_{0T_1} & x_{0A} \end{vmatrix}, \\ t_{1z} &= \begin{vmatrix} x_{1T_1} & x_{1A} \\ x_{2T_1} & x_{2A} \end{vmatrix}. \end{aligned} \quad (17)$$

Pravac  $t_2$  prolazi točkom  $C$ , a njegov smjer neka određuje jedna pomoćna točka  $T_2$  s homogenim koordinatama

$$T_2 = (x_{0T_2}, x_{1T_2}, x_{2T_2}). \quad (18)$$

Jednadžba pravca  $t_2$  koji prolazi točkama  $C$  i  $T_2$  glasi tada

$$t_{2x}x + t_{2y}y + t_{2z}z = 0, \quad (19)$$

gdje smo označili

$$\begin{aligned} t_{2x} &= \begin{vmatrix} x_{2T_2} & x_{2C} \\ x_{0T_2} & x_{0C} \end{vmatrix}, \\ t_{2y} &= - \begin{vmatrix} x_{1T_2} & x_{1C} \\ x_{0T_2} & x_{0C} \end{vmatrix}, \\ t_{2z} &= \begin{vmatrix} x_{1T_2} & x_{1C} \\ x_{2T_2} & x_{2C} \end{vmatrix}. \end{aligned} \quad (20)$$

U 3. smo poglavlju ustanovili da je

$$g^2 + \mu t_1 t_2 = 0$$

jednadžba pramena konika kojem su  $A$  i  $C$  dvostruke temeljne točke. S pomoću relacija (12)-(20) može se izvesti daje  $g^2$  oblika

$$\begin{aligned} g^2 &= F(x, y) = \\ &= a_1x^2 + 2b_1xy + c_1y^2 + 2d_1x + 2e_1y + f_1 \end{aligned} \quad (21)$$

uz oznake

$$\begin{aligned} a_1 &= g_x^2 & b_1 &= g_x g_y \\ c_1 &= g_y^2 & d_1 &= g_x g_z \\ e_1 &= g_y g_z & f_1 &= g_z^2. \end{aligned} \quad (22)$$

Analogno  $t_1 t_2$  je oblika

$$\begin{aligned} t_1 t_2 &= G(x, y) = \\ &= a_2x^2 + 2b_2xy + c_2y^2 + 2d_2x + 2e_2y + f_2 \end{aligned} \quad (23)$$

uz oznake

$$\begin{aligned} a_2 &= t_{1x}t_{2x} & b_2 &= \frac{1}{2}(t_{1x}t_{2y} + t_{1y}t_{2x}) \\ c_2 &= t_{1y}t_{2y} & d_2 &= \frac{1}{2}(t_{1x}t_{2z} + t_{1z}t_{2x}) \\ e_2 &= \frac{1}{2}(t_{1y}t_{2z} + t_{1z}t_{2y}) & f_2 &= t_{1z}t_{2z}. \end{aligned} \quad (24)$$

Pomoću izvedenih formula možemo sastaviti program za računalo koji na temelju zadanih točaka  $A$ ,  $C$ ,  $T_1$  i  $T_2$  određuje koeficijente u jednadžbi pramena

$$\begin{aligned} F(x, y) + \mu G(x, y) &= \\ &= a_1x^2 + 2b_1xy + c_1y^2 + 2d_1x + 2e_1y + f_1 + \\ &+ \mu(a_2x^2 + 2b_2xy + c_2y^2 + 2d_2x + 2e_2y + f_2) = 0. \end{aligned}$$

## 5. Primjeri

Prethodna razmatranja izvedena su 1994. radi izrade crteža za rad "Klassifikationstheorie der Kegelschnittbüschel vom Typ VI der Isotropen Ebene", V. Ščurić koji je objavljen 1996.

Za proučavanje geometrije izotropne ravnine može se npr. uzeti monografija H. Sachsa (1987). Budući da računala sama po sebi još ne poznaju geometriju izotropne ravnine, trebalo je pojedine jednadžbe "prevesti" na jezik geometrije euklidske ravnine. U tu je svrhu, na temelju u ovome radu prikazanih formula, bio sastavljen odgovarajući potprogram za računalo u Basicu, koji polazeći od zadanih homogenih koordinata točaka  $A$ ,  $C$ ,  $T_1$  i  $T_2$  određuje koeficijente u jednadžbi pripadnog pramena konika. Postupak se temelji na primjeni homogenih koordinata kako bi se i neizmjerne daleke točke ravnine mogle analitički obuhvatiti analogno kao i one u konačnosti. Točke  $A$  i  $C$  su dvostruke točke, a točke  $T_1$  i  $T_2$  pomoćne točke. Točke  $A$  i  $T_1$  definiraju jednu, a točke  $C$  i  $T_2$  drugu zajedničku tangentu svih konika pramena.

Ukoliko dvije pomoćne točke  $T_1$  i  $T_2$  padnu zajedno, tada se radi o točki  $K = T_1 = T_2$  koja istovremeno pripada pravcima  $t_1$  i  $t_2$ .

Nakon što su izračunani koeficijenti u jednadžbi pramena, primjena odgovarajućeg programa omogućila je grafičko prikazivanje pramena (Lapaine 1997). Postupak se odvija na taj način da se najprije za pojedinu krivulju pramena izračunaju koordinate niza uzastopnih točaka. Pritom se

gustoća točaka uzduž pojedine krivulje i gustoća krivulja u pramenu mogu interaktivno regulirati. Gustoća točaka uzduž pojedine krivulje bira se tako da se pri iscrtavanju ne primijeti izlomljenost linije, ali da se istovremeno prevelikom gustoćom ne preopterećuje memorija i vrijeme izvođenja. Gustoća krivulja u pramenu određuje se odabirom koraka parametra na taj način da na slici ne bude previše linija i time slika nečitljiva i na djelovima preczna, ali da se istovremeno prikažu sve karakteristične krivulje pramena.

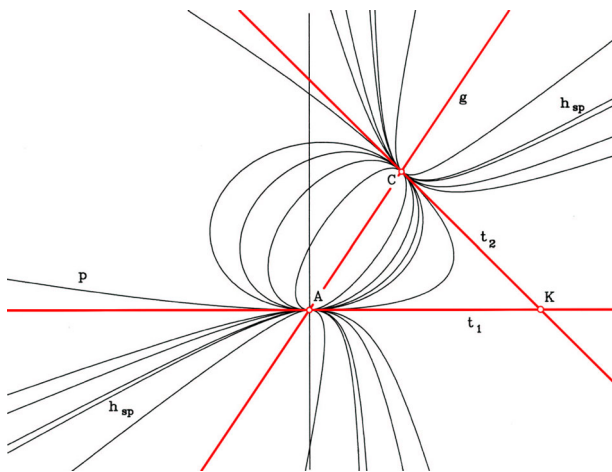
Nakon što smo zadovoljni s prikazom na ekranu monitora, slika se sprema u .DXF zapisu kako bismo je mogli učitati u AutoCAD i dalje uređivati. Tu se prvenstveno misli na opis slike, eventualno brisanje suvišnih elemenata te zadanje boje, odnosno debljine onih linija koje na slici želimo posebno istaknuti.

Primjeri koji slijede označeni su u skladu s radom (Ščurić 1996).

### Primjer 1. Pramen tipa $VI_{1a}$

Zadane su točke u homogenim koordinatama  $A(1,0,0)$ ,  $C(1,2,3)$  i pomoćna točka  $K(1,5,0)$ . Jednadžba pramena je:

$$9x^2 - 12xy + 4y^2 + \mu(xy + y^2 - 5y) = 0$$

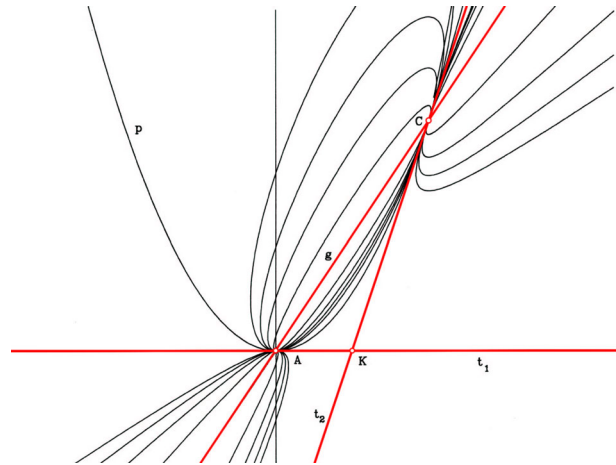


Slika 2. Pramen tipa  $VI_{1a}$

### Primjer 2. Pramen tipa $VI_{1,1a}$

Zadane dvostruke točke u homogenim koordinatama  $A(1,0,0)$ ,  $C(1,4,6)$  i pomoćna točka  $K(1,2,0)$ . Jednadžba pramena je:

$$9x^2 - 12xy + 4y^2 + \mu(3xy - y^2 - 6y) = 0$$

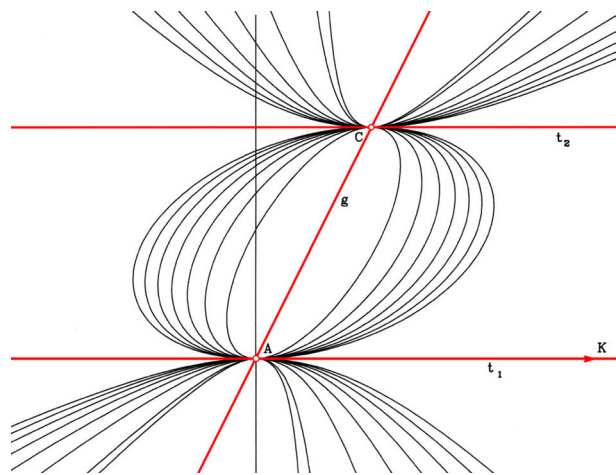


Slika 3. Pramen tipa  $VI_{1,1a}$

### Primjer 3. Pramen tipa $VI_{1c}$

Zadane dvostruke točke u homogenim koordinatama:  $A(1,0,0)$ ,  $C(1,4,8)$  i pomoćna točka  $K(0,1,0)$ . Jednadžba pramena je:

$$4x^2 - 4xy + y^2 + \mu(y^2 - 8y) = 0$$

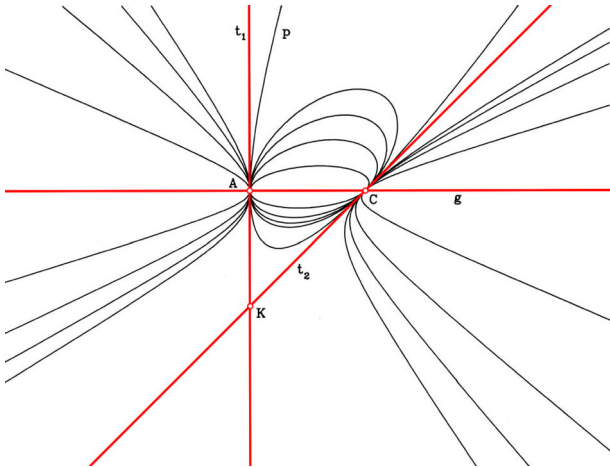


Slika 4. Pramen tipa  $VI_{1c}$

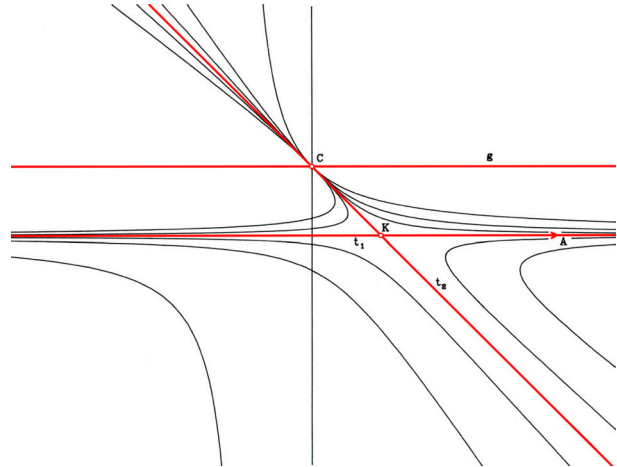
### Primjer 4. Pramen tipa $VI_{2a}$

Zadane dvostruke točke u homogenim koordinatama:  $A(1,0,0)$ ,  $C(1,4,0)$  i pomoćna točka  $K(1,0,-4)$ . Jednadžba pramena je:

$$16y^2 + \mu(-4x^2 + 4xy + 16x) = 0$$



Slika 5. Pramen tipa VI<sub>2a</sub>

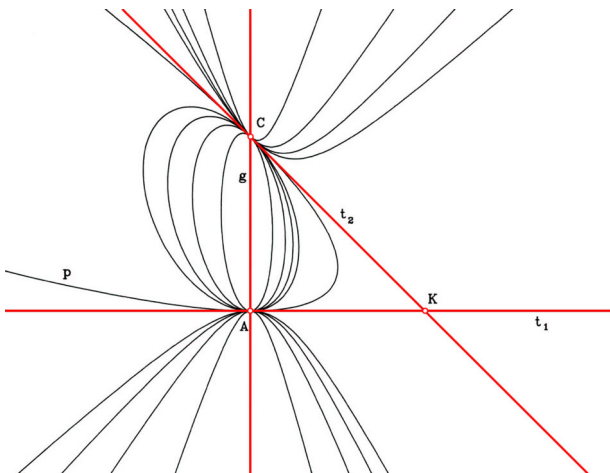


Slika 7. Pramen tipa VI<sub>4</sub>

**Primjer 5. Pramen tipa VI<sub>3a2</sub>**

Zadane dvostruke točke u homogenim koordinatama: A(1, 0, 0), C(1, 0, 6) i pomoćna točka K(1, 6, 0). Jednadžba pramena je:

$$x^2 + \mu(xy + y^2 - 6y) = 0$$

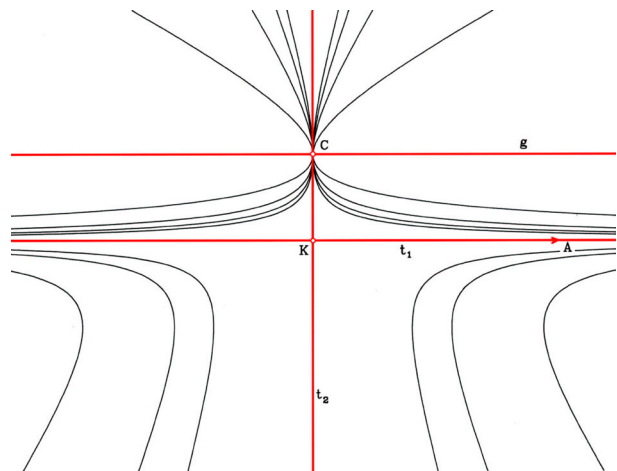


Slika 6. Pramen tipa VI<sub>3a2</sub>

**Primjer 7. Pramen tipa VI<sub>5</sub>**

Zadane dvostruke točke u homogenim koordinatama: A(0, 1, 0), C(1, 0, 3) i pomoćna točka K(1, 0, 0). Jednadžba pramena je:

$$y^2 - 6y + 9 + \mu xy = 0$$



Slika 8. Pramen tipa VI<sub>5</sub>

**Primjer 6. Pramen tipa VI<sub>4</sub>**

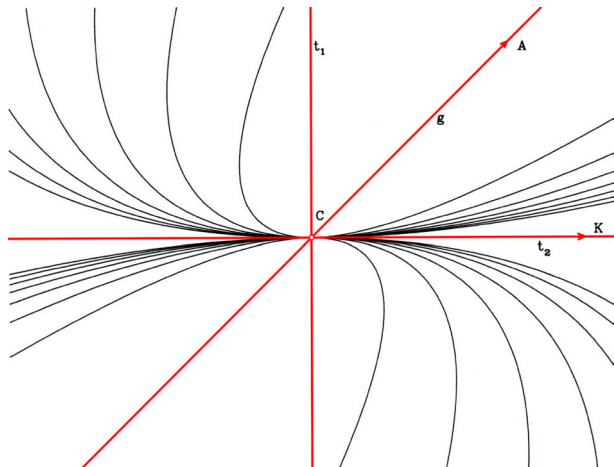
Zadane dvostruke točke u homogenim koordinatama: A(0, 1, 0), C(1, 0, 0) i pomoćna točka K(1, 6, 0). Jednadžba pramena je:

$$y^2 - 12y + 36 + \mu(xy + y^2 - 6y) = 0$$

**Primjer 8. Pramen tipa VI<sub>13</sub>**

Zadane dvostruke točke u homogenim koordinatama: A(0, 1, 1), C(1, 0, 0) i pomoćna točka K(0, 1, 0). Jednadžba pramena je:

$$x^2 - 2xy + y^2 + \mu y = 0$$

Slika 9. Pramen tipa  $VI_{13}$ 

### Literatura

CESAREC, R. (1957): Analitička geometrija linearnog i kvadratnog područja, I dio. Školska knjiga, Zagreb.

LAPAINÉ, M. (1997): Grafički prikaz pramena konika pomoću računala. KoG 2, 43-47.

LAPAINÉ, M. (1999): Pramen konika zadan pomoću jedne dvostruke i dviju jednostrukih točaka. KoG 4, 27-32.

LAPAINÉ, M., JOVIČIĆ, D. (1996): Grafički prikazi konika pomoću računala. KoG 1, 19-26.

SACHS, H. (1987): Ebene Isotrope Geometrie, Friedr. Vieweg & Sohn, Braunschweig, Wiesbaden.

ŠČURIĆ, V. (1996): Klassifikationstheorie der Kegelschnittbüschel vom Typ VI der Isotropen Ebene. Mathematica Pannonica, 7/1, 47-67.

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IVANKA BABIĆ

# Torus i neke njegove primjene u strojarstvu

## Torus i neke njegove primjene u strojarstvu

### SAŽETAK

U članku su istaknute neke karakteristike otvorene ili prstenaste torusne plohe, između ostalog i tri sistema kružnih presjeka. Torus tog oblika primjenjuje se u strojarstvu kod tvorbe strojnih dijelova. Neki strojni dijelovi nastaju presjecanjem složenih rotacijskih tijela ravninama, a neki tako da se koriste prodori dviju ili više rotacijskih ploha. Naveden je primjer oblikovanja tijela ojnice, te cijevnog odvojka koji se sastoji od valjkaste i torusne plohe koje su spojene po prodornoj krivulji. U članku se ukazuje na jednostavnost konstruiranja projekcija presječnih i prodornih krivulja u čijoj podlozi stoji geometrijska teorija.

**Ključne riječi:** apsolutna konika, cijevni odvojak, ojnice, otvoreni torus

**MSC 2000:** 51N05

## Torus and some of its Applications in Mechanical Engineering

### ABSTRACT

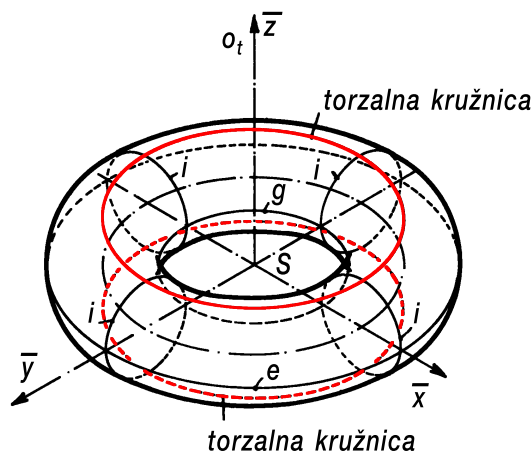
Some characteristics of the open or ring torus such as the three systems of circle sections are pointed out in the article. The torus of this shape is applied in mechanical engineering for designing certain parts of machines. Some machine parts are obtained by the intersection of a surface of revolution and a plane, while some others are obtained by the intersection of surfaces of revolution. The example presented in the article shows how the body of a thrust bar and a pipe diverging which consist of a cylinder and a torus are created. The article emphasizes the simplicity of the construction of the projection of the intersection curves which derives from the geometric theory.

**Key words:** absolute conic, pipe diverging, ring torus, thrust bar

U svim tehničkim znanostima, pa tako i u strojarstvu, neophodno je poznavanje nacrtne geometrije. Koliko se geometrije krije u naoko jednostavnim strojnim dijelovima pokazat će se na nekoliko primjera u ovom članku. Konstrukcija i izrada složenih strojnih dijelova, alata i kalupa ne može se zamisliti bez rotacijskih ploha: valjka, stošca, kugle i torusa. Jedan strojni element je sastavljen od više rotacijskih tijela, a oblikuje se presijecanjem alatima (pilama, glodalima, itd.) tj. obradom odvajanja čestica. Pri toj obradi koriste se "radionički crteži" na kojima su u projekcijama nacrtani temeljni oblici složenog rotacijskog tijela i ravnine kojima se tijelo presijeca. U većini je slučajeva torus sastavni dio pojedinih strojnih dijelova, pa je potrebno detaljnije upoznati tu plohu.

Torus je rotacijska ploha 4. reda koja nastaje rotacijom kružnice oko čvrstog pravca njezine ravnine [1]. S obzirom na međusobni položaj kružnice i osi rotacije, postoje tri oblika torusa. U strojarstvu se većinom primjenjuje *otvoreni torus* ili *prstenasta ploha*, koja nastaje rotacijom kružnice oko pravca koji ju realno ne siječe. Os rotacije je pritom ujedno i os torusa. Na otvorenom torusu posebno se ističu *grlena kružnica* i *ekvator* tj. kružnice najmanjeg i najvećeg

radijusa, koje se nalaze u ravnini okomitoj na os torusa, tzv. glavnoj simetralnoj ravnini torusa (slika 1). U sistemu ravnina okomitih na os torusa postoje dvije ravnine koje ga dodiruju duž kružnica koje su sukladne središnjoj



Slika 1.

kružnici torusa, a koje dijele “unutarnji” od “vanjskog” dijela torusa. To su *torzalne kružnice* na torusnoj plohi, tj. sve njihove točke su *paraboličkog* tipa. Sve točke na vanjskom dijelu torusne plohe su *eliptičke*, a na unutarnjem su *hiperboličke*. Torus je jedna od ploha koja nosi sve tri vrste točaka. (Općenito se kaže da je točka neke plohe *eliptičkog tipa* ako dirna ravnina plohe s diralištem u toj točki u njenom okolišu nema drugih zajedničkih realnih točaka s plohom. U *hiperboličkoj točki* plohe dirna ravnina realno siječe plohu u okolišu dirališta u krivulji, a diralište je dvostruka točka te krivulje. Točka plohe je *paraboličkog tipa* ako dirna ravnina u okolišu dirališta dira plohu duž neke krivulje koja se tada naziva *torzalnom* krivuljom.[5])

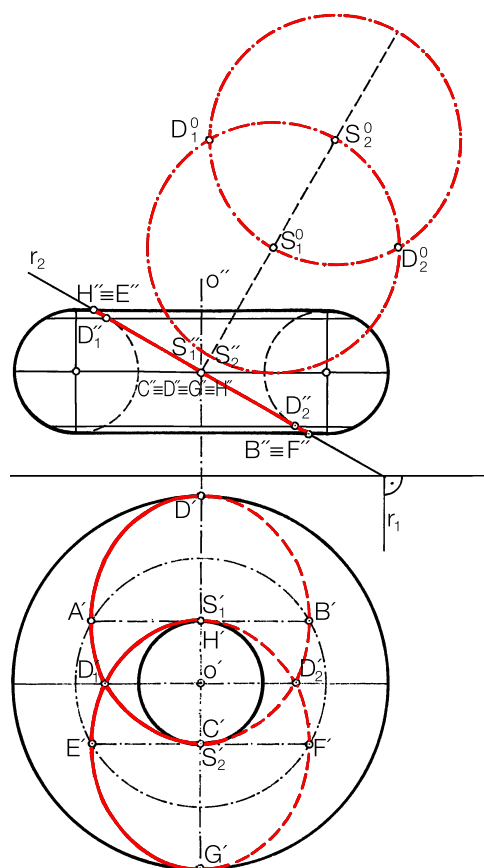
Budući da je torus algebarska ploha 4. reda, svaka ga ravnina siječe u algebarskoj krivulji 4. reda. Važno je napomenuti da postoje tri sistema presječnih ravnina koje sijeku torus u krivuljama 4. reda koje degeneriraju u par kružnica[1].

*Prvi sistem* čine sve ravnine položene kroz os torusa, a koje ga sijeku u paru sukladnih kružnica. Svaki par ovakvih kružnica čini *meridijan* torusa. Budući da se radi o pramenu ravnina koje sijeku torus u krivulji 4. reda degeneriranoj na dvije kružnice, slijedi da u svakoj od spomenutih ravnina torus ima dvije apsolutne (dvostruke) točke. Poznato je da se apsolutna konika nalazi na svakom torusu kao njegova imaginarna dvostruka krivulja [1]. Stoga su svi ravninski presjeci torusa bicirkularne krivulje 4. reda.

*Drugi sistem* kružnih presjeka torusa čine ravnine okomite na os torusa. U svakoj takvoj ravnini nalaze se dvije koncentrične kružnice, koje mogu biti realne i različite (jedna na unutarnjem, a druga na vanjskom dijelu torusa), pasti zajedno (torzalne kružnice), ili mogu biti imaginarne.

*Treći sistem* degeneriranih presjeka torusa čine tzv. dvostruke dirne ravnine torusa, tj. ravnine koje diraju torus u po dvije hiperboličke točke. Naime, svaka dirna ravnina torusa postavljena njegovom hiperboličkom točkom siječe torus u krivulji 4. reda koja u diralištu ima dvostruku točku. Uz ovu dvostruku točku ima ta krivulja i dvije imaginarne dvostruke točke na neizmjereno dalekoj apsoluti [1]. Unutar svih ravnina koje diraju torus u njegovim hiperboličkim točkama postoje i takve koje ga diraju u po dvije hiperboličke točke (tzv. dvostruke dirne ravnine). Presječna krivulja torusa s takvom ravninom ima četiri dvostruke točke i to dvije realne i dvije apsolutne. Kao što je poznato, ravninska krivulja 4. reda s četiri dvostruke točke raspada se ili na dvije konike ili pravac i krivulju trećeg reda. U ovom se slučaju ona raspada na dvije konike koje prolaze apsolutnim točkama ravnine, dakle na dvije kružnice. Naime, Villarceau-ov teorem (1848) kaže: “*Svaka dvostruka dirna ravnina torusa siječe torus*

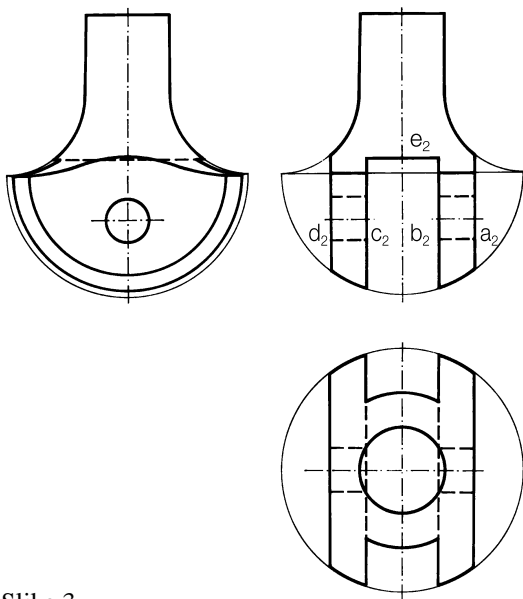
*u dvije kongruentne kružnice*” [1]. Postoji beskonačno mnogo ravnina s ovim svojstvom i one omataju dirni stožac torusa. Jedna od takvih ravnina zajedno s Villarceau-ovim kružnicama prikazana je na slici 2.



Slika 2.

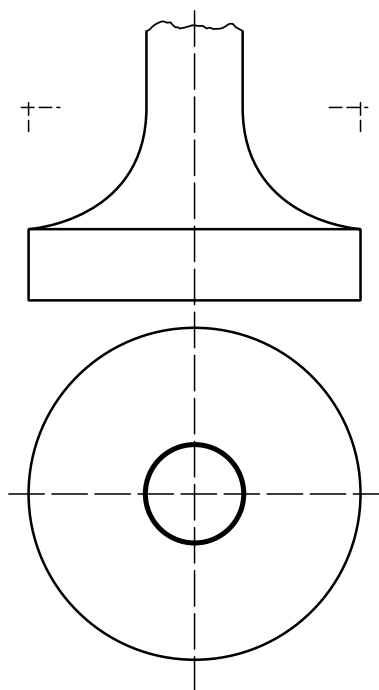
Primjenu torusa susrećemo primjerice kod oblikovanja strojnih dijelova, koji nastaju presjecanjem rotacijskih tijela ravninama. Radi se o tzv. *pogonskim ojnica* koje predstavljaju spojne elemente križne glave i rukavca koljenastog vratila [2]. Zadatak im je prenošenje snage sa svornjaka križne glave na rukavac koljenastog vratila. Ojnice se sastoje od dviju glava i tijela. Jedna glava ojnice se veže s križnom glavom, a druga s rukavcem koljenastog vratila. Dobivaju se obradom složenih rotacijskih tijela koja su uglavnom sastavljena od kugle, torusa, stošca i valjka. Na primjer, da bi se dobila viličasta glava ojnice (slika 3) složeno se rotacijsko tijelo, sastavljeno od polukugle, torusa i valjka, presijeca s pet ravnina, a za plosnatu zatvorenu glavu ojnice tijelo se presijeca s dvije ravnine. Njihova konstruktivna obrada detaljno je izvedena u sveučilišnom udžbeniku [4].





Slika 3.

Tijelo ojnice se također dobiva obradom složenog rotacijskog tijela. Postoji više oblika tijela ojnica. Na slici 4a. prikazane su projekcije složenog rotacijskog tijela, koje se sastoji od valjkaste ploče, četvrtine torusne plohe i valjka. Presijecanjem tog tijela ravninama, što se u praksi izvodi obradom odvajanja čestica, dobije se tijelo ojnice.



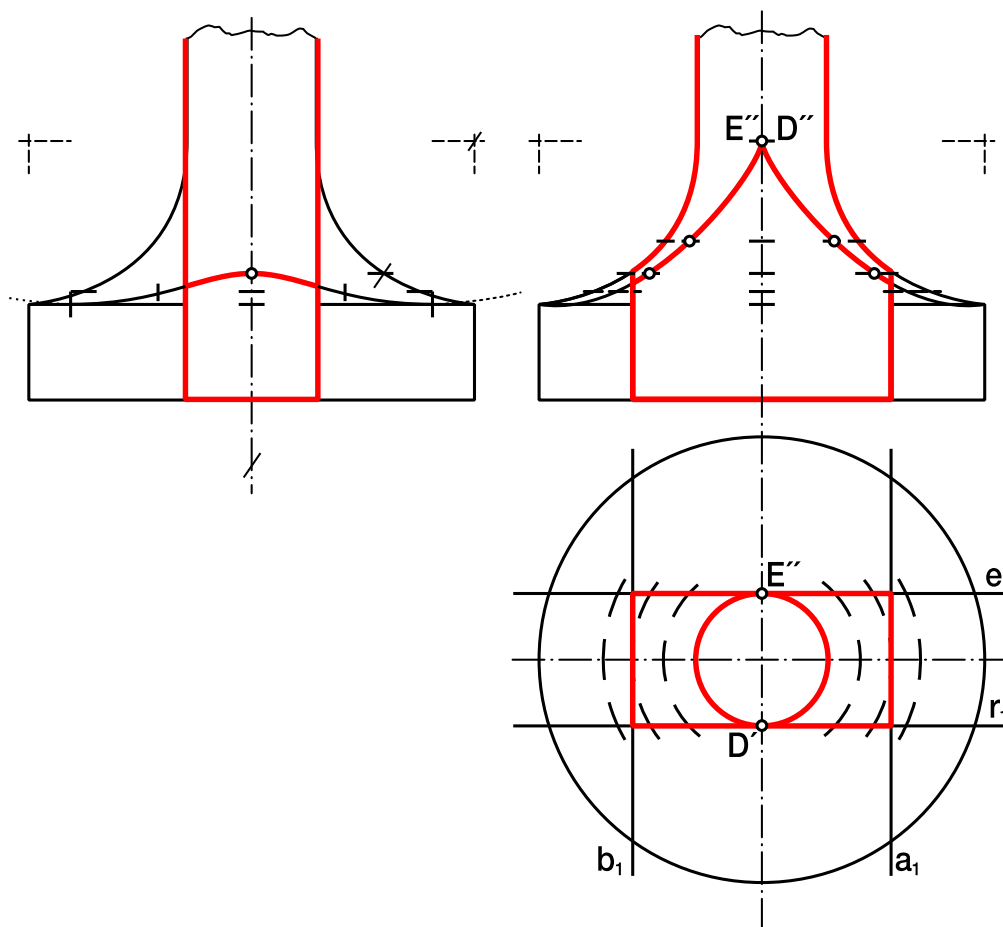
Slika 4a.

Ravnine kojima oblikujemo tijelo ojnice usporedne su s osi rotacijskog tijela i simetrično postavljene s obzirom na os tako da su dvije od njih usporedne s ravninom  $\Pi_2$ , a dvije

s ravninom  $\Pi_3$ . Postoje dvije varijante tijela ojnice: ravnine usporedne s ravninom  $\Pi_2$  dodiruju torus u točki grlene kružnice ili ga ne dodiruju. Na sl 4b. prikazana je prva varijanta.

Dijelovi presječnih krivulja koje dobivamo na torusnoj plohi su 4. reda. Naime, ravnine paralelne s osi torusa presijecaju torus u Persejevim krivuljama [7]. Postoji šest različitih oblika tih krivulja, među koje spada i meridijan torusa. Oblik krivulje ovisi o udaljenosti presječne ravnine od osi torusa. Ravnina **P**, odnosno **E**, dodiruje torus u točki grlene kružnice u kojoj presječna krivulja ima dvostruku točku, a cijela krivulja ima oblik lemniskate. Ravnine **A** i **B** sijeku torus u jednodjelnim krivuljama koje imaju po četiri infleksione točke. Točke presječnih krivulja određene su na torusnoj plohi pomoću usporednika. Ravnine kojima se presjeca torus sijeku i valjkastu ploču u pravokutnicima. Od rotacijskog tijela odstranjen je dio ispred ravnine **P**, iza ravnine **E**, desno od ravnine **A** i lijevo od **B**.

Neki strojni dijelovi nastaju i tako da se koriste *prodori* dviju ili više rotacijskih ploha. Najviše se primjenjuju prodori dviju rotacijskih ploha, čija se prodorna krivulja raspada na krivulje nižih redova. Na tom su principu izrađena tzv. *koljena* koja mogu biti sastavljena od valjkastih ili stožastih ploha. Plohe su spojene po elipsama koje su dijelovi raspadnute prodorne krivulje 4. reda. *Cijevni odvojci* su strojni dijelovi izrađeni po principu sastavljanja pojedinih ploha duž raspadnutih prodornih krivulja reda višeg od četiri, jer se uglavnom radi o prodoru torusne plohe s valjkastom, stožastom ili drugom torusnom plohom. Na slici 5. prikazan je nacrt *cijevnog odvojka* koji se sastoji od valjkaste i torusne plohe koje su spojene po prodornoj krivulji. Plohe imaju iste promjere, a zajednička im je ravnina simetrije usporedna s ravninom  $\Pi_2$ . Točke prodorne krivulje mogu se odrediti na dva načina: metodom ravnina i metodom "kliznih" kugli. U ovom primjeru to je učinjeno metodom ravnina. Umjesto tlocrta ploha korišteni su stranocrti. Ravnine su postavljene usporedno s ravninom  $\Pi_2$  te svaka siječe valjkastu plohu u po dvije izvodnice, a torus u dvije koncentrične kružnice. Sjecišta izvodnica i kružnica su točke prodorne krivulje. U svakoj ravnini se na taj način dobije osam točaka prodorne krivulje koje mogu biti realne ili su u parovima konjugirano imaginarne, a može se dogoditi da više njih padne zajedno. Prodorna krivulja je 8. reda. Budući da plohe imaju jednake promjere i zajedničku ravninu simetrije usporednu s  $\Pi_2$  imaju i dvije zajedničke dirne ravnine usporedne također s ravninom  $\Pi_2$ . Svaka od tih ravnina sadrži torzalnu kružnicu torusa i dvostruko brojevu izvodnicu valjka čija su sjecišta četiri realne dvostruke točke prodorne krivulje. Izvan zajedničkih dirnih ravnina nema više realnih točaka prodorne krivulje. No, ako se uzme u obzir da torus dva puta prolazi apsolutom, a neizmjereno

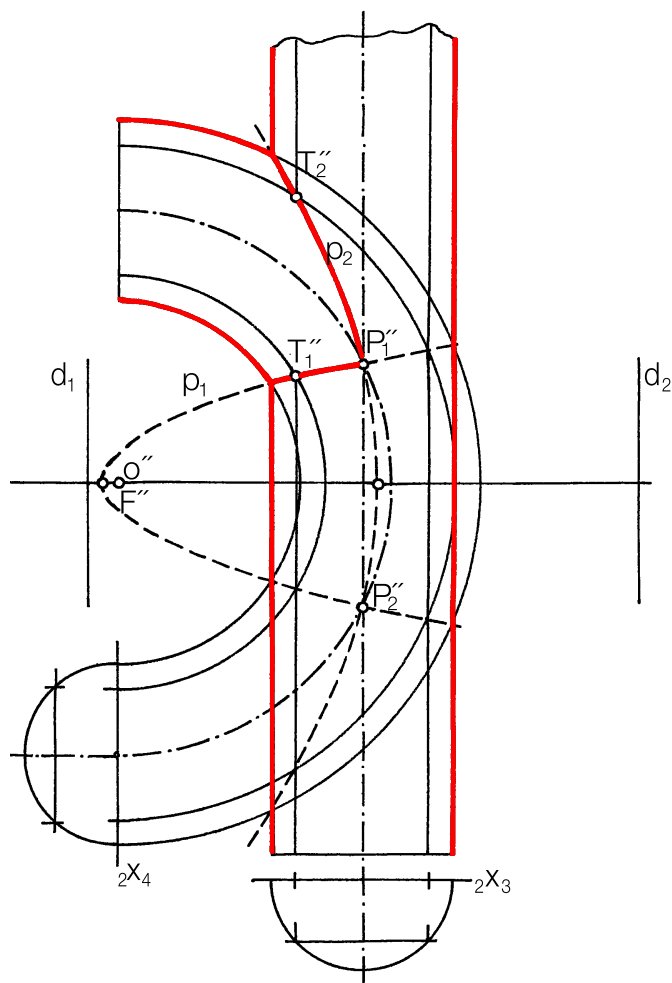


Slika 4b

daleka ravnina prostora presijeca zadani valjak u izotropnim pravcima koji diraju apsolutu [1], postoje i imaginarne dvostruke točke prodorne krivulje. Zbog prevelikog broja dvostrukih točaka prodorna se krivulja raspada na dvije krivulje 4. reda. Zbog zajedničke ravnine simetrije usporedne s ravninom  $\Pi_2$ , prodorna se krivulja projicira u nacrtu u dvije dvostruko brojene krivulje 2. reda koje se sijeku u dvostrukim točkama  $P_1$  i  $P_2$  te prodorne krivulje. Konstruktivno se može pokazati da su to dvije parabole koje imaju zajednički fokus i zajedničku os. Analitički se može dokazati da se u ovom primjeru fokus tih parabola poklapa s nacrtom projekcijom osi torusa [6], a ravnalice tih parabola su pravci usporedni s izvodnicama valjka. Ravnalice se konstruiraju tako da se polumjer usporednika, na kojem leže točke  $P_1$  i  $P_2$ , prenese na obje strane one izvodnice valjka, na kojoj se te točke nalaze. Tada se za bilo koju točku  $T_1$  krivulje  $p_1$ , odnosno  $T_2$  krivulje  $p_2$  može pokazati da vrijedi  $d(T_1F) = d(T_1d_1)$ , odnosno  $d(T_2F) = d(T_2d_2)$ , iz čega je očito da se radi o parabolama.

Posebni slučaj cijevnog odvojka nastaje kada os valjka tangira središnju kružnicu torusa. Tada se prodorna krivulja raspada na dvostruko brojenu kružnicu, koja se projicira u dužinu  $p_1$  i krivulju 4. reda, koja se projicira u parabolu  $p_2$ . Dvostruke točke  $P_1$  i  $P_2$  padaju zajedno. Analitički dokaz ove činjenice nalazi se u [6].

Postoji još niz primjera strojnih dijelova izrađenih na principu prodora rotacijskih ploha koji ukazuju na jednostavnost, ne samo projekcija, nego i samih prodornih krivulja. Na primjer *koljeno za odtjecanje čine* tri torusa čije se prodorne krivulje mogu konstruirati i kao ravninski presjeci zbog njihovog međusobnog položaja, zajedničke ravnine simetrije i jednakih promjera tih torusa [3].



Slika 5.

## Literatura

- [1] BEREIS, R., *Darstellende Geometrie I*, Akademie-Verlag GmBh, Berlin, 1964.
- [2] ELČIĆ, Z. ..., OBERŠMIT, E. ... (grupa autora), *Praktičar 3. - Strojstvo 2.*, Školska knjiga, Zagreb, 1973.
- [3] HOHENBERG, F., *Konstruktivna geometrija u tehničari*, (Niče V., prijevod s njemačkog), Građevinska knjiga, Beograd, 1966.
- [4] HORVATIĆ-BALDASAR, K., BABIĆ, I., *Nacrtna geometrija*, SAND d.o.o., Zagreb 1997.
- [5] MARKOVIĆ, Ž., *Uvod u višu analizu*, NZH, Zagreb, 1948.
- [6] STRUBECKER, K., *Nacrtna geometrija*, (Palman D., prijevod s njemačkog), Tehnička knjiga, Zagreb, 1970.
- [7] WUNDERLICH, W., *Darstellende Geometrie I*, B.I.-AG, Mannheim, 1966.

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# Opća teorija centralnih ploha 2. reda uz podršku softvera Mathematica

## Opća teorija centralnih ploha 2. reda uz podršku softvera Mathematica

### SAŽETAK

U radu je dan postupak za prepoznavanje grafova centralnih kvadratika zadanih jednadžbama u općem obliku kao i njihova klasifikacija. Napisan je program za računalo koji na osnovu početnih 10 koeficijenata opće jednadžbe plohe drugog reda izračunava svojstvene vrijednosti i svojstvene vektore matrice sužene forme plohe, ispisuje kanonski oblik jednadžbe plohe te plohu klasificira i iscrtava, ukoliko je realna, u standardnom (kanonskom) kao i u općem položaju.

**Ključne riječi:** kvadrike, linearni operatori, svojstvene vrijednosti, svojstveni vektori, dijagonalizacija, kvadratna forma

**MSC 2000:** 15A21, 53A05

## The General Theory of Central Quadrics with Mathematica

### ABSTRACT

This paper offers the procedure for recognising central quadrics graphs given by their equations in a general form as well as their classification. There has been a program made for the computer that computes eigenvalues and eigenvectors of the matrix of the associated quadratic form within the quadratic equation on the basis of ten initial general equation coefficients of the second order surface. It writes as well the canonical equation form of the surface, classifies and delineates the surface, if it is real, in the standard (canonical) as well as in the general position.

**Key words:** quadrics, linear operators, eigenvalues, eigenvectors, diagonalisation, quadratic form

## 1. Uvod

Računalni softver *Mathematica* omogućuje vrlo dobru vizualizaciju ploha drugog reda (kvadratika) upotrijebe li se njihove parametarske jednadžbe. Zadatak postaje složeniji radi li se o kvadrikama zadanim njihovim jednadžbama u općem obliku. U ovom je radu dan postupak za prepoznavanje grafa centralne (nedegenerirane) kvadrike zadane takvom općom jednadžbom te njihova klasifikacija. Napisan je i program za računalo koji na osnovu početnih 10 koeficijenata opće jednadžbe plohe drugog reda izračunava svojstvene vrijednosti i svojstvene vektore matrice sužene forme plohe, ispisuje kanonski oblik jednadžbe plohe te plohu klasificira i iscrtava, ukoliko je realna, u standardnom (kanonskom) kao i u općem položaju.

Da bismo opravdali upotrijebljene postupke potrebno je objasniti određene pojmove:

- *Linearni operatori;*
- *Slične matrice;*

- *Svojstvene vrijednosti i svojstveni vektori;*
- *Dijagonalizacija;*
- *Simetrični operator, simetrična matrica;*
- *Ortonomirana baza, ortogonalne matrice;*
- *Dijagonalizacija simetrične matrice;*
- *Kvadratna forma.*

## 2. Linearni operatori

Neka je  $\mathbf{X}$  vektorski prostor nad poljem  $\mathbf{R}$  realnih brojeva (6, str.8). Elemente vektorskog prostora nazivamo vektorima, a elemente polja skalarima. Uobičajeno je da preslikavanje koje ima domenu i kodomenu u vektorskom prostoru zovemo *operator*. Za operator  $A : \mathbf{X} \rightarrow \mathbf{X}$  kažemo da je linearan ako vrijedi

$$A(\lambda \mathbf{x} + \mu \mathbf{y}) = \lambda A(\mathbf{x}) + \mu A(\mathbf{y}) \quad (1)$$

za sve skalare  $\lambda, \mu$  i sve vektore  $\mathbf{x}, \mathbf{y} \in \mathbf{X}$ .

$A(\mathbf{x})$  ćemo zapisivati  $A\mathbf{x}$ .

Veza između linearnog operatora s prostora  $\mathbf{X}$  u  $\mathbf{X}$  i matrica ostvaruje se na osnovu činjenice da je svaki linearni operator  $A : \mathbf{X} \rightarrow \mathbf{X}$  potpuno određen svojim vrijednostima na vektorima baze prostora  $\mathbf{X}$  (6, str.60; 1, str.182).

Neka je npr.  $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$  baza (1, str.244) 3-dimenzionalnog vektorskog prostora  $\mathbf{X}$  i  $A : \mathbf{X} \rightarrow \mathbf{X}$  linearni operator određen svojim vrijednostima  $A\mathbf{e}_1, A\mathbf{e}_2, A\mathbf{e}_3$  na bazi. Rastavimo li vektore  $A\mathbf{e}_1, A\mathbf{e}_2, A\mathbf{e}_3$  po vektorima baze  $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$  dobivamo

$$\begin{aligned} A\mathbf{e}_1 &= a_{11}\mathbf{e}_1 + a_{21}\mathbf{e}_2 + a_{31}\mathbf{e}_3 \\ A\mathbf{e}_2 &= a_{12}\mathbf{e}_1 + a_{22}\mathbf{e}_2 + a_{32}\mathbf{e}_3 \\ A\mathbf{e}_3 &= a_{13}\mathbf{e}_1 + a_{23}\mathbf{e}_2 + a_{33}\mathbf{e}_3 \end{aligned} \quad (2)$$

gdje smo komponente vektora  $A\mathbf{e}_1$  u bazi  $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$  označili sa  $a_{11}, a_{21}, a_{31}$ , itd.

Matricu

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \quad (3)$$

zovemo *matricom operatora A* u bazi  $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ .

Dakle, u bazi  $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$  je svakom linearnom operatoru  $A : \mathbf{X} \rightarrow \mathbf{X}$  pridružena matrica (3) i obratno, za matricu (3) postoji jedinstveni linearni operator  $A$  takav da vrijedi (2), što je posljedica teorema o jedinstvenom prikazu vektora u bazi (6, str.59).

Jednakost linearnih operatora kao i operacije s linearnim operatorima svode se na jednakost i operacije s matricama.

Služit ćemo se sljedećim operacijama s linearnim operatorima (6, str.56):

- Zbroj linearnih operatora;
- Produkt skalara i linearnog operatora;
- Kompozicija linearnih operatora.

Navest ćemo bez dokaza svojstva tih operacija:

- Zbroj linearnih operatora je linearni operator;
- Produkt skalara i linearnog operatora je linearni operator;
- Kompozicija linearnih operatora linearni je operator (produkt operatora) (6, str.59).

Osim toga, potrebna je i operacija skalarnog produkta te pojam ranga i defekta operatora.

### Skalarni produkt

**Definicija 1** Neka su  $\mathbf{a}, \mathbf{b}$  zadani vektori iz vektorskog prostora  $\mathbf{X}$  i neka je  $\varphi$  kut među njima. Skalarni produkt (umnožak) vektora  $\mathbf{a}, \mathbf{b}$  definira se sa:

$$\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}||\mathbf{b}|\cos\varphi. \quad (4)$$

Time je definirano preslikavanje:  $\mathbf{X} \times \mathbf{X} \rightarrow \mathbf{R}$ .

Skalarni umnožak dvaju vektora danih u matricnom zapisu sa

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, \quad \mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} \quad (5)$$

glasi:

$$\mathbf{x} \cdot \mathbf{y} = \mathbf{x}^T \mathbf{y}, \text{ odnosno } \mathbf{x} \cdot \mathbf{y} = x_1y_1 + x_2y_2 + x_3y_3. \quad (6)$$

### Rang i defekt operatora. Regularni operator

Neka je  $A$  matrica pridružena linearnom operatoru  $A : \mathbf{X} \rightarrow \mathbf{X}$ . Tada je jednadžba

$$A\mathbf{x} = \mathbf{0} \quad (7)$$

ekvivalentna homogenom linearnom sustavu (3, str.63)

$$A\mathbf{x} = \mathbf{0}. \quad (8)$$

Promatramo li skup

$$\text{Ker}A = \{\mathbf{x} \in \mathbf{X} | A(\mathbf{x}) = \mathbf{0}\} \quad (9)$$

koji nazivamo *jezgrom* ili *nulpotprostorom* operatora  $A$ , vidimo da je potprostor rješenja homogenog linearnog sustava jednak jezgri operatora. Taj je potprostor razapet s  $n - r$  vektora, gdje je  $n$  dimenzija prostora  $\mathbf{X}$ , a  $r$  rang matrice  $A$  (3, str.48). Dimenziju tog potprostora odnosno jezgre nazivamo *defektom operatora* i označavamo  $k$ .

Svakom linearnom operatoru pridružen je i skup

$$\text{Im}A = \{\mathbf{y} \in \mathbf{X} | \mathbf{y} = A(\mathbf{x}), \text{ za neki } \mathbf{x} \in \mathbf{X}\}, \quad (10)$$

koji nazivamo *slikom* operatora  $A$ , a koji je i sam vektorski potprostor. Njegovu dimenziju označavamo  $r$ .

Nije teško dokazati sljedeći teorem koji je sada intuitivno jasan:

**Teorem 1** Ako je  $n$  dimenzija prostora  $\mathbf{X}$ ,  $k$  dimenzija jezgre, a  $r$  dimenzija slike operatora  $A : \mathbf{X} \rightarrow \mathbf{X}$ , tada vrijedi:  $n = r + k$ .

Dokaz (3, str.138).

Posebnu skupinu operatora tvore takozvani *regularni operatori*. To su bijektivna preslikavanja pa stoga postoji njihov inverz i mora biti  $\dim(\text{Ker}A) = 0$ . Kao što smo i ranije napomenuli, operacije s linearnim operatorima svode se na operacije s matricama pa stoga vrijedi: *Operator je regularan ako i samo ako je pripadna matrica regularna.*

### 3. Slične matrice

Linearnom operatoru su u različitim bazama pridružene različite matrice. Bit će od posebnog interesa potražiti takvu bazu u kojoj će prikaz linearnog operatora biti najjednostavniji.

Postavljamo stoga pitanje koja je veza između dviju matrica  $A$  i  $B$  koje su prikaz istog operatora  $A : \mathbf{X} \rightarrow \mathbf{X}$  ali u različitim bazama  $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$  odnosno  $\{\mathbf{e}'_1, \mathbf{e}'_2, \mathbf{e}'_3\}$ . Poznato je (3, str.129; 1, str.428) da, ako je  $A$  prikaz operatora  $A$  u bazi  $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$  prostora  $\mathbf{X}$  i  $T$  matrica prijelaza (3, str.129) iz te baze u novu bazu  $\{\mathbf{e}'_1, \mathbf{e}'_2, \mathbf{e}'_3\}$ , u novoj bazi operatoru  $A$  odgovara matrica

$$B = T^{-1}AT. \quad (11)$$

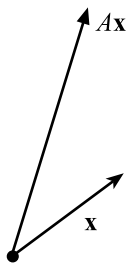
Za dvije matrice  $A$  i  $B$ , za koje postoji *regularna* (invertibilna) matrica  $T$  sa gornjim svojstvom kažemo da su slične. Dakle, operatoru  $A$  odgovaraju u različitim bazama slične matrice. Od zajedničkih svojstava sličnih matrica ovdje ćemo izdvojiti da slične matrice imaju istu determinantu (3, str.129; 6, str.195), tj.

$$\det(B) = \det(T^{-1}AT) = \det(T^{-1})\det(A)\det(T) = \det(A) \quad (12)$$

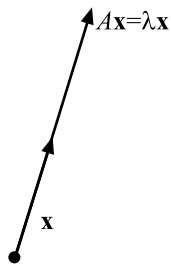
jer za regularne matrice vrijedi:  $\det(T^{-1}) = 1/\det(T)$ .

### 4. Svojsvene vrijednosti i svojsveni vektori

Neka je  $A : \mathbf{X} \rightarrow \mathbf{X}$  linearni operator, a  $\mathbf{x}$  vektor iz  $\mathbf{X}$ . Općenito djelovanje linearnog operatora prikazano je slikom 1.



Slika 1.



Slika 2.

**Definicija 2** Za realan broj  $\lambda$  kažemo da je svojsvena (karakteristična) vrijednost linearnog operatora  $A : \mathbf{X} \rightarrow \mathbf{X}$  ako postoji vektor  $\mathbf{x} \neq \mathbf{0}$  takav da je

$$A\mathbf{x} = \lambda\mathbf{x}. \quad (13)$$

Za takav vektor kažemo da je *svojsven* (karakterističan) vektor operatora  $A$  i da pripada svojsvenoj vrijednosti  $\lambda$  (slika 2).

Jednakost (13) može se napisati u obliku

$$A\mathbf{x} - \lambda\mathbf{x} = \mathbf{0}, \quad (14)$$

odnosno matricno

$$A\mathbf{x} = \lambda\mathbf{x} = \mathbf{x}, \quad (\lambda I - A) = \mathbf{0}, \quad (15)$$

gdje je  $A$  matrica pridružena operatoru  $A$ , čime se nalaženje svojstvenih vrijednosti  $\lambda$  svodi na rješavanje linearnog homogenog sustava jednadžbi. Poznato je (3, str.145) da takav sustav ima netrivialno rješenje ako i samo ako je

$$\det(\lambda I - A) = |\lambda I - A| = \begin{vmatrix} \lambda - a_{11} & -a_{12} & -a_{13} \\ -a_{21} & \lambda - a_{22} & -a_{23} \\ -a_{31} & -a_{32} & \lambda - a_{33} \end{vmatrix} = 0 \quad (16)$$

Jednadžba (16) po  $\lambda$  - *karakteristična* (svojsvena) *jednadžba* matrice  $A$  (operatora  $A$ );

$k_A = \det(\lambda I - A)$  - *karakterističan* (svojsven) *polinom* matrice  $A$  (operatora  $A$ ).

Gornju tvrdnju možemo sada izreći i ovako: Skalar  $\lambda \in \mathbf{R}$  je svojsvena vrijednost operatora  $A$  (matrice  $A$ ) ako i samo ako je  $\lambda$  korijen karakteristične jednadžbe  $k_A = 0$ .

Važno je napomenuti da karakteristični polinom ne ovisi o izboru baze. Naime, karakteristični polinom se računa preko determinante matrice  $(\lambda I - A)$ . Sama matrica ovisi o izboru baze, ali ne i njena determinanta. Svake takve dvije matrice su slične i zbog toga imaju isti svojsveni polinom tj.  $|\lambda I - A| = |\lambda I - T^{-1}BT|$ .

Kako su svojsvene vrijednosti nul-točke karakterističnog polinoma, to ni one ne ovise o izboru baze. Zato pri računanju svojsvene vrijednosti možemo uzeti povoljniju bazu za prikaz operatora  $A$ .

### 5. Dijagonalizacija

U ovom ćemo se dijelu posvetiti problemu pronalaženja baze vektorskog prostora  $\mathbf{X}$  koja se sastoji od svojstvenih vektora dane matrice  $A$  (operatora  $A$ ). Tako odabrana baza može se upotrijebiti za izučavanje geometrijskih svojstava dane matrice kao i za pojednostavljenje računa u koji je ona uključena.

**Definicija 3** Za matricu  $A$  (operator  $A$ ) kažemo da dopušta dijagonalizaciju ako postoji regularna matrica  $T$  takva da je  $T^{-1}AT$  dijagonalna matrica. Za matricu  $T$  se kaže da dijagonalizira  $A$ .

Sljedeći teorem pokazuje da je problem svojstvenih vrijednosti ekvivalentan problemu dijagonalizacije.

**Teorem 2** Za kvadratnu matricu  $A$  reda  $n$  su sljedeće tvrdnje ekvivalentne:

(i)  $A$  dopušta dijagonalizaciju.

(ii)  $A$  ima  $n$  linearno nezavisnih svojstvenih vektora.

Dokaz (1, str. 365).

Navedeni teorem osigurava da se svaka matrica reda  $n$  sa  $n$  linearno nezavisnih svojstvenih vektora može dijagonalizirati, dok je njihova linearna nezavisnost osigurana sljedećim teoremom.

**Teorem 3** *Svojstveni vektori koji odgovaraju različitim svojstvenim vrijednostima međusobno su linearno nezavisni.*

Dokaz (3, str. 147).

Važna posljedica ovog teorema: Ako su sve nul-točke karakterističnog polinoma različite, tada postoji baza prostora koju čine svojstveni vektori promatranog operatora. Neka su to, za  $n = 3$ , vektori  $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ . Budući da vrijedi  $A(\mathbf{v}_1) = \lambda_1 \mathbf{v}_1, A(\mathbf{v}_2) = \lambda_2 \mathbf{v}_2, A(\mathbf{v}_3) = \lambda_3 \mathbf{v}_3$ , njegova je matrica u ovoj bazi dijagonalna (3, str. 148).

Napomenimo, ako je  $\mathbf{v}$  svojstveni vektor koji odgovara svojstvenoj vrijednosti  $\lambda$ , odgovaraju joj i svi vektori  $\alpha \mathbf{v}, (\alpha \neq 0)$ . Osim toga, ako su  $\mathbf{v}_1, \mathbf{v}_2$  svojstveni vektori koji odgovaraju svojstvenoj vrijednosti  $\lambda$ , tada je zbog linearnosti operatora i  $\lambda_1 \mathbf{v}_1 + \lambda_2 \mathbf{v}_2$  svojstveni vektor za istu vrijednost  $\lambda$ . Drugim riječima, za svaku svojstvenu vrijednost  $\lambda$  je svaki vektor ( $\neq \mathbf{0}$ ) iz potprostora  $\text{Ker}(\lambda I - A)$  svojstveni vektor operatora  $A$ , budući da

$$(\lambda I - A)(\mathbf{v}) = \mathbf{0} \Rightarrow A(\mathbf{v}) = \lambda \mathbf{v}.$$

Potprostor  $\text{Ker}(\lambda I - A)$  nazivamo svojstveni potprostor koji pripada svojstvenoj vrijednosti  $\lambda$ .

Postupak nalaženja matrice  $T$  koja dijagonalizira matricu  $A$ , ukratko je, za  $n = 3$ , sljedeći:

**Korak 1.** Odredimo karakterističan polinom  $k_A(\lambda) = 0$  matrice  $A$ ;

**Korak 2.** Odredimo nul-točke  $\lambda_1, \lambda_2, \lambda_3$  karakterističnog polinoma, što su svojstvene vrijednosti matrice  $A$ ;

**Korak 3.** Rješavanjem homogenih sustava  $(\lambda_i I - A)\mathbf{v} = \mathbf{0}, i = 1, 2, 3$  dobiju se svojstveni vektori matrice  $A$ ;

**Korak 4.** Ukoliko matrica dopušta dijagonalizaciju formiramo matricu  $T$  kojoj su stupci svojstveni vektori  $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ ;

**Korak 5.** Matrica  $T^{-1}AT$  (operatora  $A$ ) je dijagonalna sa svojstvenim vrijednostima  $\lambda_1, \lambda_2, \lambda_3$  na glavnoj dijagonali:

$$T^{-1}AT = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}. \quad (17)$$

**Primjer:** Za danu matricu  $F$  odredimo matricu  $T$  koja ju dijagonalizira.

$$F = \begin{bmatrix} 4 & -1 & 0 \\ 2 & 2 & -1 \\ 2 & 3 & -2 \end{bmatrix}$$

**Korak 1.** Karakterističan polinom matrice  $F$  glasi

$$\lambda^3 - 4\lambda^2 + \lambda + 6 = 0.$$

**Korak 2.** Nul-točke karakterističnog polinoma, odnosno svojstvene vrijednosti matrice  $F$  su:

$$\lambda_1 = -1, \quad \lambda_2 = 2, \quad \lambda_3 = 3.$$

**Korak 3.** Rješavanjem homogenih sustava dobiju se tri linearno nezavisna svojstvena vektora:

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 5 \\ 17 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$$

**Korak 4.** Formiranje matrice  $T$ :

$$T = \begin{bmatrix} 1 & 1 & 1 \\ 5 & 2 & 1 \\ 17 & 2 & 1 \end{bmatrix}.$$

**Korak 5.** Matrica  $T^{-1}FT$  glasi:

$$T^{-1}FT = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}.$$

## 6. Simetrični operator, simetrična matrica

Jedan od specijalnih tipova linearnih operatora je *simetričan* operator. Za linearni operator  $A : \mathbf{X} \rightarrow \mathbf{X}$  kažemo da je simetričan, ako vrijedi

$$\mathbf{a} \cdot \mathbf{b} = \mathbf{a} \cdot \mathbf{A}\mathbf{b} \quad (18)$$

za sve  $\mathbf{a}, \mathbf{b} \in \mathbf{X}$ . Matrica pridružena simetričnom linearnom operatoru je simetrična s obzirom na glavnu dijagonalu.

Neka je operatoru  $A$  pridružena realna matrica  $A$ . Označimo li s  $A^T$  transponiranu matricu polazne matrice,  $A$  će biti simetrična onda i samo onda ako vrijedi

$$A^T = A. \quad (19)$$

Navest ćemo neka važnija svojstva simetrične matrice.

**Teorem 4** *Ako je  $A$  simetrična matrica, onda su sve njezine svojstvene vrijednosti realne.*

Dokaz (4, str. 147).

**Teorem 5** *Ako je  $A$  simetrična matrica, tada su svojstveni vektori koji odgovaraju različitim svojstvenim vrijednostima međusobno okomiti.*

Dokaz (4, str. 147).

## 7. Ortonormirana baza, ortogonalne matrice

Za bazu prostora  $\mathbf{X}$  kažemo da je *ortonormirana* ako su vektori baze među sobom okomiti (ortogonalni) i svaki ima duljinu (normu) jedan. Iz ortogonalnog skupa ne-nul vektora možemo dobiti ortonormirani, dijeljenjem svakog vektora njegovom normom. Cilj nam je dalje pokazati da je svaka simetrična matrica slična dijagonalnoj matrici te da postoji ortogonalna baza koju čine njeni svojstveni vektori.

Neka je  $T$  matrica čiji su stupci ortonormirani svojstveni vektori matrice  $A$ .

**Definicija 4** Za matricu  $T$  kažemo da je *ortogonalna* ako su njeni stupci ortonormirani vektori.

Iz svojstva simetrične matrice možemo zaključiti da postoji ortogonalna matrica  $T$  takva da je  $T^{-1}AT$  dijagonalna. Za ortogonalne se matrice može pokazati da vrijedi:

- 1)  $T^{-1} = T^T$ , tj. inverzna matrica matrice  $T$  jednaka je transponiranoj matrici od  $T$ ;
- 2) Determinanta ortogonalne matrice jednaka je 1 ili  $-1$ . (3, str.172)

Iz definicije ortogonalnog operatora (ortogonalne matrice) lako se može provjeriti da ortogonalni operator prevodi ortonormiranu bazu ponovno u ortonormiranu bazu. Da bi baze bile jednako orijentirane mora determinanta operatora (matrice) biti veća od nule (6, str. 42), što u jeziku ortogonalnih operatora znači da determinata mora biti jednaka 1.

## 8. Dijagonalizacija simetrične matrice

Željeli bismo dalje pokazati da je svaka simetrična matrica slična dijagonalnoj matrici. Uz teorem 4 nameće se pitanje što se događa sa svojstvenim vektorima koji odgovaraju istoj svojstvenoj vrijednosti, tj. kolika je dimenzija svojstvenog potprostora koji odgovara svojstvenoj vrijednosti  $\lambda$  čija je kratnost veća od 1. Odgovor ćemo preuzeti iz (3, str.173), naime, ima li svojstvena vrijednost kratnost  $k$ , odgovarajući svojstveni potprostor će biti  $k$ -dimenzionalan i unutar njega možemo pronaći  $k$  međusobno okomitih vektora.

Kao zaključak svega do sada rečenog, za  $n = 3$ , vrijedi:

Svaka simetrična matrica  $A$  trećeg reda posjeduje točno 3 realne svojstvene vrijednosti (brojeći njihovu višestrukost) i 3 međusobno okomita svojstvena vektora. Ona je dakle

slična dijagonalnoj matrici  $B$  tj. postoji ortogonalna matrica  $T$  i realni brojevi  $\lambda_1, \lambda_2, \lambda_3$ , tako da je

$$B = T^T A T = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}. \quad (20)$$

Odnosno, za svaku simetričnu matricu postoji ortonormirana baza koju čine normirani svojstveni vektori te matrice.

## 9. Kvadratna forma, dijagonalizacija kvadratne forme

*Linearna forma.*

Linearna forma od 3 varijable je izraz oblika

$$L(x_1, x_2, x_3) = \sum_{i=1}^n a_i x_i \equiv [a_1 \ a_2 \ a_3] \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}. \quad (21)$$

To je funkcija od 3 varijable  $x_1, x_2, x_3$  prvog stupnja, pa u razvoju ne može biti članova koji bi sadržavali umnožak varijabli.

*Kvadratna forma.*

Kvadratna forma od 3 varijable  $x_1, x_2, x_3$  je izraz koji se može zapisati kao

$$K(x_1, x_2, x_3) = [x_1 \ x_2 \ x_3] A \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, \quad (22)$$

gdje je  $A$  matrica reda 3. Uz oznaku

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix},$$

može se pisati u obliku

$$K(x_1, x_2, x_3) = x^T A x = a_{11}x_1^2 + a_{22}x_2^2 + a_{33}x_3^2 + \sum_{i \neq j} a_{ij}x_i x_j. \quad (23)$$

Članove oblika  $a_{ij}x_i x_j$ , ( $i \neq j$ ) zovemo *mješovitim članovima* kvadratne forme.

*Dijagonalizirati* kvadratnu formu podrazumijeva riješiti se mješovitih članova, čime se postiže takozvana *kanonska forma*.

## 10. Plohe drugog reda

Pod pojmom ploha drugoga reda podrazumijevamo skup točaka u prostoru koji je određen jednadžbom drugog stupnja u kartezijevim pravokutnim koordinatama  $x, y, z$  tj.

$$F(x, y, z) \equiv a_{11}x^2 + a_{22}y^2 + a_{33}z^2 + 2a_{12}xy + 2a_{13}xz + 2a_{23}yz + 2a_{14}x + 2a_{24}y + 2a_{34}z + a_{44} = 0 \quad (24)$$



gdje su koeficijenti  $a_{11}, \dots, a_{44}$  realni brojevi i barem jedan od brojeva  $a_{11}, a_{22}, a_{33}, a_{12}, a_{13}, a_{23}$  različit od nule. Funkciju  $F$  možemo zapisati matrično:

$$F(x, y, z) = [xyz1] \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{12} & a_{22} & a_{23} & a_{24} \\ a_{13} & a_{23} & a_{33} & a_{34} \\ a_{14} & a_{24} & a_{34} & a_{44} \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}. \quad (25)$$

Matrice

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{12} & a_{22} & a_{23} & a_{24} \\ a_{13} & a_{23} & a_{33} & a_{34} \\ a_{14} & a_{24} & a_{34} & a_{44} \end{bmatrix} \quad \text{i} \quad A_{44} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{12} & a_{22} & a_{23} \\ a_{13} & a_{23} & a_{33} \end{bmatrix} \quad (26)$$

su realne i simetrične.

Jednadžbu (24) moguće je zapisati i u obliku:

$$F(x, y, z) = [xyz] \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{12} & a_{22} & a_{23} \\ a_{13} & a_{23} & a_{33} \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} + 2[a_{14} a_{24} a_{34}] \begin{bmatrix} x \\ y \\ z \end{bmatrix} + a_{44} = 0. \quad (27)$$

Postupak svođenja kvadratne forme unutar jednadžbe (24) na kanonski oblik je sljedeći:

Realna i simetrična matrica  $A_{44}$  može se napisati u obliku  $A_{44} = TBT^T$  (28)

gdje su matrice  $B$  i  $T$  oblika:

$$B = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}, \quad T = [\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3]. \quad (29)$$

Napomenimo da za centralnu kvadriku matrica  $A_{44}$  mora biti regularna i da su u tom slučaju sve svojstvene vrijednosti  $\lambda_1, \lambda_2, \lambda_3$  matrice  $B$  različite od nule. Osim toga, budući da se radi o realnoj simetričnoj matrici svojstvene su vrijednosti realne.

Stupci ortogonalne matrice  $T$  komponente su jediničnih svojstvenih vektora  $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$  matrice  $A_{44}$  koji pripadaju svojstvenim vrijednostima  $\lambda_1, \lambda_2, \lambda_3$ , poredani tako da bude  $\det T = 1$ .

Matrica  $T$  predstavlja matricu prelaza iz jednog desnog kartezijevog sustava u drugi također desni kartezijev sustav. Neka su sustavi  $(O; x, y, z)$  i  $(O; x', y', z')$ .

S pravom se može postaviti pitanje kako odrediti svojstvene vektore u slučaju da svojstvene vrijednosti  $\lambda_1, \lambda_2, \lambda_3$  nisu međusobno različite. Ukoliko su sve međusobno

jednake, lako se vidi da je svaki vektor u prostoru svojstven vektor matrice  $A_{44}$ . Ako su dvije svojstvene vrijednosti međusobno jednake, na primjer  $\lambda_1 = \lambda_2 \neq \lambda_3$ , odrede se pripadni svojstveni vektori,  $\mathbf{v}_1$  i  $\mathbf{v}_3$ , dok se treći vektor dobije kao njihov vektorski produkt, uz uvjet da  $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$  čine desni sustav vektora, tj.  $\det[\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3] = 1$ .

Dakle, transformacijom

$$T = [\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3] \quad (30)$$

prelazimo na nove koordinate  $x', y', z'$ :

$$\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = T^T \begin{bmatrix} x \\ y \\ z \end{bmatrix}. \quad (31)$$

Zbog ortogonalnosti matrice  $T$  vrijedi

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = T \begin{bmatrix} x' \\ y' \\ z' \end{bmatrix}, \quad (32)$$

dok izraz (27) prelazi u

$$F(x, y, z) = [xyz] T B T^T \begin{bmatrix} x \\ y \\ z \end{bmatrix} + 2[a_{14} a_{24} a_{34}] \begin{bmatrix} x \\ y \\ z \end{bmatrix} + a_{44} = 0,$$

$$F(x', y', z') = [x' y' z'] B \begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} + 2[a_{14} a_{24} a_{34}] T \begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} + a_{44} = 0,$$

$$F(x', y', z') = [x' y' z'] B \begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} + 2[\alpha \beta \gamma] \begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} + a_{44} = 0,$$

$$F(x', y', z') = \lambda_1 x'^2 + \lambda_2 y'^2 + \lambda_3 z'^2 + 2\alpha x' + 2\beta y' + 2\gamma z' + a_{44} = 0, \quad (33)$$

gdje smo označili

$$\begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix} = T^T \begin{bmatrix} a_{14} \\ a_{24} \\ a_{34} \end{bmatrix}. \quad (34)$$

Linearni se članovi  $x', y', z'$ , u (33) mogu eliminirati translacijom koordinatnog sustava:

$$\begin{aligned} x'' &= x' + \frac{\alpha}{\lambda_1} & x' &= x'' - \frac{\alpha}{\lambda_1} \\ y'' &= y' + \frac{\beta}{\lambda_2} & y' &= y'' - \frac{\beta}{\lambda_2} \\ z'' &= z' + \frac{\gamma}{\lambda_3} & z' &= z'' - \frac{\gamma}{\lambda_3} \end{aligned} \quad (35)$$

Jednadžba kvadrike poprima **kanonski oblik**

$$F(x'', y'', z'') = \lambda_1 x''^2 + \lambda_2 y''^2 + \lambda_3 z''^2 + a'_{44} = 0 \quad (36)$$

gdje je

$$a'_{44} = a_{44} - \frac{\alpha^2}{\lambda_1} - \frac{\beta^2}{\lambda_2} - \frac{\gamma^2}{\lambda_3}. \quad (37)$$

Nije teško pokazati da vrijedi:

$$a'_{44} = \frac{\det(A)}{\det(A_{44})}. \quad (38)$$

## 11. Klasifikacija centralnih kvadraka

Kanonski oblik jednadžbe kvadrake (36) zapišimo u obliku

$$\frac{x''^2}{\frac{a'_{44}}{\lambda_1}} + \frac{y''^2}{\frac{a'_{44}}{\lambda_2}} + \frac{z''^2}{\frac{a'_{44}}{\lambda_3}} = -1 \quad (39)$$

i označimo

$$a = \sqrt{\left| \frac{a'_{44}}{\lambda_1} \right|}, \quad b = \sqrt{\left| \frac{a'_{44}}{\lambda_2} \right|}, \quad c = \sqrt{\left| \frac{a'_{44}}{\lambda_3} \right|}. \quad (40)$$

Vrijednosti  $a, b, c$  nazivamo poluosima kvadrake.

- 1) Ako je  $\text{sgn}\lambda_1 = \text{sgn}\lambda_2 = \text{sgn}\lambda_3 = \text{sgn}a'_{44}$  i  $a'_{44} \neq 0$  tada (39) možemo napisati u obliku

$$\frac{x''^2}{a^2} + \frac{y''^2}{b^2} + \frac{z''^2}{c^2} = -1 \quad (41)$$

koju nazivamo jednadžbom **imaginarnog elipsoida** s poluosima  $a, b, c$ .

- 2) Ako je  $\text{sgn}\lambda_1 = \text{sgn}\lambda_2 = \text{sgn}\lambda_3 \neq \text{sgn}a'_{44}$  i  $a'_{44} \neq 0$  tada (39) možemo napisati u obliku

$$\frac{x''^2}{a^2} + \frac{y''^2}{b^2} + \frac{z''^2}{c^2} = 1 \quad (42)$$

koju nazivamo jednadžbom **realnog elipsoida** s poluosima  $a, b, c$ .

- 3) Ako je  $\text{sgn}\lambda_1 = \text{sgn}\lambda_2 = \text{sgn}\lambda_3$  i  $a'_{44} = 0$  tada (39) možemo napisati u obliku

$$\frac{x''^2}{a^2} + \frac{y''^2}{b^2} + \frac{z''^2}{c^2} = 0 \quad (43)$$

koju nazivamo jednadžbom **točke** ili **imaginarnog stošca** s poluosima  $a, b, c$ .

- 4) Ako je  $(\text{sgn}\lambda_1) \cdot (\text{sgn}\lambda_2) = 1$  i  $(\text{sgn}\lambda_3) \cdot (\text{sgn}a'_{44}) = 1$  i  $(\text{sgn}\lambda_1) \neq (\text{sgn}\lambda_3)$  tada (39) možemo napisati u obliku

$$\frac{x''^2}{a^2} + \frac{y''^2}{b^2} - \frac{z''^2}{c^2} = 1 \quad (44)$$

koju nazivamo jednadžbom **jednoplalnog hiperboloida** s poluosima  $a, b, c$ .

- 5) Ako je  $(\text{sgn}\lambda_1) \cdot (\text{sgn}\lambda_2) = 1$  i  $(\text{sgn}\lambda_3) \cdot (\text{sgn}a'_{44}) = -1$  i  $(\text{sgn}\lambda_1) \neq (\text{sgn}\lambda_3)$  tada (39) možemo napisati u obliku

$$\frac{x''^2}{a^2} + \frac{y''^2}{b^2} - \frac{z''^2}{c^2} = -1 \quad (45)$$

koju nazivamo jednadžbom **dvoplošnog hiperboloida** s poluosima  $a, b, c$ .

- 6) Ako je  $(\text{sgn}\lambda_1) \cdot (\text{sgn}\lambda_2) = 1$  i  $(\text{sgn}\lambda_1) \neq (\text{sgn}\lambda_3)$  i  $a'_{44} = 0$  tada (39) možemo napisati u obliku

$$\frac{x''^2}{a^2} + \frac{y''^2}{b^2} - \frac{z''^2}{c^2} = 0 \quad (46)$$

koju nazivamo jednadžbom **realnog stošca** s poluosima  $a, b, c$ .

## 12. Primjeri

### Primjer 1.

Neka je ploha zadana jednadžbom:

$$4x^2 + 5y^2 + 6z^2 - 4xy + 4yz + 4x + 6y + 4z + 30 = 0.$$

Karakteristična jednadžba:

$$\lambda^3 - 15\lambda^2 + 66\lambda - 80 = 0$$

Svojtvene vrijednosti:

$$\lambda_1 = 5, \lambda_2 = 8, \lambda_3 = 2.$$

Svojtveni vektori:

$$\mathbf{v}_1 = \begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} -1 \\ 2 \\ 2 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} -2 \\ -2 \\ 1 \end{bmatrix},$$

$$\det[\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3] = 1.$$

$$a'_{44} = 25 \Rightarrow \text{sgn}\lambda_1 = \text{sgn}\lambda_2 = \text{sgn}\lambda_3 = \text{sgn}a'_{44} \text{ \& } a'_{44} \neq 0.$$

Radi se o imaginarnom elipsoidu.

$$\text{Kanonski oblik jednadžbe: } 5x''^2 + 8y''^2 + 2z''^2 + 25 = 0.$$

### Primjer 2.

Neka je ploha zadana jednadžbom:

$$4x^2 + 5y^2 + 6z^2 - 4xy + 4yz + 4x + 6y + 4z - 27 = 0.$$

$$\text{Karakteristična jednadžba: } \lambda^3 - 15\lambda^2 + 66\lambda - 80 = 0$$

$$\text{Svojtvene vrijednosti: } \lambda_1 = 5, \lambda_2 = 8, \lambda_3 = 2.$$

Svojtveni vektori:

$$\mathbf{v}_1 = \begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} -1 \\ 2 \\ 2 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} -2 \\ -2 \\ 1 \end{bmatrix},$$

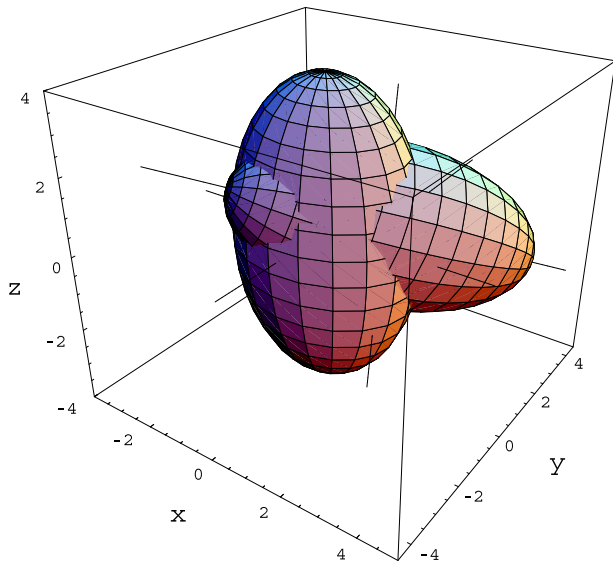
$$\det[\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3] = 1.$$

$$a'_{44} = -32 \Rightarrow$$

$$\text{sgn}\lambda_1 = \text{sgn}\lambda_2 = \text{sgn}\lambda_3 \neq \text{sgn}a'_{44} \text{ \& } a'_{44} \neq 0.$$

Radi se o realnom elipsoidu.

$$\text{Kanonski oblik jednadžbe: } 5x''^2 + 8y''^2 + 2z''^2 - 32 = 0.$$

**Primjer 3.**

Neka je ploha zadana jednadžbom:

$$4x^2 + 6y^2 + 6z^2 + 4xy + 6xz + 8x + 4y + 6z + 4 = 0.$$

Karakteristična jednadžba:  $\lambda^3 - 16\lambda^2 + 71\lambda - 66 = 0$

Svojtvene vrijednosti:

$$\lambda_1 = 1.25834, \lambda_2 = 6, \lambda_3 = 8.74166.$$

Svojtveni vektori:

$$\mathbf{v}_1 = \begin{bmatrix} 0 \\ -3 \\ 2 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} -1.58055 \\ 0.666667 \\ 1 \end{bmatrix}, \mathbf{v}_3 = \begin{bmatrix} 0.913886 \\ 0.666667 \\ 1 \end{bmatrix},$$

$$\det[\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3] = 1.$$

$$a'_{44} = 0 \Rightarrow \operatorname{sgn}\lambda_1 = \operatorname{sgn}\lambda_2 = \operatorname{sgn}\lambda_3 \text{ \& } a'_{44} = 0.$$

Radi se o imaginarnom stošcu.

Kanonski oblik jednadžbe:

$$1.25834x''^2 + 6y''^2 + 8.74166z''^2 = 0.$$

**Primjer 4.**

Neka je ploha zadana jednadžbom:

$$4x^2 - 2y^2 + 6z^2 + 10xy + 2xz - 8yz + 8x + 10y + 2z + 4 = 0.$$

Karakteristična jednadžba:  $\lambda^3 - 8\lambda^2 - 38\lambda + 300 = 0$

Svojtvene vrijednosti:

$$\lambda_1 = -6.14143, \lambda_2 = 6, \lambda_3 = 8.14143.$$

Svojtveni vektori:

$$\mathbf{v}_1 = \begin{bmatrix} -1.42008 \\ 2.68034 \\ 1 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} -0.579916 \\ 0.680336 \\ 1 \end{bmatrix}, \mathbf{v}_3 = \begin{bmatrix} 4 \\ 1 \\ 3 \end{bmatrix},$$

$$\det[\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3] = 1.$$

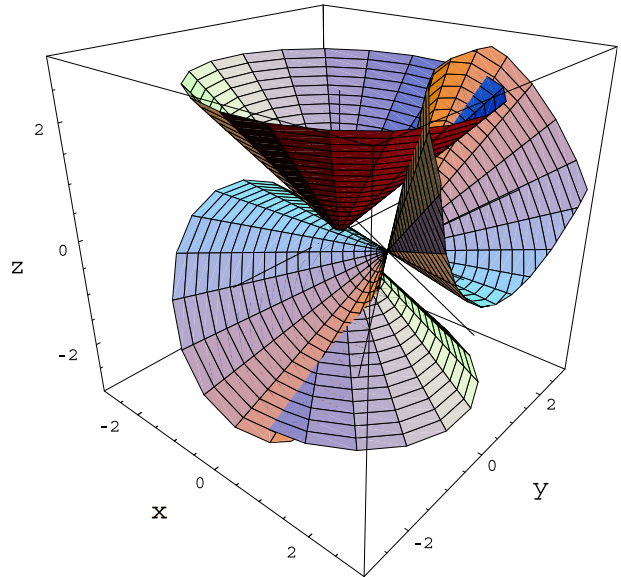
$$a'_{44} = 0 \Rightarrow$$

$$(\operatorname{sgn}\lambda_1) \cdot (\operatorname{sgn}\lambda_2) = 1 \text{ \& } \operatorname{sgn}\lambda_1 \neq \operatorname{sgn}\lambda_3 \text{ \& } a'_{44} = 0.$$

Radi se o realnom stošcu.

Kanonski oblik jednadžbe:

$$6x''^2 + 8.74166y''^2 - 6.14143z''^2 = 0.$$

**Primjer 5.**

Neka je ploha zadana jednadžbom:

$$6x^2 - 2y^2 + 6z^2 + 4xz + 8x - 4y - 8z + 1 = 0.$$

Karakteristična jednadžba:  $\lambda^3 - 10\lambda^2 + 8\lambda + 64 = 0$

Svojtvene vrijednosti:  $\lambda_1 = 4, \lambda_2 = 8, \lambda_3 = -2.$

Svojtveni vektori:

$$\mathbf{v}_1 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \mathbf{v}_3 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix},$$

$$\det[\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3] = 1.$$

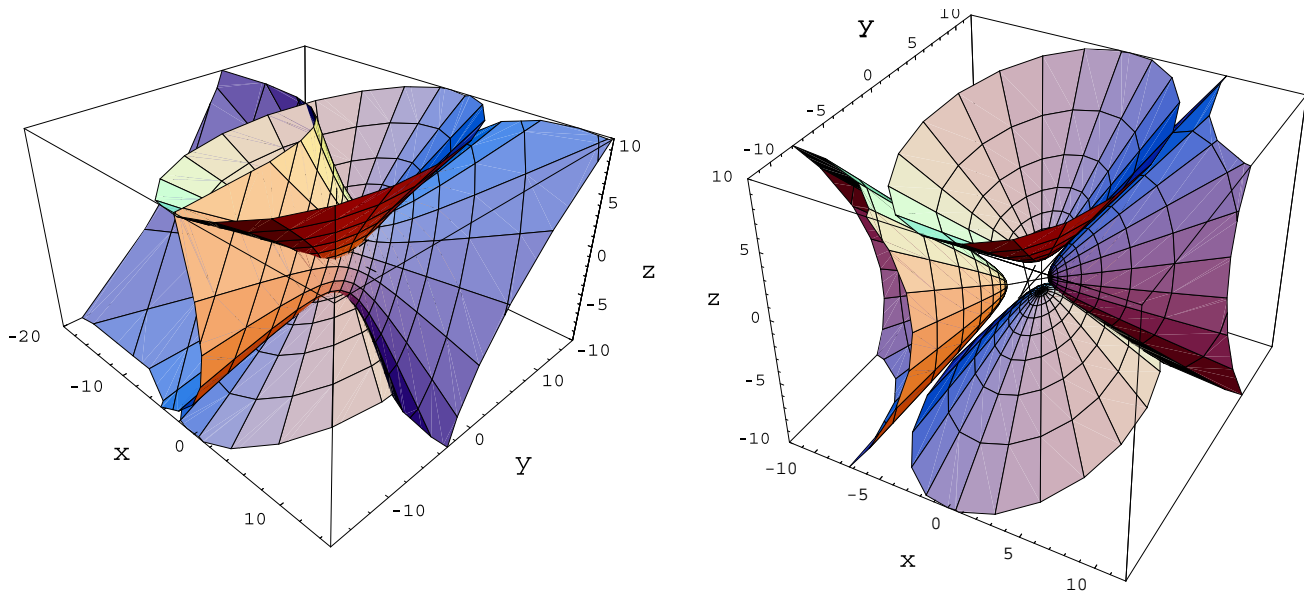
$$a'_{44} = -5 \Rightarrow$$

$$(\operatorname{sgn}\lambda_1) \cdot (\operatorname{sgn}\lambda_2) = 1 \text{ \& } (\operatorname{sgn}\lambda_3) \cdot (\operatorname{sgn}a'_{44}) = 1 \\ \text{ \& } \operatorname{sgn}\lambda_1 \neq \operatorname{sgn}\lambda_3.$$

Radi se o jednoplošnom hiperboloidu.

Kanonski oblik jednadžbe:

$$4x''^2 + 8y''^2 - 2z''^2 - 5 = 0.$$

**Primjer 6.**

Neka je ploha zadana jednadžbom:

$$x^2 + y^2 + 5z^2 - 6xy + 2xz - 2yz - 4x + 8y - 12z + 14 = 0.$$

Karakteristična jednadžba:  $\lambda^3 - 7\lambda^2 + 36 = 0$

Svojtvene vrijednosti:  $\lambda_1 = 3, \lambda_2 = 6, \lambda_3 = -2$ .

Svojtveni vektori:

$$\mathbf{v}_1 = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}, \mathbf{v}_3 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix},$$

$$\det[\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3] = 1.$$

$$a'_{44} = 6 \Rightarrow (\operatorname{sgn}\lambda_1) \cdot (\operatorname{sgn}\lambda_2) = 1 \text{ \& } (\operatorname{sgn}\lambda_3) \cdot (\operatorname{sgn}a'_{44}) = -1 \text{ \& } \operatorname{sgn}\lambda_1 \neq \operatorname{sgn}\lambda_3.$$

Radi se o dvoplošnom hiperboloidu.

Kanonski oblik jednadžbe:  $3x'^2 + 6y'^2 - 2z'^2 + 6 = 0$ .

**Literatura**

- [1] ANTON, H., RORRES, C.: *Elementary Linear Algebra*, John Wiley & Sons, Inc., New York, 1994.
- [2] BLANUŠA, D.: *Viša matematika*, 1 dio, drugi svezak, Tehnička knjiga, Zagreb, 1965.
- [3] ELEZOVIĆ, N.: *Linearna algebra*, Element, Zagreb, 1995.
- [4] GURSKIJ, E. I.: *Osnove linearne algebre i analitičke geometrije*, (na ruskom), Višejšaja škola, Minsk, 1982.
- [5] HORVATIĆ K.: *Linearna algebra*, II dio, Matematički odjel PMF-a, Sveučilište u Zagrebu, Zagreb, 1995.
- [6] KUREPA, S.: *Uvod u linearnu algebru*, Školska knjiga, Zagreb, 1978.
- [7] LAPAINE, M., JOVIČIĆ, D.: *Grafički prikaz konika pomoću računala*, KOG, broj 1, Zagreb, 1996.

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# Crtanje prodornih krivulja pomoću AutoCAD-a

## Crtanje prodornih krivulja pomoću AutoCAD-a

### SAŽETAK

U radu je dan analitički izvod na temelju kojega je izrađen program za crtanje prodornih krivulja kosih kružnih stožaca i valjaka u AutoCAD-u. Navedena je internet adresa na kojoj se taj program može podići, a dano je i uputstvo za njegovu upotrebu. Pomoću tog programa iscrtani su neki primjeri.

**Cljučne riječi:** AutoCAD, prodorne krivulje

**MSC 2000:** 65D17, 65D18

## Drawing Intersection Curves by AutoCAD

### ABSTRACT

This article presents the analytical derivation which was the base for designing the computer program for drawing the intersection curves of oblique cones and cylinders with AutoCAD. The internet address where you can download this program and the instructions can be found in the article. There are some examples which are drawn by using this program.

**Key words:** AutoCAD, intersection curves

## 1. Uvod

Upotreba računala za tehničko crtanje svakako ima jednu veliku i vrlo uočljivu prednost, a to je preciznost. Za svaki crtež ispravno nacrtan pomoću računala možemo sa sigurnošću tvrditi da je točniji od ručno nacrtanog. Međutim, upotreba računala ponekad se izbjegava. Neki se crteži mogu, navodno, puno brže izraditi ručno. To prije svega ovisi o crtaču i o programskom paketu kojeg koristi. Računalo je sposobno samostalno obavljati većinu zadaća koje prilikom ručnog crtanja oduzimaju puno vremena. Primjerice, šrafitiranje ili podjela dužine na nekoliko jednakih dijelova. Programski paketi poput AutoCAD-a imaju mnoštvo gotovih programskih rutina za crtanje raznih linija, kružnica, pisanje teksta i sl. Naravno, nikad nemaju sve ono što bi korisnik htio. Zbog toga je većina programskih paketa tehničke struke "otvorenog tipa", korisnik može vrlo lako nadograditi paket odnosno napisati programsku rutinu koja mu je potrebna.

## 2. Prodori u AutoCAD-u

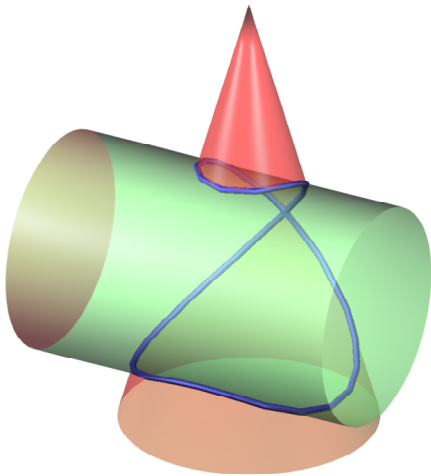
U AutoCAD-u od verzije 14 postoje "solid modeli" odnosno čvrsta tijela s kojima je moguće vršiti logičke operacije presjeka ( naredba INTERSECT), unije (UNION) i oduzimanja (SUBTRACT). Tako, primjerice, možemo

kocku probušiti valjkom i sl. Ukoliko napravimo uniju dvaju tijela dobit ćemo jedno tijelo na kojemu se vidi prodorna krivulja. Pomoću naredbe EXPLODE možemo to tijelo rastaviti na krivulje i plohe od kojih je sastavljeno, a među ostalim će tu biti i prodorna krivulja. AutoCAD prodornu krivulju obliha ploha aproksimira sa 100 točaka koje su spojene neprekinutom glatkom krivuljom. Koordinate točaka krivulje računalo određuje tako da oble plohe podijeli na konačan broj poligona i skup svih presječnica tih poligona je aproksimacija prodorne krivulje, koja je utoliko točnija što je veći broj podjela.

AutoCAD ima mogućnost crtanja rotacijskih te uspravnih eliptičkih stožaca (CONE) i valjaka (CYLINDER). Stoga je vrlo lako nacrtati prodornu krivulju nekog stošca i valjka pod uvjetom da su uspravni. AutoCAD crtež može se prebaciti u neki program za vizualizaciju te dobiti crtež kao na slici 1.

## 3. Prodori kosih stožaca i valjaka

Kako nacrtati kosi stožac i valjak? Budući da znamo da je kosi kružni stožac ustvari uspravni eliptički stožac presječen ravninom i da u AutoCAD-u postoji naredba za presjek tijela ravninom (SLICE), zaključujemo da je moguće nacrtati i kosi stožac.



Slika 1.

### 3.1. Analitička podloga za izradu datoteke

Neka je kosi kružni stožac određen vrhom  $V$ , središtem ( $S$ ) i polumjerom ( $r$ ) baze, te ravninom baze. Radi lakšeg proračuna postavljamo koordinatni sustav tako da mu je ishodište u vrhu,  $xy$  ravnina paralelna s bazom, a  $xz$  ravnina sadrži os stošca (slika 2).

Da bismo nacrtali kosi stožac potrebno je definirati elipsu koja čini bazu pridruženog eliptičkog uspravnog stošca. Ta je elipsa perspektivnom kolineacijom pridružena kružnici koja čini bazu kosog kružnog stošca [1, str.102]. Središte te perspektivne kolineacije je vrh stošca, a os je presječnica ravnina elipse i kružnice. Točke u polju kružnice označit ćemo slovima  $A, B, C, \dots$ , njima pridružene točke u polju elipse  $A', B', C', \dots$ , a uređene trojke njihovih koordinata  $(A_x, A_y, A_z), \dots$ . Budući da znamo vrh stošca, za definiciju elipse dovoljno je odrediti koordinate njezina središta  $O'$  te polusoje  $r_1$  i  $r_2$ .

Točka  $A'$  podudara se s točkom  $A$ , zbog toga što se nalazi na osi kolineacije. Koordinate točke  $B'$  možemo odrediti iz uvjeta da je duljina izvodnice  $\overline{VB'}$  jednaka duljini izvodnice  $\overline{VA'} = \overline{VA}$ . Ako duljinu izvodnice  $\overline{VB'}$  označimo s  $d_1$ , a duljinu izvodnice  $\overline{VB}$  s  $d_2$ , onda su koordinate točke  $B'$  dane sa

$$B'_x = B_x \frac{d_1}{d_2}, \quad B'_y = B_y = 0, \quad B'_z = B_z \frac{d_1}{d_2}. \quad (1)$$

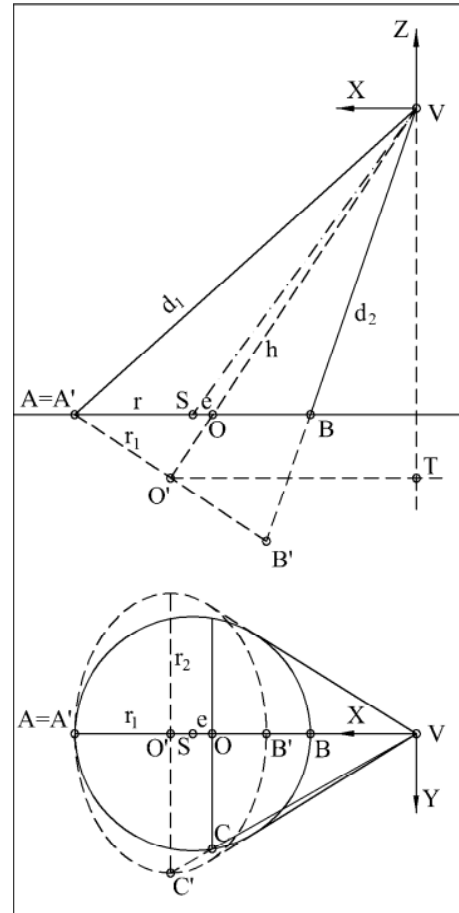
Središte elipse  $O'$  polovište je dužine  $\overline{A'B'}$  pa vrijedi

$$O'_x = \frac{A_x + B'_x}{2}, \quad O'_y = \frac{A_y + B'_y}{2}, \quad O'_z = \frac{A_z + B'_z}{2}. \quad (2)$$

Duljina prve polusoje  $r_1$  jednaka je duljini dužine  $\overline{A'O'}$ .

Duljina druge polusoje  $r_2$  jednaka je duljini dužine  $\overline{C'O'}$ , a kako je  $O'_y = 0$  zaključujemo da je

$$r_2 = |C'_y|. \quad (3)$$



Slika 2.

Zbog sličnosti trokuta vrijedi  $C'_y = C'_x / C_x$ , a budući da je  $C_x = O_x$  i  $C'_x = O'_x$  vrijedi i sljedeća relacija

$$C'_y = C_y \frac{O'_x}{O_x}. \quad (4)$$

Koordinatu  $C_y$  možemo dobiti iz jednadžbe kružnice:

$$C_y = \sqrt{r^2 - e^2}, \quad (5)$$

gdje je  $e$  udaljenost točkaka  $O$  i  $S$ .

Budući da je  $O_y = S_y$  i  $O_z = S_z$ , znači da je  $e = S_x - O_x$ .

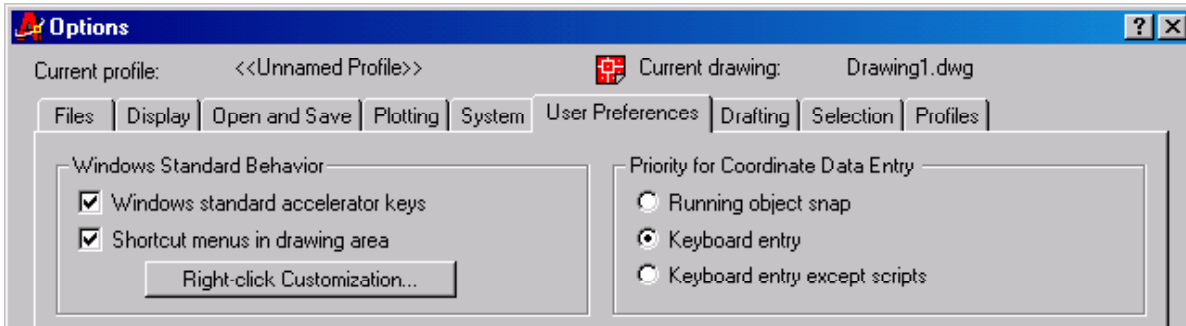
Ako duljinu dužine  $\overline{VO'}$  označimo s  $h$ , tada zbog sličnosti  $\triangle AO'O$  i  $\triangle VO'T$  vrijedi

$$e + r = \frac{r_1 h}{|O'_z|} \Rightarrow e = \frac{r_1 h}{|O'_z|} - r. \quad (6)$$

Koordinatu  $O_x$  možemo jednostavno izračunati iz relacije

$$O_x = S_x - e. \quad (7)$$

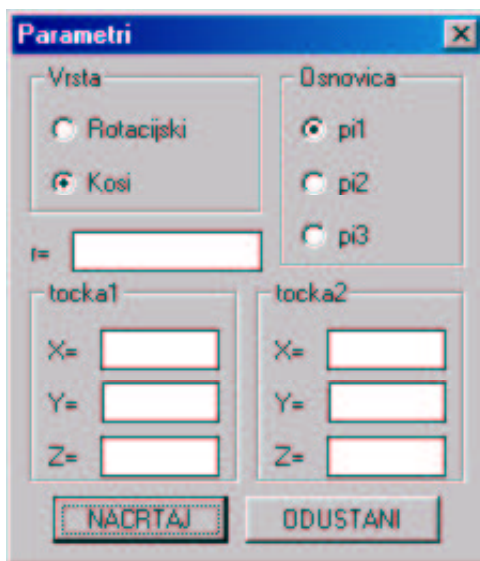
Sada nam relacije (2), (6), (5), (7), (4) i (3) omogućuju izračunavanje duljine druge polusoje  $r_2$ .



Slika 3a

Pomoću  $O'$ ,  $r_1$  i  $r_2$  može se nacrtati uspravni eliptički stožac koji presječen ravninom daje polazni kosi kružni stožac.

Valjak je znatno jednostavniji od stošca, budući da se kod njega točke  $O$  i  $S$  podudaraju, pa je  $r_2 = r$ . Valjak treba presjeći s dvije ravnine, jer ima dvije osnove.



Slika 3b

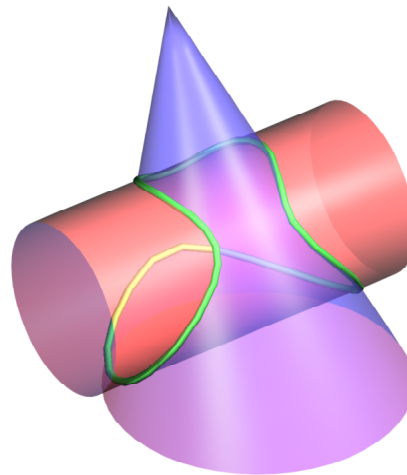
### 3.2. Program za crtanje kosih stožaca i valjaka

Znajući sve ovo, možemo u AutoLISP-u napisati jednostavan program za crtanje kosih stožaca i valjaka. Autor ovoga članka napravio je dvije datoteke koje se mogu naći na adresi [www.grad.hr/nastava/geometrija](http://www.grad.hr/nastava/geometrija).

Da bi program funkcionirao potrebno je imati AutoCAD verziju 14 ili noviju, te napraviti sljedeće: snimiti crtež (može i prazan) u direktorij gdje se nalazi datoteka PRODOR.DCL, pomoću naredbe APPLOAD učitati PRODOR.LSP, u opcijama AutoCAD-a odabrati karticu "User Preferences" i opciju "Priority for Coordinate Data Entry" podesiti na "Keyboard entry" kao što je pokazano

na slici 3a. Tada se pomoću naredbi STOZAC i VALJAK može crtati kose stošce i valjke (slika 3b).

Ovako se može vrlo jednostavno dobiti prodorna krivulja bilo kojeg stošca i valjka. Ta krivulja može poslužiti za lakše shvaćanje prodora i kao podloga za rješavanje zadataka.



Slika 4.

### Literatura

- [1] I. BABIĆ, S. GORJANC, A. SLIEPČEVIĆ, V. SZIROVICZA: *Konstruktivna geometrija - Vježbe*, IGH, Zagreb, 1994.
- [2] A. PREVAREK: *AutoCAD u profesionalnoj primjeni*, Znak d.o.o., Zagreb, 1995.

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**ANA SLIEPČEVIĆ**

# Eine Anwendung der perspektive Kollineation

## Jedna primjena perspektivne kolineacije

### SAŽETAK

Cilj je ovog rada da, podsjećanjem na neke stare geometrijske konstrukcije izvedene pomoću perspektivne kolineacije i afineteta, ponovno aktualizira crtež u geometriji, čija je izrada danas olakšana upotrebom računala. Zadaci se rješavaju metodom perspektivno kolinearne pridruženih figura, tj. svaki se problem u vezi s konikom perspektivnom kolineacijom preslika u elementarno rješiv zadatak u vezi s kružnicom, što rezultira geometrijski točnim rješenjem početnog problema.

**Cljučne riječi:** konika, perspektivna kolineacija, perspektivna afinost

## One Application of Perspective Collineation

### ABSTRACT

The aim of this paper is to remind us of some previous geometrical constructions derived by the means of a perspective collineation and an affinity. It should refresh the drawing in geometry. Nowadays it is much easier by using the computer. The tasks are solved with the method of the collinear corresponded figures. Each problem connected with a conic by using the method of a perspective collineation can be transformed into the elementary problem connected with a circle. It results with the geometrically correct solution of the initial problem.

**Key words:** conic, perspective collineation, affinity

**MSC 2000:** 51N05

Die im folgenden initiierte Reihen unternimmt den Versuch, einige Themen aus dem "Erdgeschoß der Geometrie" wiederzubeleben. Es sind alte, bekannte Sachverhalte, die noch immer gefallen finden können. Sie sollten aber vor allem die Leserschaft des KoG anregen und ermuntern "ihre Lieblingsjuwelen" aus dem Erdgeschoß der Geometrie im KoG vorzustellen. Vielleicht hat jemand in der Tischlade eine besondere elegante Konstruktion oder weiß von einer modernen Anwendung alten konstruktiven Materials? Wenn jemand dieser Einladung nachkommt und so eine Reihe von Veröffentlichungen initiiert wird, die zu einer Sammlung interessanten konstruktiven Materials führt, dann ist das Ziel der Autorin erreicht.

Die folgenden, mit konstruktiv einfachen und bekannten Mitteln lösbaren Aufgaben betreffen *Kegelschnitte*, also eine in der Technik sehr häufig angewendete Kurvenklasse. Diese Kurven sind bekanntlich dadurch ausgesuchend, daß sie sämtlich untereinander kollinear sind. Sie können also durch eine lineare Bijektion, die sogenannte *Kollineation* oder im Spezialfall eine *Affinität* ineinander übergeführt werden. Insbesondere läßt sich jeder Kegel-

schnitt der Anschauungsebene als kollineares Bild eines (euklidischen) Kreises auffassen, wobei die Kollineation, bzw. Affinität, speziell durch eine perspektive Kollineation (bzw. perspektive Affinität) realisiert werden kann.

Der Schauplatz für die konstruktive (und zweckmäßige analytische) Behandlung der Kegelschnitte ist demgemäß die *projektiv abgeschlossene Anschauungsebene P*, wie sie vom Umgang mit der Zentralprojektion (Perspektive) vertraut ist. Für Ellipsen reicht hingegen schon die Anschauungsebene als Schauplatz der Behandlung aus. Im CAGD kommt unter anderen die Aufgabe vor, durch eine Anzahl von Stützstellen, also Punkten in  $\Pi$ , eine möglichst glatte Kurve möglichst niedriger Ordnung hindurchzulegen. Durch fünf solche Punkte allgemeiner Lage ist eine solche glatte Kurve ein eindeutig bestimmter Kegelschnitt. Durch technisch-physikalischen Erfordernisse kann dabei die Frage nach den Stellen extremaler Krümmung, also den Scheiteln dieser "Splein-Kurve", motiviert sein.

Im folgenden werden einige solchen Aufgabe konstruktiv dadurch gelöst, daß die Angabepunkte (bzw. Anga-



betangenten) des Kegelschnittes mittels einer geeigneten perspektiven Kollineation in Punkte (bzw. Tangenten) eines Kreises transformiert werden. Diese Methode setzt nur die Kenntnisse der Perspektivdarstellung eines Kreises, also elementar geometrische Überlegungen - eventuell räumlich interpretierbar - voraus. Andere Methoden zur Lösung solcher Aufgaben setzen Kenntnisse der Projektiven Geometrie voraus, die dem Konstruktor unter Umständen nicht zur Verfügung stehen. Vom Standpunkt "geometrischen Abbildungsmethoden" (siehe [2]) werden im folgenden vorgeführten Aufgaben mittels der *Methode kollinearier bzw. affiner Hilfsfiguren* gelöst und elementargeometrisch interpretiert.

### Aufgabe 1.

Man bestimme die Schnittpunkte einer Geraden  $p$  mit jenem Kegelschnitt  $c$ , der durch fünf allgemein liegenden Punkte  $A, B, C, D, E$  hindurch geht.

*Lösungsansatz:*

Unter allen möglichen Kollineationen, die  $c$  in einen Kreis  $c'$  überführen, wählen wir eine perspektive Kollineation  $\kappa$ , die z. B.  $A, B, C, D$  in ein Rechteck  $A'B'C'D'$  überführen soll. Der fünfte Punkt  $E$  muß dabei in einen Kreisbogen  $E'$  übergehen, sodaß  $E'$ , verbunden mit gegenüberliegenden Rechteckspunkten  $B', D'$  zwei orthogonale Geraden liefert (Thales).

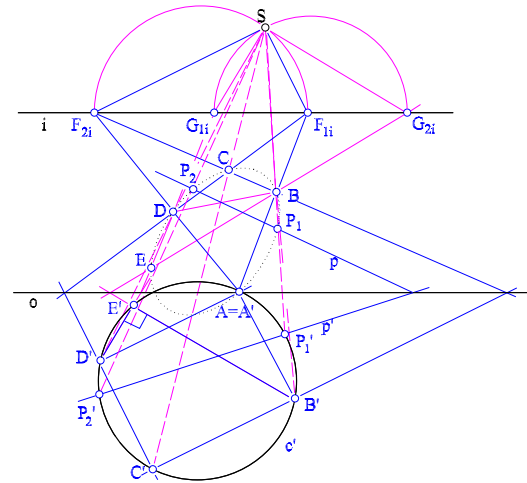
*Konstruktive Lösung:*

$AB \cap CD = F_{1i}$ ,  $BC \cap DA = F_{2i}$ . Diese beiden Punkte müssen unter  $\kappa$  in Fernpunkte  $F'_{1i}, F'_{2i}$ , orthogonaler Richtungen übergehen, also die Verschwindungsgerade  $i$  der gesuchten perspektiven Kollineation  $\kappa$  aufspannen. Die Geraden  $EB$  und  $ED$  schneiden  $i$  in  $G_{1i}$ , und  $G_{2i}$ , die zu einem weiteren Paar von Fernpunkten orthogonaler Richtungen gehören. Aus dem gesuchten Kollineationszentrum  $S$  müssen sich also  $F_{1i}$  und  $F_{2i}$ , sowie  $G_{1i}$ , und  $G_{2i}$ , durch orthogonale Geradenpaare projizieren, weshalb  $S$  der Schnittpunkt der beiden Thales-Kreise über  $(F_{1i}, F_{2i})$  bzw.  $(G_{1i}, G_{2i})$  sein muß.

Nun kann noch die Kollineationsachse  $o$  beliebig parallel zu  $i$  ( $o \neq i$ ) gewählt werden. In Figur 1. ist  $o$  durch  $A$  gelegt worden, was  $A = A'$  zur Folge hat.

Als Ergebnis müssen  $A', B', C', D'$  ein Rechteck bilden, also einen Umkreis  $c'$  besitzen der auch  $E'$  enthält.

Mittels der Umkehrung  $\kappa^{-1}$  der Kollineation  $\kappa$  kann nun jeder weitere Punkt  $X'$  von  $c'$  auf einen Punkt  $X$  von  $c$  übertragen werden. Insbesondere kann man so die Schnittpunkte  $P_1$  und  $P_2$  der Geraden  $p$  mit dem Kegelschnitt  $c$  als Urbilder der Schnittpunkte von  $p'$  mit  $c'$  bestimmen.



Figur 1.

### Aufgabe 2.

Bestimmung der konjugierten Durchmesser des Kegelschnittes  $c$  aus Aufgabe 1.

*Lösungsansatz:*

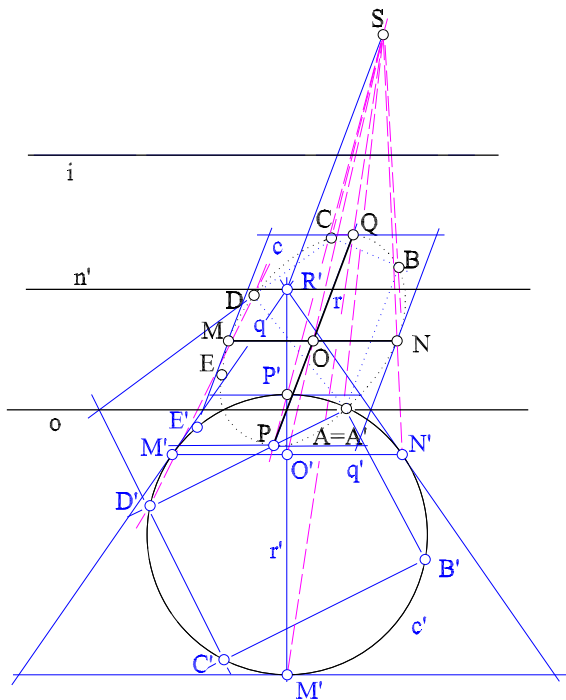
Zwei Durchmesser eines Kegelschnittes heißen konjugiert, wenn die Tangenten in den Endpunkten des einen parallel zum anderen sind.

Die Bilder der Fernpunkte dieser Durchmesser gehören der Fluchtgeraden  $n'$  von  $\kappa$  an. Je nach der Lage von  $n'$  zu  $c'$  (Passante oder Tangente oder Sehne), ist  $c$  eine Hyperbel, Parabel oder Ellipse. Für die Fluchtgerade  $n'$  ergibt die konstruktive Vervollständigung von  $\kappa$  die Gleichheit der (orientierten) Abstände  $d(i, S)$  und  $d(o, n')$ . (Die elementargeometrischen Lösungsstrategien für die Bestimmung etwa der Achsen von  $c$  sind vom Kegelschnitt-Typs abhängig und werden in den nächsten Aufgaben vorgeführt.)

*Konstruktive Lösung:*

Der Schnittpunkt  $R'$  der Fluchtgerade  $n'$  mit dem zur  $o$  orthogonalen Kreisdurchmesser  $r'$  ist jedenfalls dem Fernpunkt eines Kegelschnittsdurchmesser  $r$  zugeordnet. Dabei geht  $r$  durch den Schnittpunkt  $r' \cap o$  und ist parallel zum Kollineationstrahl  $SR'$ . Existieren durch den Punkt  $R'$  zwei reelle Kreistangenten, also  $c$  eine *Ellipse*, so sind diese den zum Durchmesser  $r$  parallelen Ellipsentangenten zugeordnet, und die Verbindungsgerade  $q'$  ihrer Berührungspunkte  $M', N'$  ist deshalb dem zur Durchmesser  $r$  konjugierten Durchmesser zugeordnet. Der Schnittpunkt  $r' \cap q' = O'$  entspricht dem Mittelpunkt  $O$  des Kegelschnittes.

Unter Benützung der der Projektiven Geometrie zugehörigen Begriffswelt “Polarsystem”, “Spurinvolution”, “Paare konjugierter Punkte” etc. würde eine einheitliche Formulierung der typenabhängigen Konstruktion möglich sein: Paaren konjugierter Durchmesser von  $c$  entsprechen dabei durch  $O'$  gehende Sehnenpaare von  $c'$ , deren Schnittpunkte mit  $n'$  in der Spurinvolution von  $c'$  in  $n$  gekoppelt sind. Damit ist jedes Paar konjugierter Durchmesser, insbesondere die Achsen, bestimmbar. In den Aufgaben 3, 4, 5 ist Achsenkonstruktion aber weitergehend elementar gelöst.



Figur 2.

**Aufgabe 3.**

Eine Ellipse  $c'$  sei durch konjugierte Durchmesser  $M'N'$  und  $P'Q'$  gegeben. Konstruiere eine perspektive Kollineation  $\kappa$  und einen Kreis  $c$  so daß es gilt  $\kappa(c) = c'$ .

*Lösungsansatz*

Es bestehen unendlich viele Möglichkeiten, um eine solche perspektive Kollineation und den zugehörigen Kreis  $c$  zu bestimmen. Bekanntlich bildet sich nur ein Kreisdurchmesser  $r$  unter  $\kappa$  in einen Ellipsendurchmesser  $r'$  ab. Die Tangenten in den Endpunkten solcher zugeordneter Durchmesser müssen dabei zur Kollineationachse  $o$  parallel sein.

Sei  $r' = P'Q'$  nicht der zu  $o$  parallele Ellipsendurchmesser; dann steht  $r$  senkrecht zum dazu konjugierten Durchmesser  $M'N'$  und nach willkürlicher Wahl von  $o$  gilt  $r \cap r' \in o$ .

Es gibt ein tangential gleichschenkelige Kreistrapez das sich unter  $\kappa$  in ein tangential Ellipsenparallelogramm abbildet.

Das der Ellipse  $c'$  eingeschriebene Dreieck  $N'P'Q'$  muss dabei Kollineationsbild eines dem Kreis  $c$  eingeschriebenen rechtwinkligen Dreiecks  $NPQ$  sein.

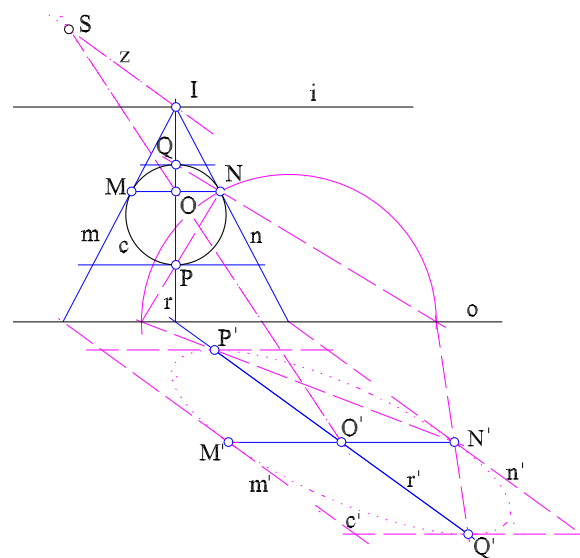
*Konstruktive Lösung*

Die Kollineationachse  $o$  und die Verschwindungsgerade  $i$  wählt man beliebig zum Durchmesser  $M'N'$  parallel. Der Kreisdurchmesser  $r$  geht dann durch  $o \cap r'$  und steht zur  $o$  senkrecht.

Der Punkt  $I = i \cap r$  muß in den Fernpunkt  $I'$  der Geraden  $r'$  übergehen sodaß ein Kollineationstrahl  $z$  parallel zu  $r'$  durch  $I$  festgelegt ist. Die Urbilder  $m, n$  der Ellipsentangenten  $m', n'$  schneiden diese auf  $o$  und gehen den Punkt  $I$  durch. Der Kreis  $c$  ist mit den Tangenten  $m$  und  $n$  und Berührungspunkt  $N$  bestimmt.

Die Schnittpunkte  $o \cap N'P'$  und  $o \cap N'Q'$  sind die Endpunkte des Thaleskreis-Durchmessers, der auf der Kreistangente  $n$  den Berührungspunkt  $N$  ausschneidet. Das Urbild des Ellipsendurchmessers  $M'N'$  ist die zur  $o$  parallele Kreissehne  $MN$ , und der Punkt  $O = MN \cap r$  ist das Urbild des Ellipsenzentrum =  $O'$ .

Der Schnittpunkt der Strahlen  $z$  und  $OO'$  ist das Kollineationszentrum  $S$ .



Figur 3.

**Aufgabe 4.**

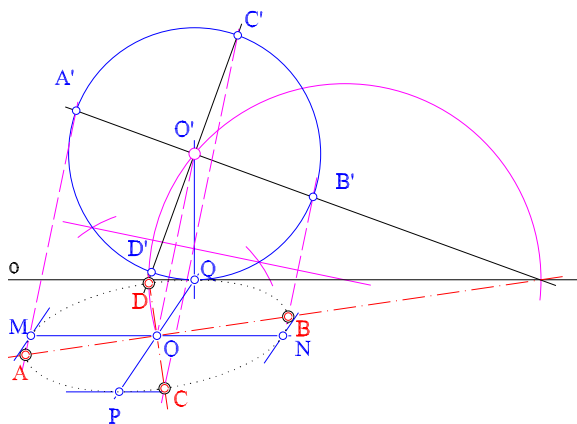
Die Bestimmung der Achsen der durch konjugierten Durchmesser  $MN, PQ$  gegebene Ellipse mittels perspektiven Affinität.

*Lösungsansatz:*

Ist der Kreis in den vorgegangenen Aufgaben so beschaffen, daß sein Durchmesser gleich jenem Ellipsendurchmesser ist, welcher zur Kollineationsachse  $o$  parallel ist, so werden die Kurven  $c$  und  $c'$  in *perspektiv affinen Verwandtschaft* gekoppelt sein. Die Achsen von  $c$  stammen dann, wie alle Paare konjugierter Durchmesser, von rechtwinkligen Kreisdurchmessern und schneiden diese natürlich auf der Kollineationsachse. Wählt man die Kollineationsachse nicht durch  $O$ , so kann für die Bestimmung der einander entsprechenden rechten Winkel in  $O$  und  $O'$  ein geeigneter Thaleskreis herangezogen werden.

*Konstruktive Lösung:*

Insbesondere kann im Sinne des Lösungsansatzes die (zum einen Ellipsendurchmesser parallele) Affinitätachse mit der Ellipsentangente in einem Punkt des anderen Durchmessers (etwa in  $Q$ ) zusammenfallend gewählt werden. Der Kreisdurchmesser ist dem Ellipsendurchmesser  $d(MN)$  gleich. Die Verbindungsgerade  $OO'$  der Mittelpunkte ist der Affinitätstrahl. Der durch die Punkte  $O$  und  $O'$  gehenden Thaleskreis, dessen Mittelpunkt an der Achse  $o$  liegt, schneidet  $o$  in Punkten der gesuchten Ellipsenachsen.



Figur 4.

**Aufgabe 5.**

Eine perspektive Kollineation  $\kappa$  ist mit dem Zentrum  $S$ , der Achse  $o$  und Verschwindungsgerade  $i$  gegeben. Man soll die Parabel als kollineares Bild eines  $i$  berührendes Kreises  $c$  konstruieren und deren Achse und Scheitelpunkt bestimmen.

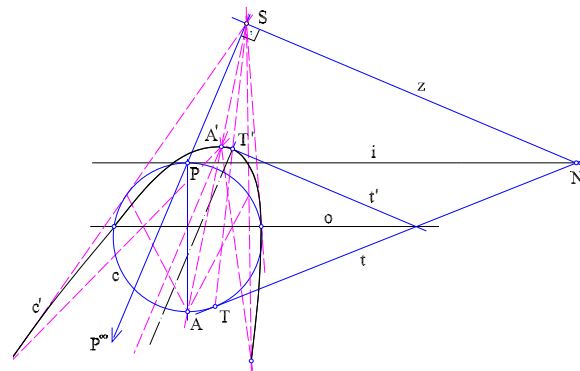
*Lösungsansatz:*

Der Berührungspunkt  $P$  des Kreises mit der Verschwindungsgerade  $i$  bildet sich in den Fernpunkt der Parabel ab. Somit ist die Parabelachse parallel zu dem Strahl  $SP$ . Der

zu  $SP$  orthogonale Strahl durch  $S$  legt den Fernpunkt der Scheiteltangente der Parabel fest. Mittels der Kollineation  $\kappa$  wird dieser Fernpunkt in den Punkt  $N$  von  $i$  übergeführt, wobei die aus  $N$  an  $c$  legbare Resttangente dann das Urbild der Scheiteltangente ist.

*Konstruktive Lösung:*

Der Berührungspunkt  $T$  der aus dem Punkt  $N$  legbaren, von  $i$  verschiedenen Kreistangente  $t$  geht in den Scheitelpunkt  $T'$  der Parabel über. Die Parabelachse geht durch diesen Punkt und ist zu  $SP$  parallel. Für die Feststellung des Brennpunktes der Parabel kann die elementare Parabelei-genschaft, dass nämlich der Lotfußpunkt aus dem Brennpunkt auf jede Parabeltangente in ihren Schnittpunkt mit der Scheiteltangente fällt.



Figur 5.

**Aufgabe 6.**

Eine perspektive Kollineation  $\kappa$  ist durch ihr Zentrum  $S$ , der Achse  $o$  und Verschwindungsgerade  $i$  gegeben. Man soll eine Hyperbel als kollineares Bild eines  $i$  schneidenden Kreises  $c$  konstruieren und die Achsen und Scheitelpunkte der Bildhyperbel bestimmen.

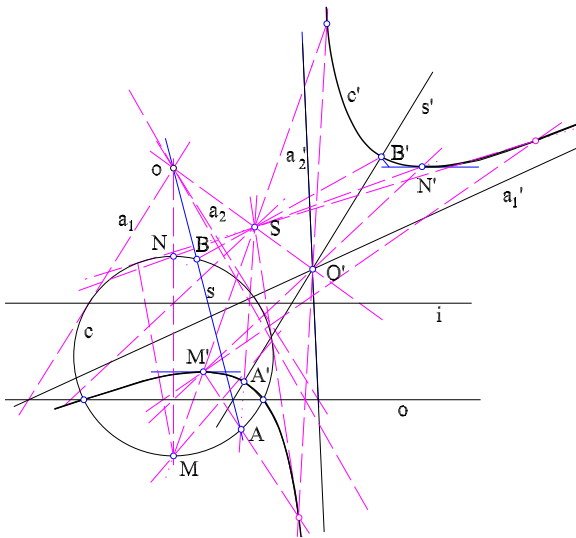
*Lösungsansatz:*

Die Schnittpunkte  $A_1, A_2$  des Kreises  $c$  mit Verschwindungsgerade  $i$  bilden sich in die Fernpunkte der Hyperbel ab und die in diesen Punkten existierenden Kreistangenten  $a_1, a_2$  in die Hyperbelasymptoten. Die Achsen der Hyperbel fallen dann in die Winkelsymmetralen dieser Asymptoten.

*Konstruktive Lösung:*

Der Schnittpunkt  $O$  der Kreistangenten  $a_1$ , und  $a_2$  bildet sich in den Mittelpunkt  $O'$  der Hyperbel ab und ist ein Außenpunkt von  $c$ . Der Punkt  $O'$  kann als Schnittpunkt der Asymptoten  $a'_1 = \kappa(a_1)$  und  $a'_2 = \kappa(a_2)$  bestimmt werden. Die Hyperbel-Hauptachse  $s'$  ist jene Symmetrale des Asymptotenwinkels, der von einer Sehnes von  $c$  stammt.

Die Schnittpunkte  $A, B$  der Gerade  $s = \kappa^{-1}(s')$  mit dem Kreis  $c$  bilden sich in Scheitelpunkte der Hyperbel ab.



Figur 6.

**Aufgabe 7.**

Ein Kegelschnitt  $k$  sei durch fünf Tangenten  $a, b, c, d, e$  gegeben. Bestimme eine perspektive Kollineation  $\kappa$ , dadurch bestimmten Kegelschnitt in einen Kreis  $k'$  überführt.

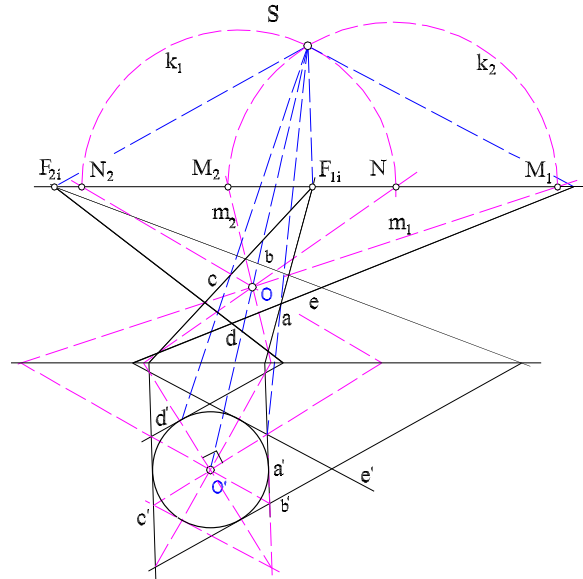
*Lösungsansatz:*

Unter allen möglichen Kollineationen, die  $k$  in einen Kreis  $k'$  überführen, wählen wir eine solche perspektive Kollineation  $\kappa$ , die das Tangentenvierseit  $abcd$  in einen Kreistangenten-Rhombus überführen soll. Die fünfte Tangente  $e$  muß dabei in eine Kreistangente  $e'$  übergehen, so daß  $e'$  mit gegenüberliegenden Rhombuseiten  $a'$  und  $c'$  einen neuen Rhombus bestimmt.

*Konstruktive Lösung:*

Die Geradenpaare  $a, c$  und  $b, d$  müssen in parallelen Geraden übergehen, so daß die Punkte  $a \cap c = F_{1i}$  und  $b \cap d = F_{2i}$  die Verschwindungsgerade  $i$  der gesuchten perspektiven Kollineation  $\kappa$  aufspannen. Der Schnittpunkt  $O$  der Diagonalen  $m_1$  und  $m_2$  der tangentialen Vierseiten  $abcd$  geht in den Kreismittelpunkt  $O'$  über. Die Geraden  $m_1$  und  $m_2$  schneiden  $i$  in Punkten  $M_1$  und  $M_2$ , die zu einem Paar von Fernpunkten orthogonale Richtungen gehören, wie auch

die Schnittpunkte  $N_1$  und  $N_2$  der Diagonalen der zweiten tangentialen Vierseits. Das Kollineationszentrum  $S$  bestimmt man als der Schnittpunkt der Thales-Kreise  $k_1$  und  $k_2$  über  $(M_1, M_2)$  bzw.  $(N_1, N_2)$ , da sich diese Paare aus  $S$  rechtwinklig projizieren müssen. Die Kollineationachse wählt man beliebig zu  $i$  parallel.



Figur 7.

**Literatur**

- [1] CESAREC, R.: *Analitička geometrija linearnog i kvadratnog područja I*, Školska knjiga, Zagreb, 1957.
- [2] MÜLLER, E.: *Vorlesungen über Darstellende Geometrie, I. Bd.*, Leipzig-Wien, 1923.

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SANJA FILIPAN  
SONJA GORJANC  
HRVOJE KVASNIČKA

# Natkrivanje paraboličkim konoidom

## Natkrivanje paraboličkim konoidom

### SAŽETAK

U radu je za jedan tip paraboličkog konoida 3. stupnja dan način konstrukcije te izvod jednadžbi u Kartezijevim koordinatama. U programu *Mathematica* definirane su funkcije za crtanje i izrađeni crteži konoida, grafovi funkcija Gaussove i srednje zakrivljenosti te primjeri moguće primjene konoida pri natkrivanju pravokutnoga tlocrta.

**Ključne riječi:** parabolički konoid, Mathematica

## Roofing with a Parabolic Conoid

### ABSTRACT

This article presents the construction and equations for one type of 3th degree conoid. We defined with *Mathematica* the functions for drawing and draw the pictures of the conoid, the graphics presentations of Gaussian and mean curvatures and the examples for roofing over the rectangular plan with a parabolic conoid.

**Key words:** parabolic conoid, Mathematica

**MSC 2000:** 68U05, 51N20

Studenti druge godine Građevinskog fakulteta u Zagrebu, Sanja Filipan i Hrvoje Kvasnička izradili su tijekom školske godine 2000/2001., pod mentorskim vodstvom nastavnice geometrijskih predmeta, rad *Natkrivanje paraboličkim konoidom*. Rad je nagrađen Rektorovom nagradom. Neznatno izmijenjeni dijelovi toga rada, uz dodatak prikaza Gaussove i srednje zakrivljenosti, sadržaj su ovoga članka.

## 1. Uvod

Postoji više načina konstruktivnog izvođenja pravčastih ploha. Na Građevinskom fakultetu u Zagrebu, u okviru kolegija Primijenjena geometrija pravčaste se plohe izvode kao sistemi pravaca koji sijeku tri prostorne krivulje  $k_1$ ,  $k_2$  i  $k_3$ . Prostorne krivulje  $k_1$ ,  $k_2$  i  $k_3$  nazivamo *ravnalicama*, a pravce koji ih sijeku *izvodnicama* pravčaste plohe. Pravčaste plohe kojima je jedna ravnalica beskonačno daleki pravac nazivamo *konoidima*. Ravnine koje sadrže takvu beskonačno daleku ravnalicu nazivamo *direkcijskim ravninama* plohe.

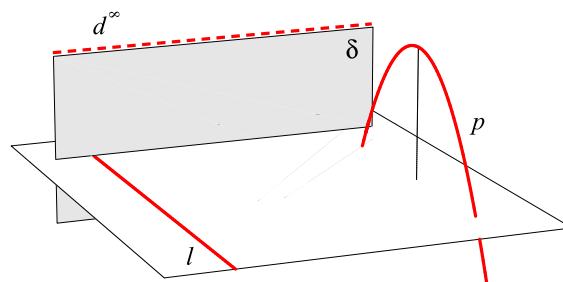
Ako se za ravnalice odaberu algebarske krivulje, koje su karakterizirane redom i razredom, nastaje *algebarska pravčasta ploha*. Red i razred takve plohe uvijek se podudaraju, tj. algebarske pravčaste plohe imaju *stupanj*.

Čitatelja koji se želi detaljnije upoznati s općom teorijom takvog načina izvođenja pravčastih ploha upućujemo na literaturu [6], [7], [4], [2] i [1].

## 2. Parabolički konoid

### 2.1. Zadavanje paraboličkog konoida

Za konstruktivnu i analitičku obradu odabran je parabolički konoid koji će kasnije poslužiti za natkrivanje pravokutnoga tlocrta. Jedna njegova ravnalica je parabola  $p$ , druga je beskonačno daleki pravac  $d^\infty$  određen direkcijskom ravninom  $\delta$  koja je paralelna s osi parabole, a treća je ravnalica pravac  $l$  okomit na direkcijsku ravninu i paralelan s ravninom parabole (slika 1).



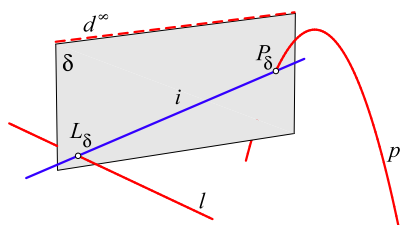
Slika 1. Ravnalice konoida.

Budući da se parabola  $p$  i pravac  $d^\infty$  sijeku u beskonačno dalekoj točki njezine osi, taj je konoid 3. stupnja (prema [7, str. 276]). Prema [1, str. 42] postoji samo jedna višestruka linija na njemu, a to je dvostruki pravac  $d^\infty$ .

**2.2. Konstrukcija paraboličkog konoida**

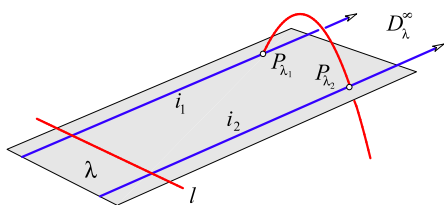
Izvodnice konoida konstruiraju se u ravninama pramena  $[d^\infty]$  ili  $l$ .

1. Svaka ravnina  $\delta \in [d^\infty]$  (pramen paralelnih ravnina) siječe pravac  $l$  u točki  $L_\delta$ , a parabolu  $p$  u njezinoj beskonačno dalekoj točki te u konačnoj točki  $P_\delta$ . Spojnica  $L_\delta P_\delta$  je izvodnica konoida (slika 2).

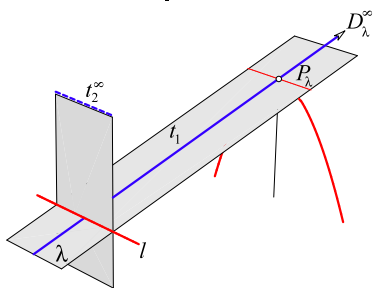


Slika 2. Izvodnica konoida u ravnini pramena  $[d^\infty]$ .

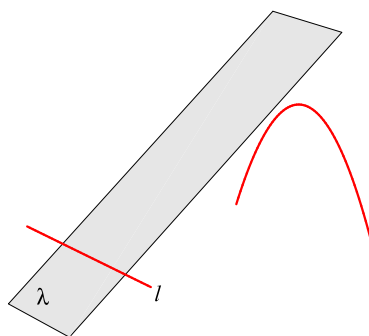
2. Svaka ravnina  $\lambda \in [l]$  siječe parabolu  $p$  u točkama  $P_{\lambda_1}$  i  $P_{\lambda_2}$ , a pravac  $d^\infty$  u točki  $D_\lambda^\infty$ . Spojnice  $D_\lambda^\infty P_{\lambda_1}$  i  $D_\lambda^\infty P_{\lambda_2}$  izvodnice su konoida koje mogu biti realne i različite (slika 3a), mogu se podudarati (slika 3b), a mogu biti i konjugirano imaginarne (slika 3c). Ravnine u kojima se točke  $P_{\lambda_1}$  i  $P_{\lambda_2}$  podudaraju nazivamo *torzalnim ravninama*, a izvodnice u tim ravninama *torzalnim* pravcima plohe (slika 3b).



Slika 3a. Par realnih i različitih izvodnica u ravnini pramena  $[l]$ .

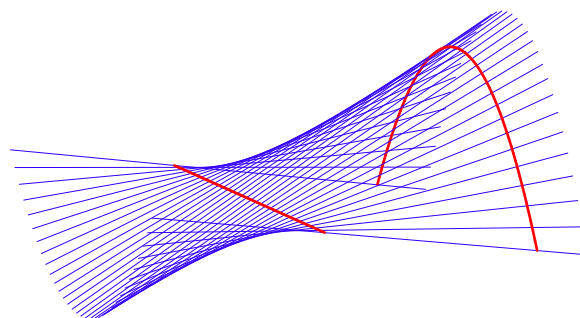


Slika 3b. Torzalne ravnine i torzalni pravci konoida.



Slika 3c. Ravnina pramena  $[l]$  koja konoid siječe u paru konjugirano imaginarnih izvodnica.

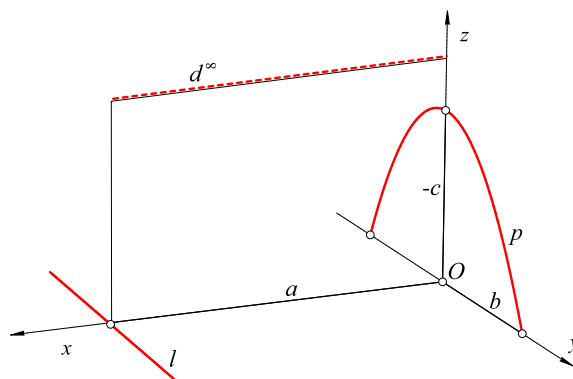
Na bilo koji od ova dva načina možemo konstruirati po volji mnogo izvodnica konoida (slika 4).



Slika 4: Izvodnice konoida.

**2.3. Jednadžbe paraboličkog konoida**

Ravnalice konoida smjestimo u Kartezijev koordinatni sustav  $O(x,y,z)$  kao na slici 5.



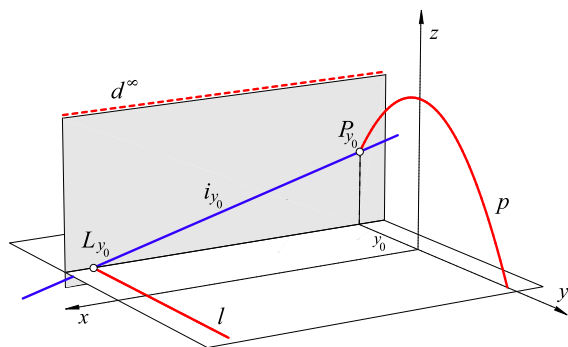
Slika 5. Položaj ravnalica u Kartezijevom koordinatnom sustavu.

Ravnalice konoida (parabola  $p$ , jednostruki pravac  $l$  i direkcijske ravnine  $\delta$  pramena dvostrukog pravac  $d^\infty$ ) mogu se sada odrediti njihovim jednadžbama.<sup>1</sup>

$$\begin{aligned} p \cdots & \begin{cases} x = 0 \\ z = \frac{c}{b^2}(y^2 - b^2), \quad b, c \in R, b, c \neq 0 \end{cases} \\ l \cdots & \begin{cases} x = a \\ z = 0, \quad a \in R, a \neq 0. \end{cases} \\ \delta \cdots & \dots y = y_0, \quad y_0 \in R. \end{aligned} \quad (1)$$

(Napomena: Bez smanjenja općenitosti mogli smo za bilo koji od parametara  $a$ ,  $b$  ili  $c$  odabrati jedinicu. To bi nam, međutim, pri kasnijem natkrivanju umanjilo mogućnosti za slaganje različitih konoida.)

U svakoj ravnini  $y = y_0$  pramena  $[d^\infty]$  leži izvodnica  $i_{y_0}$ . Ona je spojnica točke  $L_{y_0}(a, y_0, 0)$  na pravcu  $l$  i točke  $P_{y_0}(0, y_0, \frac{c}{b^2}(y_0^2 - b^2))$  na paraboli  $p$  (slika 6).



Slika 6. Izvodnica u ravnini  $y = y_0$ .

Jednadžbe izvodnice su:

$$i_{y_0} \cdots \begin{cases} y = y_0 \\ z = \frac{c}{ab^2}(y_0^2 - b^2)(a - x), \quad a, b, c \in R, a, b, c \neq 0 \end{cases} \quad (2)$$

Prođe li parametar  $y_0$  skupom realnih brojeva opisati će izvodnice  $i_{y_0}$  parabolički konoid. Stoga je

$$z = \frac{c}{ab^2}(y^2 - b^2)(a - x), \quad a, b, c \in R, a, b, c \neq 0 \quad (3)$$

eksplicitni oblik jednadžbe konoida.

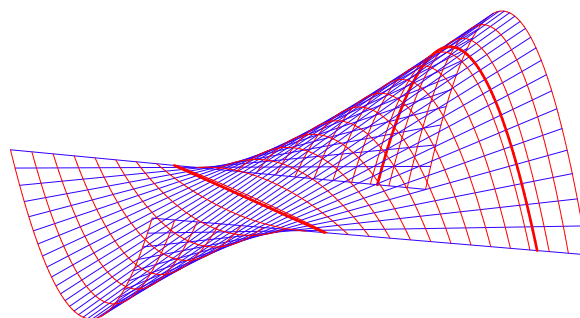
Za prikaze u programu *Mathematica* prikladnije su parametarske jednadžbe ploha [5, str. 229]. Za odabrani konoid, na temelju jednadžbe (3) prirodno se izvodi sljedeća

<sup>1</sup>Da bismo i pravac  $d^\infty$  odredili jednadžbama morali bismo koristiti homogene Kartezijeve koordinate  $(x:y:z:w)$ . Njegove bi jednadžbe tada bile  $y = y_0, w = 0$ .

parametrizacija.

$$\begin{aligned} x(u, v) &= u \\ y(u, v) &= v \\ z(u, v) &= \frac{c}{ab^2}(y^2 - b^2)(a - u), \quad u, v \in R, \\ & a, b, c \in R, a, b, c \neq 0 \end{aligned} \quad (4)$$

Za tu parametrizaciju parametarske krivulje konoida su pravci  $i_{v_0}$  i parabole  $p_{u_0}$ . Naime, svaka ravnina  $u = u_0$ ,  $u_0 \in R$  siječe konoid po paraboli  $z = \frac{c}{ab^2}(v^2 - b^2)(a - u_0)$  i beskonačno dalekom torzalnom pravcu. Za slučaj  $u_0 = a$  ravnina je torzalna pa se presječna krivulja raspada na pravac  $l$  i dvoznačni torzalni beskonačno daleki pravac plohe (slika 3b). Svaku od spomenutih parabola  $p_{u_0}$ ,  $u_0 \neq a$  možemo odabrati za ravnalicu.



Slika 7. Parametarske krivulje konoida koje odgovaraju parametrizaciji (4).

### 3. Prikazi paraboličkog konoida u programu *Mathematica* 4.0

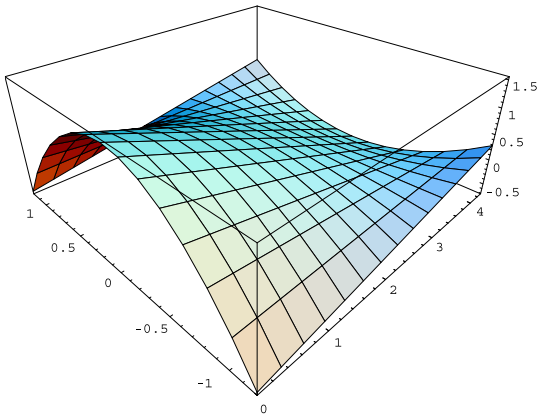
*Mathematica* je softverski paket koji se upotrebljava kao numarički i simbolički kalkulator, programski jezik, sistem za vizualizaciju funkcija i podataka, platforma na kojoj se grade paketi za specifične primjene, kreiraju interaktivni dokumenti u kojima se pojavljuju tekst, animacija i zvuk, itd. Ovdje ga prvenstveno koristimo za grafički prikaz funkcija u trenutno aktualnoj verziji 4.0.

Na temelju jednadžbi (3) i (4) definiramo funkcije  $Kon[a, b, c] : R^2 \rightarrow R$  i  $Konoid[a, b, c] : R^2 \rightarrow R^3$ . Za konkretne vrijednosti brojeva  $a$ ,  $b$  i  $c$  graf funkcija  $Kon[a, b, c]$  je parabolički konoid kojeg obrađujemo. Funkciju  $Konoid[a, b, c]$  definiramo kao listu parametarskih jednadžbi konoida određenog realnim brojevima  $a$ ,  $b$  i  $c$ . To je funkcija čije vrijednosti odgovaraju Kartezijevim koordinatama točaka konoida. U programu *Mathematica* definicije tih funkcija zapisujemo na sljedeći način.

```
Kon[a_,b_,c_][x_,y_] := c/(a*b^2) (y^2-b^2) (a-x)
Kon[a_,b_,c_][u_,v_] := {u,v,c/(a*b^2) (v^2-b^2) (a-u)}
```

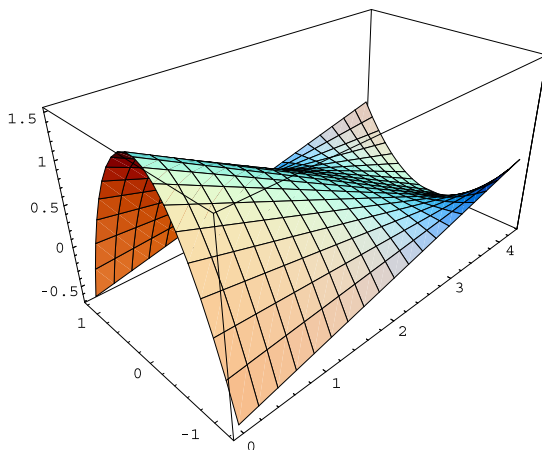
Sada za konkretni parabolički konoid ( $a = 3$ ,  $b = 1$ ,  $c = -1.5$ ) možemo, korištenjem naredbi *Plot3D* i *ParametricPlot3D*, dobiti sljedeće prikaze.

```
Plot3D[Kon[3,1,-1.5][x,y],{x,0,4.3},{y,1.2,1.2},
ViewPoint->{-1,-1,0.8}]
```



Slika 8.

```
ParametricPlot3D[Evaluate[Konoid[3,1,-1.5][u,v]],
{u,0,4.3},{v,1.2,1.2},ViewPoint->{-1,-1,0.8}]
```

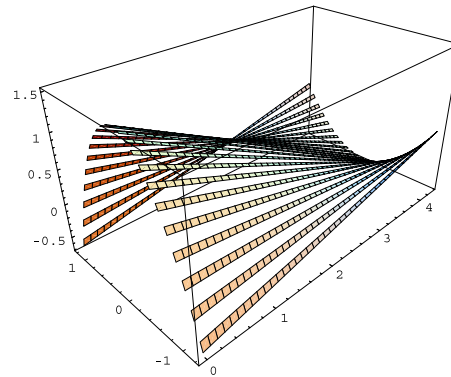


Slika 9.

Pored tih standardnih prikaza možemo izraditi i nešto zanimljivije. Tako, primjerice, korištenjem naredbi *Show* i *Table* možemo dobiti niz dijelova plohe koji kad se prikažu na istoj slici daju *prugastu* ili *rešetkastu* sliku konoida. Kontrolu koraka parametara  $u$  i  $v$  omogućuje nam učitavanje standardnog *Mathematica* paketa *ParametricPlot3D*.

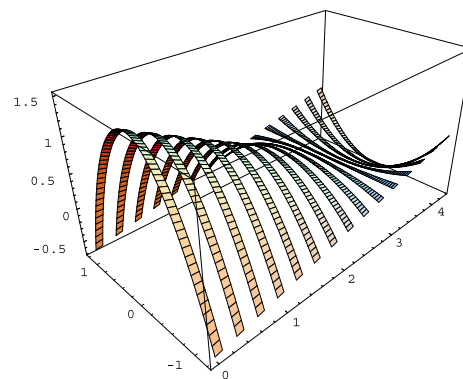
```
<<Graphics`ParametricPlot3D`
```

```
Show[Table[
ParametricPlot3D[Evaluate[Konoid[3,1,-1.5][u,v]],
{u,0,4.3,0.1},{v,i,i+2.4/67,2.4/67},
DisplayFunction->Identity],
{i,-1.2,1.2-2.4/67,7.2/67}],
DisplayFunction->${DisplayFunction}]
```



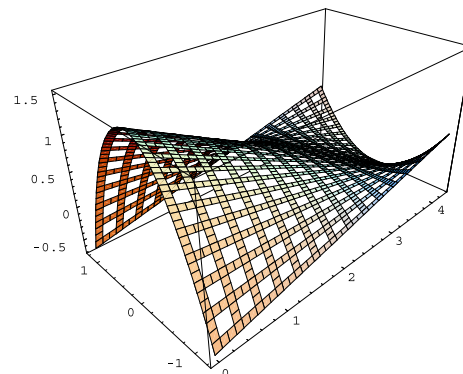
Slika 10.

```
Show[Table[
ParametricPlot3D[Evaluate[Konoid[3,1,-1.5][u,v]],
{u,i,i+0.1,0.1},{v,-1.2,1.2,2.4/67},
DisplayFunction->Identity],
{i,0,4.2,0.3}],
DisplayFunction->${DisplayFunction}]
```



Slika 11.

```
Show[%,%]
```

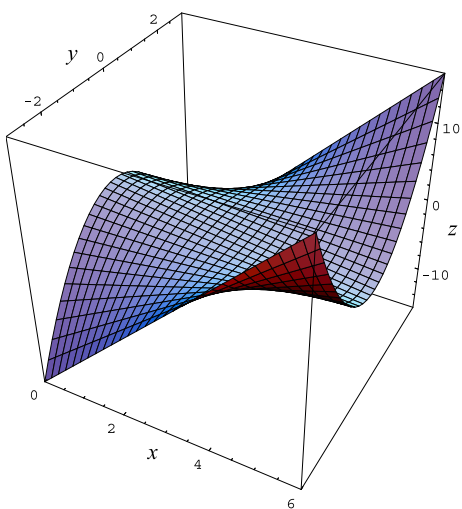


Slika 12.

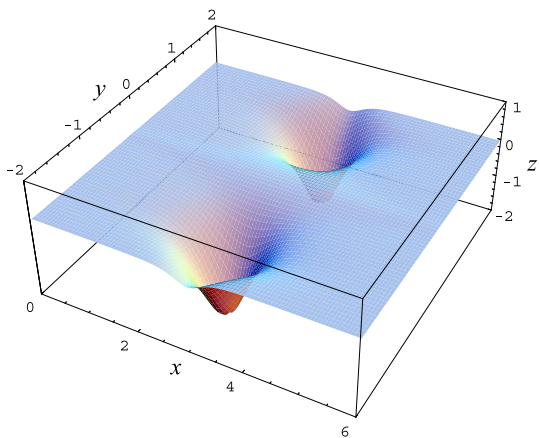


#### 4. Prikazi Gaussove i srednje zakrivljenosti paraboličkog konoida

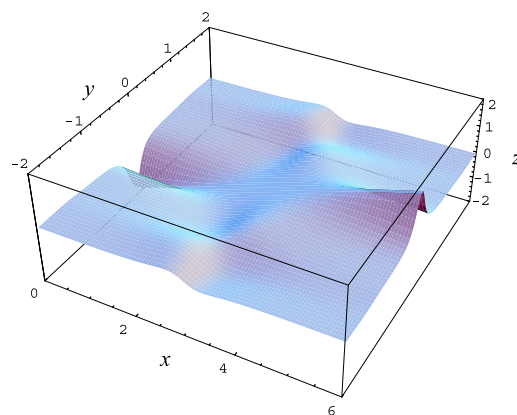
Gaussova i srednja zakrivljenost u regularnoj točki plohe važni su pojmovi diferencijalne geometrije ploha. Glavne zakrivljenosti u nekoj regularnoj točki plohe su ekstremne zakrivljenosti onih krivulja na plohi koje prolaze tom točkom, a leže u ravninama koje sadrže normalu plohe. U točki plohe Gaussova je zakrivljenost jednaka *produktu*, a srednja *polovini zbroja* glavnih zakrivljenosti. Za njihovo izračunavanje koriste se I. i II. diferencijalna forma [5, str. 252], [3, str. 373-380]. U knjizi [3, str. 394] definirane su, u jeziku *Mathematica*, funkcije `gcurvature` i `mcurvature` koje za bilo koju plohu zadanu parametarskim jednadžbama računaju vrijednosti Gaussove i srednje zakrivljenosti u svakoj njezinoj regularnoj točki. Te su definicije upotrebljene pri izradi sljedećih crteža.



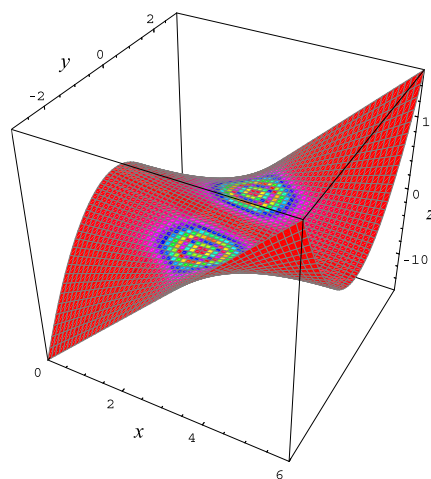
Slika 13. Graf funkcije `Kon[3, 1, -2]` u području  $[0, 6] \times [-1, 1]$ .



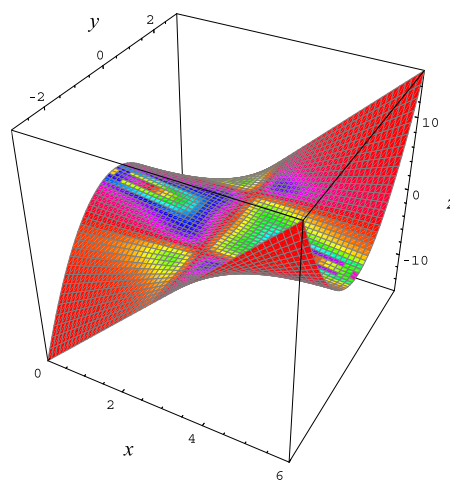
Slika 14. Graf funkcije `gcurvature[Konoid[3, 1, -2]]` u području  $[0, 6] \times [-1, 1]$ .



Slika 15. Graf funkcije `mcurvature[Konoid[3, 1, -2]]` u području  $[0, 6] \times [-1, 1]$ .



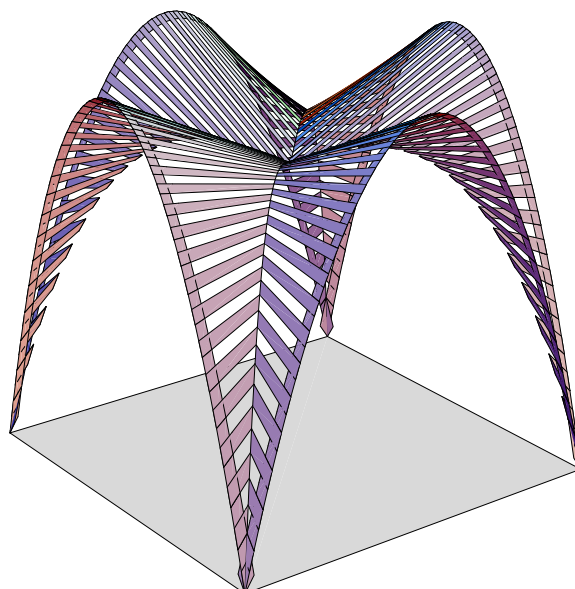
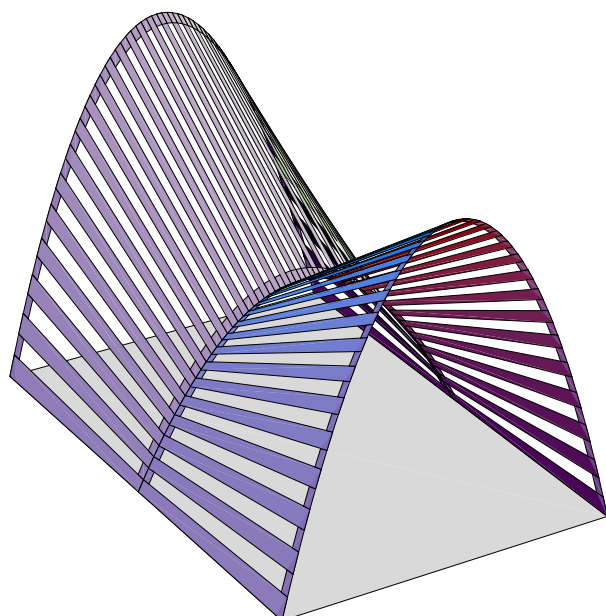
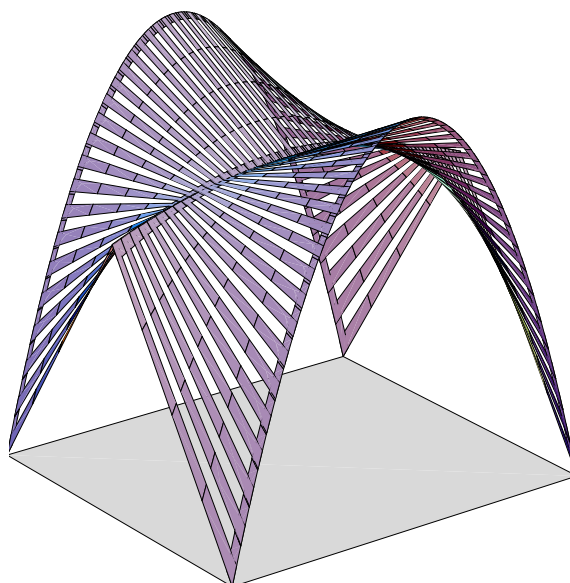
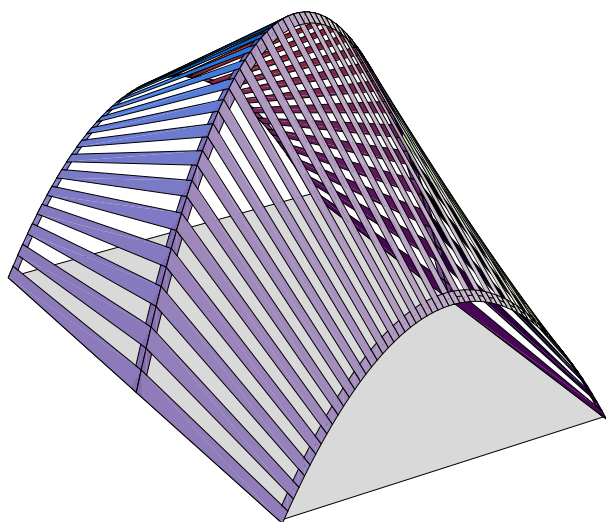
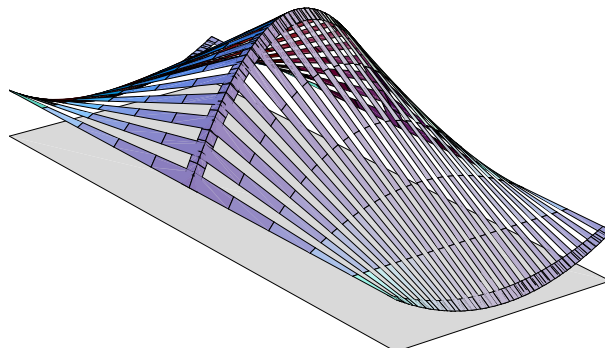
Slika 16. Graf sa slike 13 obojan bojom koja je funkcija Gaussove zakrivljenosti.

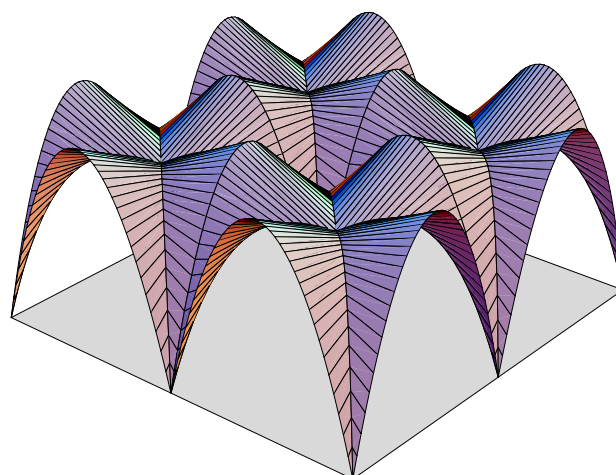
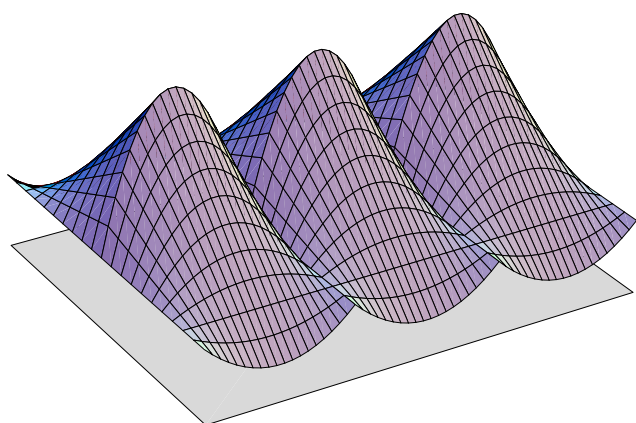
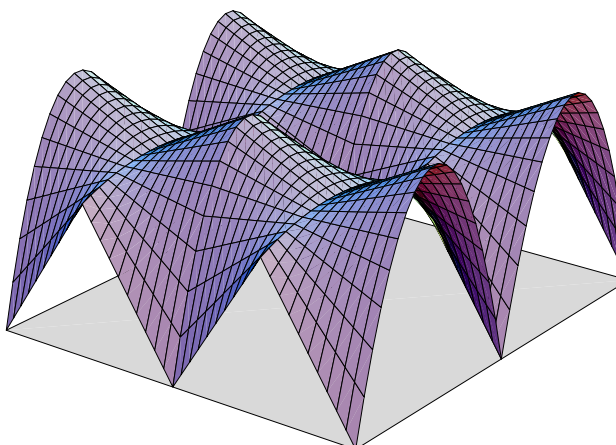
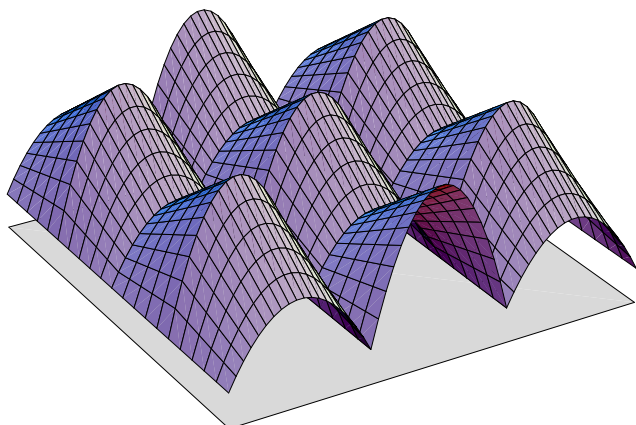
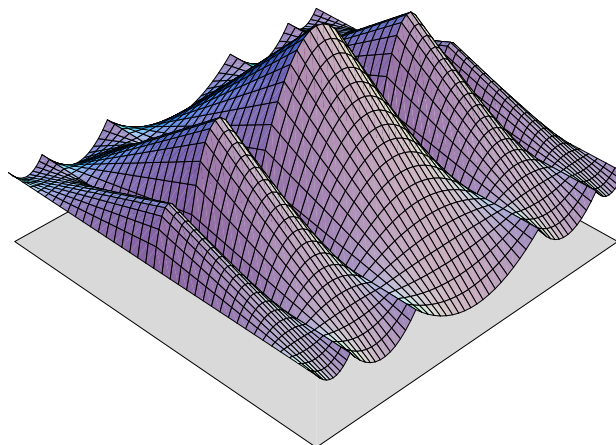
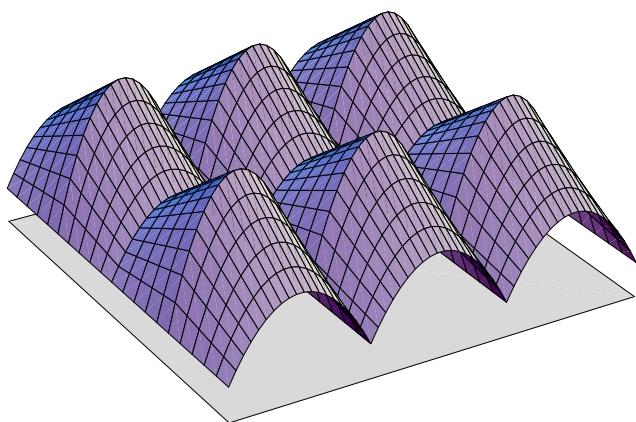


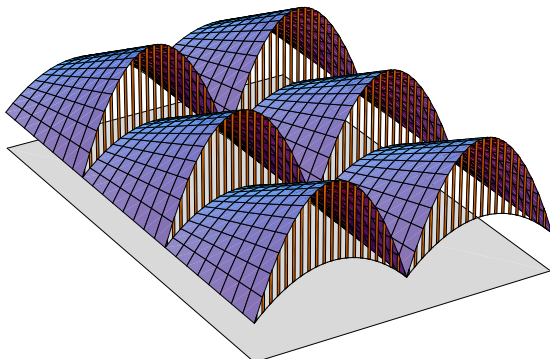
Slika 17. Graf sa slike 13 obojan bojom koja je funkcija srednje zakrivljenosti.

## 5. Natkrivanje pravokutnoga tlocrta

Parabolički konoid može se za natkrivanje koristiti na nebrojeno mnogo načina. Elementi su razni izrezi plohe koji se jednostavno slažu u nizove. Na sljedećim je crtežima prikazano nekoliko ideja za natkrivanje pravokutnoga tlocrta.







## Literatura

- [1] S. GORJANC: *The Generation of Ruled Cubics by Using Mathematica 3.0*. Proceedings of 8th ICECGDG, Austin, Texas, USA, 1998.
- [2] S. GORJANC: *Izvođenje pet tipova pravčastih ploha 4. stupnja* KoG No. 2, 57-67, 1997.
- [3] A. GRAY: *Modern Differential Geometry of Curves and Surfaces with Mathematica*. CRC Press, Boca Raton, 1998.
- [4] B. KUČINIĆ, O. KRISTOFOROVIĆ, I. SALER: *Oble forme u graditeljstvu*. Građevinar, Zagreb, 1992.
- [5] Ž. MARKOVIĆ: *Uvod u višu analizu, II. dio*. Školska knjiga, Zagreb, 1952.
- [6] E. MÜLLER, J. L. KRAMES: *Konstruktive Behandlung der Regelflächen*. Franc Deuticke, Leipzig und Wien, 1931.
- [7] V. NIČE: *Deskriptivna geometrija II*. Školska knjiga, Zagreb, 1980.
- [8] S. WOLFRAM: *Mathematica 4.0*.

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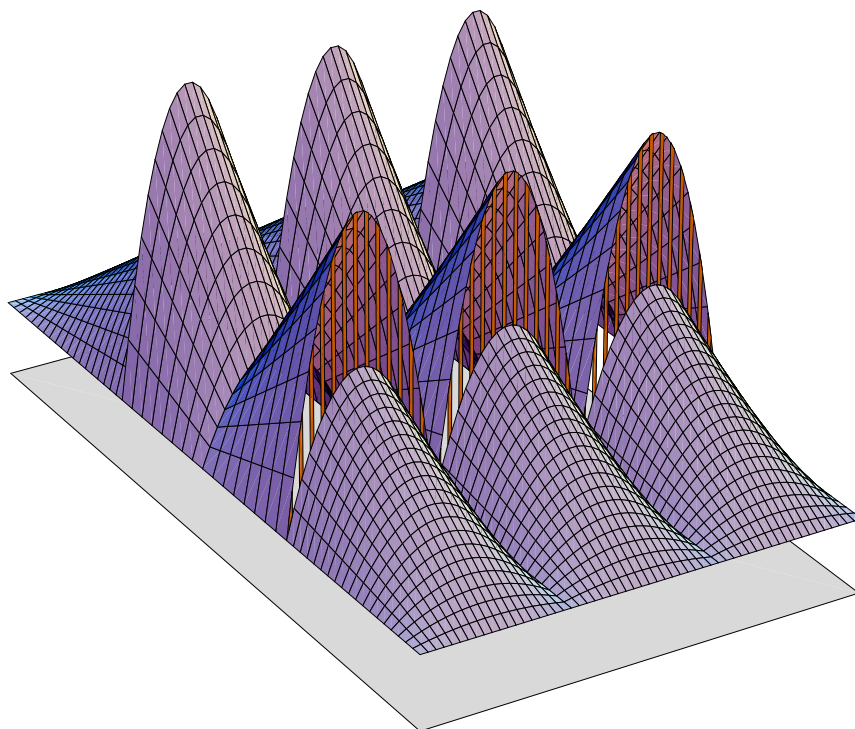
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# Visual Communication Curricula for the Global Engineers

## Nastava vizualnog komuniciranja za svjetske inženjere

### SAŽETAK

Uvođenje studenata u inženjersku profesiju počinje razvijanjem sposobnosti komunikacije, posebno grafičke. U ovom radu autori iz različitih dijelova svijeta uspoređuju svoju sadašnju praksu predavanja sadržaja inženjerske grafike (nacrtna geometrija, ručno tehničko crtanje, crtanje slobodnom rukom i računalna grafika te CAD-programi: Auto CAD...) i pokušavaju zacrtati put koji bi mogao slijediti nastavni plan u budućnosti. Oni predlažu što treba ostati kao princip vizualne komunikacije, a što treba mijenjati.

**Cljučne riječi:** inženjerska grafika, računalna grafika, nastavni plan, budući inženjeri svijeta

**MSC 2000:** 97B40

## Visual Communication Curricula for the Global Engineers

### ABSTRACT

The first step in introducing the engineering profession to students is to develop good communication skills, especially graphics communication. In this paper authors from different parts of the world compare their current practices in delivering Engineering Graphics units (Descriptive Geometry, Manual Technical Drafting, Freehand Drawing Computer Graphics and CAD programs: AutoCAD.) in an attempt to map a path which the future curriculum in this area may follow. They propose what will stay as principles language and what is going to change.

**Key words:** engineering graphics, computer graphics, curriculum, future global engineers

## 1 Introduction

Apart from the ability for excellent written and oral communication, engineers have had their own visual language - drawings. Computer technology has significantly changed the teaching philosophy of engineering graphics over the past two decades. We envisage that visual communication abilities of future engineers will continue to evolve in the next twenty years with an emphasis towards more interdisciplinary skills. A greater volume of information will be provided in a visual form with the use of drawings, charts, and diagrams. This will often be accompanied by the use of sound and motion.

Engineers today are the creators of the physical world, not just in their own country but of the whole world which is like a huge site, where they move from one project to an-

other, from one country to another. To identify the profile of a global engineer and to create a curriculum which will produce such an engineer is a task of high priority of universities around the world.

Engineering futurists believe that change is occurring at such a rapid pace that the knowledge we teach our students today will be outdated by the time they graduate. Furthermore, as engineers, they will need to change their occupation several times throughout their working life. Australian writer Peter Ellyard argued that 70 per cent of professions that will exist 20 years from now do not as yet exist. We have already experienced the rapid development of new engineering fields (such as mechatronics and biomedical engineering) and seen a significant drop in enrolment in some of the more traditional engineering disciplines.

The whole industry will shift as a result of the move into a globalised, post-industrial society. This is a transition that we have to understand and adapt to. We have to go beyond the point of self-interest and envisage the future shape of an engineer. In so doing, we can create a new teaching founded on the basic principles underlying each engineering discipline.

Much research has been undertaken worldwide to outline a set of criteria that will be required of the future global engineer. The recent UNESCO report [1] pinpointed the need for dramatic changes. It suggested four points that must be reinforced as part of the reformation of education in the next millennium. According to this report, learning must satisfy the following criteria:

- 1) enhance the quantity/quality of knowledge,
- 2) develop initiative,
- 3) nurture the development of self and
- 4) encourage teamwork.

Yerlici [2] proposed a similar philosophy; that higher education should accomplish the following:

- 1) improve the ability to question and seek answers;
- 2) sharpen attention to detail;
- 3) refine the mind for greater sophistication at interpreting data; and
- 4) encourage independent thinking.

In addition, Yerlici stated that while providing basic knowledge, good teachers should stimulate the minds of students in the direction of critical thinking and creativity.

Professor Evan Pretty suggested the following in *Vision 2020- Education in next millennium* [3]:

- 1) Active learning should be emphasised over passive teaching to promote deep rather than surface learning;
- 2) Curricula and assessment systems should be re-designed to enhance critical thinking and problem solving skills as well as develop creativity and communication skills;
- 3) The role of the university teacher must undergo change from a 'lecturer' to a manager of the education process - in other words, the sage on the stage will become the guide on the side; and
- 4) Students will have a wide choice of courses that may be undertaken 'at a distance', therefore flexible courses for resource based learning will become a necessity.

The student/staff ratio has been increasing rapidly in past few years. However, the importance of the professor's job will not decline [4]. There will always be a need for professors who can guide and motivate students in their course.

## 2 Engineering graphics curricula – present situation

Good communication skills are emphasized in all new strategies for creating a curriculum that will build a global engineer. Drawings are, and will continue to be, a major tool for engineers worldwide. The prevalence of computer technology has had crucial implications for teaching engineering graphics.

At present, significant differences exist globally in the delivering of knowledge in this area. The developed nations, which are well equipped with information technology infrastructure, have been exposed to permanent changes in teaching philosophy. This is in contrast to many educational systems, especially in the developing world, that have stagnated largely in traditional frames. In some of those systems, the instructor is still seen as an expert who imparts knowledge to students who, in turn, are expected to absorb the information without much critical thinking. The relationship between students and the lecturer(s) are rigid. Those systems are also characterized by overloaded curriculums and an emphasis on individual work in the learning process. Some of the didactical models are too mechanistic and therefore will not satisfy today's and especially tomorrow's more active learners.

In other developing countries education is modernized. There is no more unidirectional imparting of knowledge by an "expert" on a particular discipline

In Europe the classical knowledge are appreciated but computer sciences are developing as well, in many institutes. New disciplines are appearing, like computational geometry, CAGD [8], and the old ones are transforming. Ideas for the future were suggested in SEFI seminars [9] and the links between geometry and CAD are expected to increase in importance. Very sophisticated packages for construction calculations are reality in engineering. Actual development of engineer constructions is due to theory of constructions development and computer technologies, new materials and new architectural and constructive results [6]. In Europe engineers use of many professional CAD packages. Professors use software for scientific and professional work, like *Mathematica*, *Cabri*, *Maple* and educational software, made by professors at universities.<sup>1</sup>

<sup>1</sup>Prof. Seybold introduces Computer aided geometry (CAG), which is dimension independent. O. Giering, H. Seybold: KONSTRUKTIVE INGENIEURGOMETRIE, Carl Hanser Verlag, München.

Use of more powerful software requires more knowledge in users, including geometric knowledge.

Computer graphics and CAD are taught at universities in different ways<sup>2</sup>, [6]. In Slovenia is active Society for higher didactic<sup>3</sup>. In Croatia, University of Rijeka the researches<sup>4</sup> are made on the quality of university teaching [7] aimed to adapt to the rapid change of technologies. Knowledge is considered as open, dynamic system. University organise many scientific tribunes on education with guests from Universities in Croatia, England, Italy, Slovenia and University of Connecticut, USA.

**Table 1** illustrates four different curriculums from four parts of the world.

### 3 Future Directions

Our upcoming students will expect more active ways of seeking knowledge. New techniques have to implement more experimentation, exploration, testing of ideas in reality, exchanging of ideas with peers in small group activities, simulation, project based learning, life-long learning, critical thinking, and research skills. Collaboration and teamwork are powerful forces in the learning process and must form a part of every curriculum. Teachers are expected to undertake the role of facilitators in the learning process. The learning process is moving towards a more informal style of learning or self-directed learning where students are encouraged to teach themselves. It will increase the student's responsibility for individual learning results and empower the learner not to be dependent on the instructor.

A new, less formal approach can be developed to teach visual communication skills through the critique of established views. Instead of teaching in isolated units, knowledge can be disseminated through real design projects in a unit that students from different disciplines may enrol in. Good visualization skills will empower new engineers to be creative in their designs and influence their vision of the future world in the new millennium.

### 4 Principles of visual communication

What are the principles of visual communication language that will remain which engineering graduates need to un-

derstand in order to be equipped for professional practice in the new millennium?

The principles that we believe future teaching philosophy must incorporate are as follows:

- 1) Principles of visual science: geometric projections, space perception;
- 2) International Standards for technical drafting for different disciplines; and
- 3) Drawing techniques: freehand sketching and computer aided drafting.

First, **Principles of visual science** are taught in Descriptive Geometry. The main knowledge students should gain from this unit include good visualisation skills and how to present 3D object on 2D media and vice versa. Computer technology renders the teaching of unfriendly, tedious geometrical construction methods obsolete. Information technology has the potential to solve many of these problems. Education in the future will undertake an interdisciplinary approach to equip students with a wider base of knowledge so that an understanding of all parts is gained rather than a narrow specialisation in a particular area.

The very basics of Descriptive Geometry - the philosophy of projection, types of projections, 2D presentations, 3D presentations, sections, and details - will be sufficient for students to learn some techniques in freehand sketching and to start to draft using computers.

In Europe, principles of visualization will be constructive geometrical thinking, CAG, CAGD. In Europe many lecturers think that before CAD, future engineers will need general geometric ideas, concepts and methods, the ability of 3D thinking and geometric reasoning, to be able to solve new problems, using new technology. The theoretical 3D solution has to be finished in the mind beforehand.

Second, **International Standards** for computer aided technical drafting for different disciplines do not as yet exist. It is our hope that in the new globalized engineering arena international standards will be created in the next 20 years, which educators can introduce to students worldwide.

Third, **Drawing techniques**. Engineers today do not draft. Engineers sketch ideas while designing, or to provide explanations to other people. On some occasions, engineers do simple drawings using the computer, but the main job

Prof. dr. Stachel's programs CADDG, CAD3D, CAD2D for TU Win are permeated and mentioned in the book: G. Glaeser, H. Stachel: OPEN GEOMETRY, Springer Verlag, 1999.

<sup>2</sup>Through geometric and informatic education, CAD courses, or through introduction in computer graphics: Zámožik J., Richtáriková D., COMPUTER AND GEOMETRY, Special SEFI European Seminar on Engineering Education, Bratislava, 1997.

<sup>3</sup>"Slovensko društvo za visokošolsku didaktiku" you can find on <http://www.fe.uni-lj.si/sdvd/welcome.html>

<sup>4</sup>"The Quality of Teaching in Higher Education" project: <http://www.pefri.hr/Projekti/00913/>

Table 1. Different curricula from four parts of the world

	University of Western Australia Perth, Australia	Bogazici University Bebek Istanbul, Turkey	University of Santo Tomas España, Manila	Ateneo de Manila University Loyola Heights, Quezon City	The University of Rijeka, Croatia
COMMON TO FIRST YEAR	<ul style="list-style-type: none"> <li>- Civil</li> <li>- Mechanical</li> <li>- Materials</li> <li>- Environmental</li> </ul>	<ul style="list-style-type: none"> <li>- Civil</li> <li>- Mechanical</li> <li>- Electrical</li> <li>- Computing</li> <li>- Industrial</li> <li>- Chemistry</li> </ul>	<ul style="list-style-type: none"> <li>- Civil</li> <li>- Chemical</li> <li>- Mechanical</li> <li>- Electrical</li> <li>- Electronics and Communications</li> <li>- Industrial</li> </ul>	<ul style="list-style-type: none"> <li>- Electronics</li> <li>- Communications and Computers</li> </ul>	<ul style="list-style-type: none"> <li>- Mechanical</li> <li>- Shipbuilding</li> </ul> <p>similar first year</p> <ul style="list-style-type: none"> <li>- Electrical</li> <li>- Civil</li> </ul>
INTRODUCTION TO ENGINEERING PROFESSION UNIT (which comprises visual communication and oral/written communication units)	<p><b>Project Engineering 115 (PE 115):</b></p> <ul style="list-style-type: none"> <li>-Lectures 1hr/week/2 sems (given by professors from each discipline);</li> <li>- <b>Engineering Drawing</b> 4hrs/week/1 semester</li> <li>- <b>Written/Oral Communication</b> 1hr/week/2 semesters (on how to write technical reports, resumes, how to give oral presentations);</li> <li>- <b>Practical Project Work</b> (projects on each department-duration of each project 5 weeks).</li> </ul>	<p><b>Introduction to Engineering</b> (offered by each department separately):</p> <ul style="list-style-type: none"> <li>3hrs/week/1 semester</li> <li>-<b>Graphics (ENGG 110)</b> 3hrs/week/1 semester</li> <li>-<b>Turkish (TK 221, TK 222)</b> 2hrs/week/2 semesters</li> </ul>	<p>(There is no <i>Introduction to Engineering Profession</i> or <i>Introduction to Engineering unit</i>. Visual communication units are separate units).</p> <p><i>Oral/Written Communication Unit:</i></p> <p>First semester:</p> <ul style="list-style-type: none"> <li>- <b>Comm. Skills 1 (Eng 101a)</b> Lectures, 3 hrs/week</li> </ul> <p>Second semester:</p> <ul style="list-style-type: none"> <li>- <b>Comm. Skills 2</b> Lectures, 3hrs/wk/sem</li> </ul>	<p>(There is no <i>Introduction to Engineering Profession</i> or <i>Introduction to Engineering unit</i>. Visual communication units are separate units).</p> <p><i>Oral/Written Communication Unit:</i></p> <ul style="list-style-type: none"> <li>- <b>Communication Across the Curriculum I &amp; II</b> (En 11 En 12) Lectures, 3hrs/wk/sem</li> </ul>	<p>(There is no <i>Introduction to Engineering Profession</i> or <i>Introduction to Engineering unit</i>. Visual communication units are separate units).</p> <p><i>Oral/Written Communication Unit:</i></p> <ul style="list-style-type: none"> <li>- <b>English Language</b></li> </ul>
VISUAL COMMUNICATION UNITS	<p><b>Engineering Drawing</b> (as part of PE115):</p> <ul style="list-style-type: none"> <li>-Lectures, 1hr/week/1 sem</li> </ul> <p>-Tutorials:</p> <ul style="list-style-type: none"> <li>. freehand sketching and manual drafting</li> <li>2hrs/week/1 semester;</li> <li>. AutoCAD</li> <li>1hr/week/1 semester</li> </ul>	<p><b>Graphics (ENGG 110)</b></p> <ul style="list-style-type: none"> <li>-Tutorials: 3hrs/week/1 semester (Lectures incorporated into tutorials)</li> </ul>	<p><b>Engineering Drawing DRAW 111/CAD</b></p> <ul style="list-style-type: none"> <li>- Tutorials, 3hrs/week/1sem (Lectures incorporated into tutorials)</li> <li>. mechanical drafting 2hrs/week/1st semester</li> <li>. AutoCAD R14 1hr/week/1st semester</li> </ul>	<p><b>Technical Drawing/Drafting (ECE 11)</b></p> <ul style="list-style-type: none"> <li>- Tutorials, 3hrs/week/1 sem (Lectures incorporated into tutorials)</li> <li>. mechanical drafting 3hrs/week/1st semester</li> </ul>	<p>Mechanical</p> <ul style="list-style-type: none"> <li>- <b>Engineering Drawing</b> 3hrs/week/1 sem</li> <li>- <b>CAD</b> 2hrs/week/2 sem</li> </ul>



<p>VISUAL COMMUNICATION UNIT'S OUTLINE</p>	<ul style="list-style-type: none"> <li>- Basics of Descriptive Geometry</li> <li>- Introduction to technical documentation</li> <li>- Australian Standards</li> <li>- Technical Illustrations and Diagram</li> <li>- Fast sketching</li> <li>- Artistic Drawing</li> <li>- Computer Graphics-AutoCAD</li> </ul>	<ul style="list-style-type: none"> <li>- Multi-view projects</li> <li>- Sectioning</li> <li>- Dimensioning</li> <li>- Fasteners</li> <li>- Isometric</li> <li>- Assembling</li> <li>- Computer graphics-AutoCAD</li> </ul>	<p><b>- Engineering Drawing 121 (Draw 121/CAD)</b> Prerequisite: Draw111/CAD</p> <ul style="list-style-type: none"> <li>- Tutorials, <i>3hrs/week</i> (lectures are incorporated into tutorials)</li> <li>. mechanical drafting</li> <li>. <i>2hrs/week/2 sem</i></li> <li>. AutoCAD R14</li> <li>. <i>1hr/week/1 sem</i></li> </ul>	<p><b>- CAD and Drafting (ECE12)</b> Prerequisite: ECE 11</p> <ul style="list-style-type: none"> <li>- Tutorials, <i>4hrs/week/summer</i></li> </ul>	<p>Civil</p> <ul style="list-style-type: none"> <li>- <b>Descriptive Geometry</b></li> <li>- Lectures: <i>3hrs/ week/1 sem</i></li> <li>- Tutorials: <i>2hrs/week/1 sem</i></li> <li>- <b>Applied Geometry and CAD</b></li> <li>- Lectures: <i>2hrs/week/2 sem</i></li> <li>- Tutorials: <i>2hrs/week/2 sem</i></li> </ul>
<p>ASSESSMENT</p>	<p>There is <b>no final exam</b>. There is a visualisation test at the beginning and at the end of semester. <i>Final mark is based on:</i></p> <ul style="list-style-type: none"> <li>- Weekly Assessment (on the work produced each week in tutorials)</li> <li>- Average mark for the Assignments</li> <li>- Mark for the final Portfolio</li> <li>- Self-assessment</li> </ul>	<p>There is a <b>final exam</b>.</p>	<p>There is a <b>comprehensive exam</b>. <i>Final grade is based on:</i></p> <ul style="list-style-type: none"> <li>- weekly activity or exercise produced each meeting</li> <li>- comprehensive examination</li> <li>- hands-on examination in AutoCAD</li> </ul>	<p>There is a <b>comprehensive exam</b>. <i>Final grade is based on:</i></p> <ul style="list-style-type: none"> <li>- weekly activity-work produced each week</li> <li>- comprehensive examination</li> <li>- hands-on examination in AutoCAD and project incorporated with another subjects, Engineering Practices and Workshop</li> </ul>	<p>There is a <b>final exam</b> (written and oral). <i>Final mark is based on:</i></p> <ul style="list-style-type: none"> <li>- weekly activity-work</li> <li>- individual exercise</li> <li>- written and oral exam</li> <li>- hands-on examination in AutoCAD or Design CAD.</li> </ul> <p>Written exam can be replaced by two partial exams-colloquiums.</p>

is the responsibility of draft-persons. However, an understanding of drawings is crucial for the engineer for their own visualisation purposes and so as to clarify to others; for example, clients or builders.

In some countries engineers still draft, if necessary. In Europe they use specialized software. But when they use some general CAD program solving new engineering problems, they combine and adapt their knowledge and ideas in the virtual 3D-computer space. Draft persons can not do it.

## 5 Teaching philosophy

Based on the principles of visual communication, the teaching philosophy for the future curriculum must incorporate the following issues:

- 1) Self-directed learning and life-long learning skills
- 2) Flexible curriculum
- 3) Face to face/"Virtual" education
- 4) Multidisciplinary approach
- 5) Collaboration

**Self-directed learning and life-long learning skills.** Future developments are difficult to predict and continuous learning, a life long process, is one of the solutions to ensuring that an individual can adapt when new situations arise. Life-long learning includes formal and informal experiences, such as attending formal courses, and self-directed learning through the Internet. Information technology (IT) can be harnessed to foster greater interaction and collaboration among students and teachers as well as between students themselves. IT has the potential to provide a more varied, effective and efficient learning environment [4].

Volitional control of the learning process by students rather than by the teacher will develop in students a more serious attitude towards the course. They will voluntarily seek more information. Having the freedom to choose the time and the place for learning will enable students to take greater responsibility over their own learning. In this manner, students will be nurtured to become more active and creative learners.

**Flexible curriculum.** As an environment for learning, universities today are at a crossroad. An increasing number of universities are adopting flexible forms of delivering knowledge. A flexible curriculum is accompanied by a shift towards outcomes-based teaching and assessment where the emphasis is on what skills and knowledge are learned rather than the content taught. The curriculum

should be designed to develop critical understanding, analytical and problem solving skills, as well as a capacity for creativity. Students will have to be more involved in the programmes; for example, students will be encouraged to partake in the creation of the new curriculum. The objective is to lessen student reliance on being taught exactly what is going to be examined. The flexible curriculum is supposed to be dynamic and must incorporate direct and easy access to the faculty.

**Face to face/"Virtual" education.** Academic institutions today are in transition. There is an increasing shift towards the use of the Internet in courses, which is evident from the development of virtual universities. Online teaching provides the possibility to expand the scope and the content of the curriculum. The new model will be more interactive than the basic form of computer-assisted education. It will allow students to post comments to discuss areas of interest or concern on the World Wide Web. It may also include chat rooms, where all participants may log on to a course site and interact with each other in real time. The aim is to develop students into more active and creative learners.

A wide range of data can be placed on the Internet, including the theory of Descriptive Geometry, and the description of drawing techniques (freehand drawing, computer graphics-software, etc.). Face-to-face classrooms or lectures have to be used to convey general instructions and provide examples, and for projects in which students will have a chance to apply their acquired knowledge.

**Multidisciplinary approach** must be incorporated into undergraduate courses. Communication skills will be fostered by means of project work but will also be taught formally by means of assessed reports and presentations that will implement elements from other related areas of visual science. It will incorporate not only knowledge in technical freehand sketching and computer aided drafting in accordance with standards, but also the capability and the creativity to understand areas such as architectural graphics, technical illustration-graphs and charts, presentation graphics, artistic drawing, multimedia and similar elements [5]. This approach will provide an opportunity for students from different faculties, who deal with visual data, to learn together about principles of visual language in a course that will have an interdisciplinary approach. It can result in joint projects of students from different engineering disciplines, architecture, fine arts, agriculture, medicine, geography, chemistry and/or physics.

**Collaboration** is the key to enriching engineering education. Faced with greater competition from nontraditional providers in the industry, each university must continually reinvent itself and collaborate with industry to develop flexible solutions using IT [4].

Universities must maintain an enhanced environment that is in touch with the requirements of the real world. Universities must be in permanent collaboration and interaction with industry in order to assist students in developing the necessary skills needed to meet future employment requirements. There are three levels of possible collaboration, which are stated as follows:

- 1) Collaboration among students - team work;
- 2) Collaboration with other units from within the faculty or between faculties around the world; and/or
- 3) Collaboration with government and/or industry.

The Internet may be used as a medium for providing students with the instructions and requirements of the unit; including information on projects, basic concepts and theory, and the assessment structure. The main form of student learning will derive from working on the projects that will highlight the importance of teamwork. The projects may be modeled upon projects from other units that incorporate drawings, or real projects from industry. As practice, students can volunteer their services to different design offices to produce some drawings. This creates a demand to place the visual communication unit in the second year of undergraduate study, instead of where it is currently offered (the first semester in the first year of undergraduate study). In first year, students lack knowledge about the engineering profession and basic engineering concepts; therefore are less equipped to tackle real projects.

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## 6 Conclusion

Globalisation made possible through the revolution of IT has resulted in a new breed of engineers with multidisciplinary abilities. This multidisciplinary approach forms the foundation for the basic outline of the future curriculum for the Visual Communication unit proposed. It will involve the use of a combination of the traditional classroom and the classroom in cyberspace.

- Lectures used primarily to convey general information though the teacher will continue to define course content and drive the course (teacher-manager);
- Student-centered learning approach via broad on-line subject matter, where students will be able to pursue their own interests;
- Face-to-face tutorials based on specialised real projects completed in teams;
- Unit placed in second year - requires some basic knowledge about the course; and
- Flexible curriculum open to students from a wide range of different faculties interested to learn to communicate visually (all branches of engineering, Architecture, Geography, Agriculture, Medicine, Physics, Chemistry, etc.)

The overall teaching philosophy must remain dynamic and ever changing.

## References

- [1] UNESCO, *Learning: The treasure within*, UNESCO Publishing, Paris, 1998. Twigg, C. *The Need for National Learning Infrastructure* Education Review, Sept/Oct. 1994b, 29(5).
- [2] YERLICI, V.: *The place of teaching and research in engineering education*, (*Ingenieur Pedagogik Brucke Zwischen Lehre and Forshung*), A. Melezinek, G. Kurz (eds), Leuchtturm-Verlag, Germany, 297-300, 1993.
- [3] PETTY, E.: *Vision 2020-Education in the next millennium*, SEFI Annual Conference, Ireland, 1999, <http://www.ul.ie/2020/ewp.html>.
- [4] LEE, S.: *Computer Aided Instruction in Engineering Mechanics*, The University of Western Australia, 1999.
- [5] OSTROGONAC-Š, R.: *Broadening the engineering graphics curriculum at the University of Western Aus-*

- tralia*, 10th Australasian Conference on Engineering Education, Gladstone, 1998.
- [6] PLETENAC, L.: *Geometric CAD modelling in education*, Proceedings, Special SEFI European Seminar on Engineering Education, Bratislava, 1997.
- [7] LEDIĆ, J., AND OTHERS: *The quality of teaching in higher education: the croatian case*, University of Rijeka, School of Education, HR-51000 Rijeka, Croatia
- [8] VELICHOVÁ, D.: *Geometria s podporu pocitacov*, Seminar on computational geometry, STU Bratislava, 1995.
- [9] JEŽEK, F., VELICHOVÁ D.: *Variational geometry and invariance*, Proceedings, Special SEFI European Seminar on Engineering Education, Bratislava, 1997.

ANA SLIEPČEVIĆ

## Iz prizemlja više geometrije

*“Ako je istina, a istina jest, da je matematika kraljica znanosti, onda je isto tako istina da je geometrija njezina princeza.”*

Tako se nekad govorilo.

A kako je danas? Matematika se možda još uvijek drži kao kraljica, a kakva nam je princeza? Moglo bi se reći, da u zadnje vrijeme “princeza” i nije jako privlačna. Nekako je hladna i nemaštovita, formuličava i tabličava, anemična i neslikovita. Današnja geometrija, koja geometrizira formulom, jednadžbom i tablicom umjesto crtežom, kojoj su oduzeli sav šmuk i šarm kojim je vjekovima napajala maštu znanstvenika, danas i nije nešto osobita. Osiromašili su je, izolirali, zanemarili oni neki “veliki” koji su izrastali na njezinim temeljima. Treba joj pomoći, treba ju uljepšati i učiniti toplijom. Ta stara dama bi mogla doživjeti renesansu, ako se u njoj ponovo probudi crtež - konstrukcija, ako ju se “obuče” u obnovljeno geometrijsko ruho, koje danas može biti, zavaljujući kompjutorima, još puno bogatije i ljepše nego nekad. Netko mora pokrenuti, pridignuti i obnoviti tu nekadašnju ljepoticu dok još nije potpuno zaboravljena. A tko će to učiniti, ako ne mi, njezini malobrojni obožavatelji i štovatelji? A za to je potrebno samo malo snage i puno volje i ljubavi.

Potrebno je zaviriti u veliku matematičku zgradu koja je pradavno sagradjena na čvrstim geometrijskim temeljima. Ova je zgrada vjekovima podnosila više ili manje uspješne nadogradnje, pa danas ima svojih visokih etaža do kojih ne

može doprijeti baš svatko. Matematički zanesenjaci često su, težeći visokoj znanosti i apstrahirajući pri tome ono prizemno i razumljivo, stvorili podosta toga nerazumljivoga, nedostižnoga, apstraktnoga i većini nezanimljivoga, što je matematiku učinilo tako odbojnom i sivom. No, ne treba zaboraviti da u našoj zgradi postoji i prizemlje, koje izrasta iz samih temelja, gdje obitava i naša stara, gotovo zaboravljena dama.

Zasigurno bi mnogima bilo zanimljivo zaviriti u to prizemlje. Bilo bi tu za svakoga po nešto. Mnoge su vrednote ovdje patinirale, pokrila ih je debela prašina, ali nisu izgubile vrijednost i nekadašnji sjaj. Zaslužile su da ih se izvuče iz zaborava, obriše prašina, pa će zablistati novim sjajem. Stare su to i poznate stvari, zapostavljene i gurnute u zaborav, ali sigurno još mnogima dopadljive, koje bi možda mogle pokrenuti nečiju geometrijsku maštu i potaknuti ga da objavi neku crticu iz ovog prizemlja. Ako se to dogodi, postignut je cilj autorice ovih redaka.

Ako ste skloni sličnim razmišljanjima, ako ponekad geometrizirate za svoju dušu, ako vam se čini da ste otkrili novu ili smatrate da možete osuvremeniti neku staru “nedodirljivu” konstrukciju, ako imate simpatičnu geometrijsku ideju, ne oklijevajte, javite se, sigurno ćete nekoga obradovati. Rezultat jednog takvog geometriziranja je i članak objavljen u ovom broju pod naslovom *Eine Anwendung der perspektive Kollineation*.

Prevela i obradila: JELENA BEBAN-BRKIĆ

## Brojevi što život znače

**Matematički jezik je izuzetna pojava, pomalo i tajnovita, koja nas bez prestanka iznenađuje.**

Čitajući talijanski dnevni list *La Repubblica* zadržala sam se na rubrici *Cultura & Spettacoli* (Kultura i Zabava) kojoj se u toj novini daje puno prostora. Jedan me članak privukao svojim naslovom: *Numeri per non morire*. Malo slobodniji prijevod bi glasio kao i naš naslov. Zanimljiv je toliko da ga vrijedi izložiti u cijelosti:

“Razvoj znanosti se napokon udružio s vjerovanjem u djelotvornost matematike pri opisu *Univerzuma* u kojem živimo i prihvaćena je uloga matematike kao tajne koja se polako otkriva i otvara put našem shvaćanju i mogućnosti rukovanja snagama i strukturama *Prirode*.

Fizičari i ostali prirodnjaci oduvijek su bivali zatečeni mogućnošću upotrebe matematike pri opisivanju stvarnoga svijeta. Više puta se pokazalo da davno sročeni ezoterični tekstovi takozvane *čiste matematike* od strane matematičara koji su bili zainteresirani isključivo za harmoniju i unutarnju eleganciju matematike za sebe, postaju jezik kojim se na najprirodniji način opisuje neka nedavno otkrivena prirodna pojava.

Ono što se naučilo o sveprisutnosti i koristi matematike jest da se *Priroda* izražava i matematičkim jezikom. To je jezik koji se u bitnim točkama razlikuje od jezika kao što je na primjer engleski. Napravi li se gramatička pogreška, engleski jezik ne postaje neupotrebljiv, komunikacija je i dalje moguća. Međutim, prekinu li se logička pravila u jeziku matematike, sve se gubi. Možda je čak uputno, u prvim fazama učenja, predavati engleski na manje strog način, zažmirivši na ortografske greške i gramatičke nedostatke. Ishod bi bio katastrofalan prihvatiti li se tako ležeran stav predavajući matematiku.

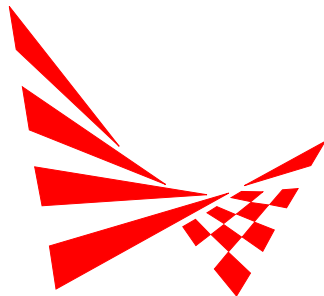
Često je pitanje: što proučavaju matematičari? Moglo bi se odgovoriti: u osnovi sve, ali baš sve moguće modele, sve odnose koji se mogu zamisliti među stvarima. Neke od tih stvari su prepoznatljive, dok su druge samo zamišljene. Matematičari proučavaju modele lišene spe-

cijalnog sadržaja. Zainteresirani su za općenitost koja se kasnije može po potrebi prilagoditi posebnim situacijama.

Matematički jezik se često izučava odvojeno od drugih vidova naše kulture. Mnogo se gubi zbog ne povezivanja s povijesti i antropologijom. Na primjer, simboli brojeva, koji se i danas rabe, prvi put su upotrijebljeni u Indiji da bi se kasnije uselili u Europu preko arapske civilizacije. Oni predstavljaju jedno od najvećih intelektualnih otkrića u povijesti ljudske rase. Puno su univerzalniji od slova feničkog alfabeta kojima je napisana ova rečenica. Oni su remek djelo ekonomično sakupljenih informacija. S 10 simbola lakih za pamćenje može se prikazati bilo koja veličina. Tajna je u vrijednosti položaja na kojem se nalazi pojedini simbol, što predstavlja izvor informacija. U jednom drukčijem sustavu, kao što su to na primjer rimski brojevi, položaj simbola nam ne govori ništa o vrijednosti koju predstavlja. Imamo neprikladan sustav u kojemu je čak i množenje dvaju brojeva, primjerice CVII i LXIII (107 i 63) vrlo složen zadatak.

Može se činiti čudnim, ali matematika može biti pitanje sačuvanja vlastitog života. U vrijeme ruske revolucije, fizičar i matematičar Igor Tamm bio je zaustavljen od grupe razbojnika na svom putu za jedno selo u okolici Odesse gdje se namjeravao snabdjeti hranom. Zazirući od njegovog građanskog izgleda, zarobili su ga i odveli vođi. Na pitanje čime se u životu bavi odgovorio je: matematikom. Njihov vođa, sumnjičav, rekao mu je da ispiše pogrešku pri aproksimaciji funkcije Taylorovim polinomom. Ako uspije, oslobodit će ga, u protivnom će ga ubiti. Tamm je kleknuo i polako, u prašini drhtavim rukama ispisao traženo. Kad je dovršio, vođa je pogledao i dao mu znak da je slobodan. 1958. godine Igor Tamm je dobio Nobelovu nagradu za fiziku, ali nije nikad uspio otkriti identitet zagonetnog ispitivača. Ipak, kaže da je našao način da zadrži pažnju svojih studenata na praktičnoj važnosti matematike.”

# Vijesti, izvješća i najave



## GODIŠNJA SKUPŠTINA HDKGIKG-A

U Opatiji je, u hotelu *Imperijal*, 26. rujna 2000. održana godišnja skupština HDKGIKG-a.

U uobičajenom Dnevnom redu valja istaknuti iscrpan izvještaj predsjednice Vlaste Ščurić o aktivnostima Društva od samog njegovog osnutka. Uz sve probleme koji su bili prisutni pri osnivanju i registraciji Društva na samom početku, istaknute su zasluge i upornost pojedinaca. *“Da nije Ivanka Babić na početku bila tako uporna, da joj nisu pomagale Ana Sliepčević i Vlasta Szirovicza, te da Sonja Gorjanc nije prihvatila dužnost urednice časopisa KoG, naglašava predsjednica, zasigurno nas ne bi bilo tu gdje smo danas. Ta čvrsta četvorka s Građevinskog fakulteta pokreće rad Društva.”*

Od djelatnosti Društva, ističe se redovito održavanje znanstveno-stručnih kolokvija svake godine u rujnu, plenarnih sastanaka u veljači, te stručnih skupova u lipnju. Prvi je Znanstveno-stručni kolokvij održan 1995, a već 1997. godine dobiva međunarodni karakter. Naši su gosti bili eminentni europski geometričari: prof. dr. Emil Molnár iz Budimpešte (1998.),

prof. dr. Hellmuth Stachel iz Beča (1999.), te prof. dr. Gunter Weiss iz Dresdena (2000.).

Mora se istaknuti neobično velika zasluga kolegice S. Gorjanc za izlazak prvog broja časopisa KoG, ali isto tako ostaje otvoreno pitanje da li bi bilo i ostalih brojeva bez njezinog udjela. Za vrlo dobar grafički dizajn zaslužan je M. Ambruš-Kiš, poznati kolumnist časopisa *Vidi*. Brojne pohvale koje stižu iz zemlje i inozemstva ukazuju da je časopis napravljen vrlo profesionalno u sadržajnom i estetskom smislu.

Ako se svemu doda činjenica da su članice Društva autorice dvaju sveučilišnih i dvaju srednjoškolskih udžbenika iz Nacrtna geometrije, da su aktivno sudjelovale na mnogim domaćim i inozemnim znanstvenostručnim skupovima, te da uspješno iniciraju podmlađivanje članstva, mora se ostati zadovoljan. Kao visokostručna udruga za područje geometrije, imali smo i neke manje lijepe aktivnosti. Zatražili smo zabranu upotrebe izričito lošeg sveučilišnog udžbenika izdanog na FESB-u u Splitu, te reagirali na neregularno raspisan natječaj za predmet Nacrtna geometrija na istom fakultetu s polučinjenim uspjehom. Na kraju se samo može konstatirati da je osnivanjem udruge učinjen pravi potez, da je Društvo danas preraslo u respektabilnu udrugu, koju čine ljudi kompetentni na

svom području, te da je bilo pravo zadovoljstvo predsjednikovati tijekom proteklih šest godina postojanja.

Izvješće tajnice odnosilo se na financijske probleme Društva. Financiranje Društva provodi se iz proračuna Ministarstva znanosti i tehnologije (potpora za časopis KOG, znanstveni skup, djelatnost udruge), članarina i sponzorstava. Godišnji prihodi i rashodi su gotovo isti, pa poslujemo pozitivno.

Slijedila su još izvješća Izdavačkog savjeta i Nadzornog odbora koja su, kao i prethodna, jednoglasno prihvaćena.

Na kraju su razriješeni dužnosti članovi u tijelima upravljanja, te su izabrani novi kako slijedi. Novi članovi *Upravnog odbora* su: I. Babić, Z. Božikov, G. Đarmati-Pavlič, J. Kos-Modor, D. Perše, L. Pletenac, A. Sliepčević, N. Sudeta, V. Ščurić; *Nadzornog odbora*: J. Beban-Brkić, B. Hajsig, K. Horvatić-Baldasar; *Izdavačkog savjeta*: S. Gorjanc, M. Lapaine, V. Szirovicza, a uredništvo časopisa KoG ostalo je nepromijenjeno.

Nakon Godišnje skupštine održan je sastanak Upravnog odbora u novom sastavu, na kojemu je za novu predsjednicu Društva izabrana Ana Sliepčević, a za dopredsjednicu Lidija Pletenac. Tajnica Društva i dalje je Ivanka Babić, a njezina zamjenica Nikoleta Sudeta.

**Ivanka Babić**



Prof. Vlasta Ščurić, predsjednica HDKGIKG-a od 1994.-2000.



## 6. ZNANSTVENO-STRUČNI KOLOKVIJ HDKGIK-A OPATIJA, 25-26. RUJNA 2000.

U Opatiji je 25. i 26. rujna 2000. održan 6. znanstveno-stručni kolokvij Hrvatskog društva za konstruktivnu geometriju i kompjutorsku grafiku. Sponzori skupa bili su Građevinski fakultet u Zagrebu, Građevinski fakultet u Rijeci, te poduzeće za grafičke usluge SAND d.o.o. iz Zagreba. Uz dvadesetak domaćih sudionika skupu su prisustvovali sljedeći gosti: prof. dr. Gunter Weiss iz Dresdenu, prof. dr. Alojz Muhić i dipl. ing. Domen Kušar iz Ljubljane, te mr. sc. Karmela Miletić iz Mostara. Održana su sljedeća znanstvena i stručna predavanja:

*M. Lapaine, N. Frančula:*

Rješenje problema o točki unutar poligona

*Z. Čerin:*

Zajednička karakterizacija osam centralnih točaka trokuta

*S. Gorjanc:*

Klasifikacija nožišnih ploha (1,2) kongruencija

*G. Weiss:*

The Rise and Decline of Visualisation Abilites

*A. Sliepčević:*

Kružnice višeg reda u hiperboličnoj ravnini

*K. Potaček, J. Pavlek:*

Primjena računala u prirodnim i tehničkim znanostima (prikaz fraktala)

*A. Muhić:*

Kružne kompozicije u antičkoj arhitekturi

*J. Beban-Brkić:*

Utjecaj metričke klasifikacije konika u  $I_2$  na analizu kvadraka u  $I_3^{(2)}$

*K Jurasić:*

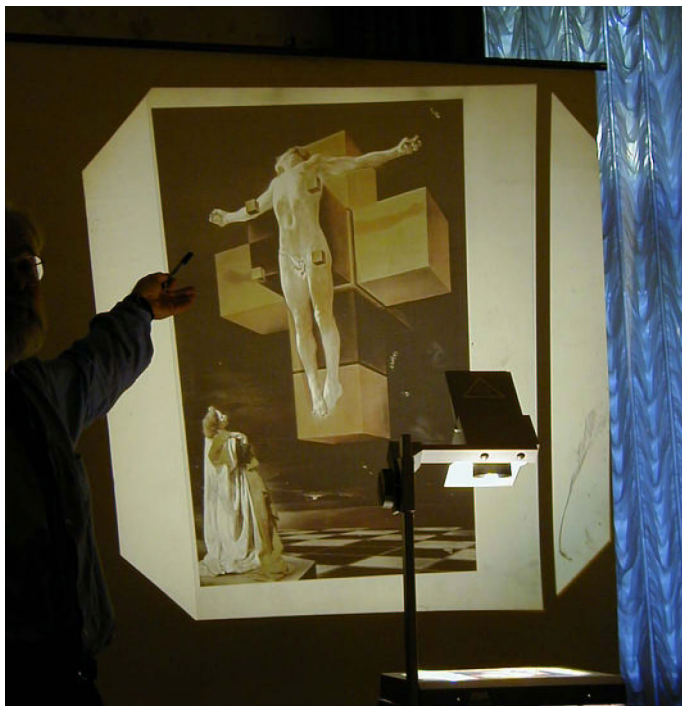
Krivulje i njima pridružene krivulje pomoću programa *Mathematica*

*B. Hajsig:*

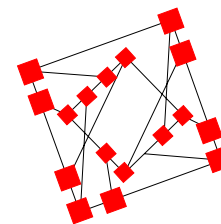
O testu percepcije sa razredbenog postupka na Arhitektonskom fakultetu u Zagrebu

Od spomenutih gostiju posebno treba istaknuti prof. dr. Guntera Weissa poznatog europskog geometričara, bečkog profesora, a sada direktora Tehničkog fakulteta u Dresdenu, koji je nositelj većeg broja geometrijskih kolegija s posebnim naglaskom na primjenu geometrije u arhitekturi, medicini, mehanici, biologiji. ... Kao vrhunski znanstvenik voditelj je mnogih doktoranata, savjetnik i suradnik znanstvenicima iz mnogih zemalja s posebnim naglaskom na istočnoeuropske zemlje. Predavač je po pozivu u mnogim sveučilišnim centrima diljem Europe. Predsjednik je europske sekcije Internacionalnog društva za geometriju i grafiku ISGG.

**Ana Sliepčević**



Sa izlaganja prof. Guntera Weissa u hotelu *Imperijal* u Opatiji.



## O RADU INŽENJERSKE SEKCIJE HMD-A U AKAD. GOD. 2000/01

Cilj Hrvatskoga matematičkog društva (HMD) je unapređivanje i promicanje matematičke znanosti, nastave matematike na svim razinama, primjene matematike u drugim disciplinama te unapređivanje društvenog položaja matematičara i matematike u cjelini.

Inženjerska se sekcija HMD-a bavi unapređivanjem i promicanjem dostignuća o primjenama matematike u drugim područjima i informiranje o njima.



Prvi sastanak Inženjerske sekcije HMD-a, od kada je njen voditelj M. Lapaine, održan je u lipnju 1999. Rad sekcije nastavio se na jesen te godine i od tada se sastanci održavaju neprekidno u pravilu svakog drugog četvrtka u mjesecu u vijećnici AGG fakulteta u Kačićevoj 26 u Zagrebu. Osim obavijesti i dogovora o daljnjem radu, na programu je uvijek i jedno predavanje. Akademске godine 2000/01 održana su sljedeća predavanja:

**12. 10. 2000.**

*Akademik Vladimir Devidé:* Matematika u grafici Mauritsa Cornelisa Eschera

**9. 11. 2000.**

*Prof. dr. sc. Bogdan Zelenko:* O numeričkom rješavanju parcijalnih diferencijalnih jednadžbi

**14. 12. 2000.**

*Dr. sc. Krešimir Fresl:* Neke numeričke aproksimacije minimalnih ploha

**18. 1. 2001.**

*Prof. Mladen Pejaković:* Zlatni rez u likovnim umjetnostima

**8. 2. 2001.**

*Prof. dr. sc. Ivan Ivanšić:* Blanušin graf

**8. 3. 2001.**

*Prof. dr. sc. Zvonimir Šikić:* Matematika i muzika

**19. 4. 2001.**

*Mr. sc. Božica Hajsig i doc. Tonči Žarić:* O testu percepcije s razredbenog postupka na Arhitektonskom fakultetu

**10. 5. 2001.**

*Akademik Žarko Dadić:* Položaj matematike u školstvu u Hrvata u 18. stoljeću

**14. 6. 2001.**

Inženjerska sekcija HMD-a 1999-2001

Sažeci svih predavanja održanih akad. god. 1999/2000 skupljeni su i zajedno s kratkim životopisima predavača objavljeni u Knjizi sažetaka. Isto tako će se učiniti i s predavanjima iz akad. god. 2000/01.

Osim redovitih sastanaka s predavanjem, Inženjerska sekcija HMD-a organizirala je i dva tečaja. Prvi je bio tečaj programskog jezika za slog i prijelom teksta  $\text{\LaTeX}$ . Tečaj se odvijao jednom tjedno po 2 sata, a ukupno 12 sati, u studenom i prosincu 2000. god. u računaonici Građevinskog fakulteta u Zagrebu. Voditelj je bio dr.

sc. Krešimir Fresl, a sudjelovalo je 10 polaznika.

Nakon uspješno organiziranog tečaja  $\text{\LaTeX}$ -a, organiziran je tečaj crtanja krivulja i ploha u programu *Mathematica*. Tečaj se odvijao jednom tjedno po 2 sata, ukupno 10 sati, u ve-ljači i ožujku 2001. god., također u računaonici Građevinskog fakulteta u Zagrebu. Voditeljica je bila dr. sc. Sonja Gorjanc, a sudjelovalo je 15 polaznika.

Sudjelovanje u radu Inženjerske sekcije HMD-a omogućeno je svima, bez obzira bili članovi HMD-a ili ne. Nečlanovima HMD-a preporuča se da postanu članovi uplatom godišnje članarine od 120 kn na žiro račun 30102-678-79162. U iznos članarine uključeno je primanje jednog od časopisa koje društvo izdaje.

Ukoliko želite primati obavijesti o radu i sastancima Inženjerske sekcije, potrebno je voditelju dostaviti svoju adresu, po mogućnosti elektroničku (e-mail). O sastancima Inženjerske sekcije mogu se naći nove obavijesti na internetu, na adresi Hrvatskoga matematičkog društva <http://www.math.hr/hmd>.

voditelj Inženjerske sekcije HMD-a

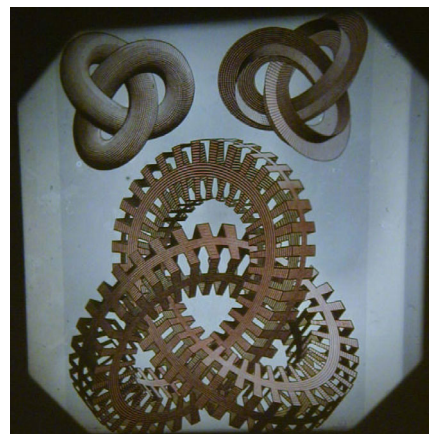
**Miljenko Lapaine**

tel.: 45 61 273, faks: 48 28 081

e-mail: [mlapaine@geof.hr](mailto:mlapaine@geof.hr)



Sa izlaganja akademika Vladimir Devidéa u vijećnici AGG-a.





## SYMPOSIUM

### DARSTELLENDEN

### GEOMETRIE

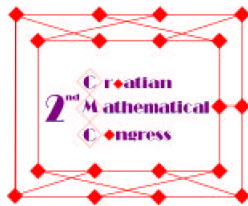
DRESDEN, 15-17. LIPNJA 2000.

U organizaciji Tehničkog univerziteta Dresden u Dresdenu je od 15-17. lipnja 2000. održano europsko zasjedanje stručnjaka iz područja Nacrtna geometrije. Skup je bio posvećen isključivo problematici obrazovanja iz područja nacrtna geometrije na tehničkim i umjetničkim studijima. Prisustvovalo je osamdesetak sveučilišnih nastavnika koji se bave podučavanjem geometrijskih predmeta, a koji su po obrazovanju većinom matematičari, ili manjim dijelom arhitekti. Iz 47 predavanja i prezentacija eminentnih stručnjaka iz Njemačke, Austrije, Mađarske, Češke, Poljske, Slovenije i Hrvatske moglo se zaključiti da problem kontinuiranog smanjenja broja sati nastave geometrije, što nerijetko dovodi i do ukidanja predmeta, ne muči samo nas u Hrvatskoj. Veliki sveučilišni centri, kao što su Berlin, Dresden, Beč... susreću se s lošim, ali na žalost sve češćim, nastojanjima "amerikanizacije" geometrije. Vrlo često je maćehinski odnos prema geometriji posljedica neadekvatne kompjutorizacije nastave geometrije što dovodi do sukoba geometričara-teoretičara i onih drugih koji smatraju da temelji geometrije više nisu potrebni u sveopćoj kompjutorizaciji, te tako polako ali sigurno pokopavaju ovu lijepu disciplinu. Nastavi li se ovaj trend i Europa će uskoro "ekshumirati geometriju" kao što se to može vidjeti na karikaturi u jednom američkom časopisu gdje Coxeter ekshumira geometriju.

Kroz velik broj predavanja provlačila se slična misao o spašavanju geometrije, a možda ju je najjasnije izrazio prof. Horst Sonderman iz Berlina čije mišljenje ovdje navodim:

*"Kompjutorska i digitalna izrada slika danas je izuzetno pogodna za predstavljanje i prezentiranje arhitektonskih projekata. Pogotovo stoga, što daje nove mogućnosti trodimenzionalnog modeliranja kompleksnih struktura i predočavanja nematerijalnih oblika kao što su osvjetljenja i sjene. Ali mi moramo insistirati na tome da bez jakih matematičkih znanja o euklidskom prostoru nije moguć suvereni rad s novom tehnologijom. Ova se znanja i danas mogu postići samo intenzivnim učenjem nacrtna geometrije..."*

Ana Sliepčević



## 2. HRVATSKI MATEMATIČKI KONGRES ZAGREB, 15-17. LIPNJA 2000.

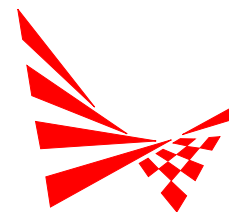
U Zagrebu je od 15. do 17. lipnja 2000., u organizaciji Hrvatkog matematičkog društva i Matematičkog odsjeka PMF-a, održan 2. hrvatski matematički kongres (2nd Croatian Congress of Mathematics).

U četiri sekcije, kroz plenarna, pozivna i petnaestminutna izlaganja, svoje je radove izložilo više od sto sudionika. Sažeci izlaganja objavljeni su u Zborniku koji se može naći i na Internet stranici [www.math.hr/congress/](http://www.math.hr/congress/).

U radu kongresa sudjelovalo je pet članova HDKGIKG-a: Jelena Beban-Brkić, Zdravka Božikov, Sonja Gor-

janc, Miljenko Lapaine i Lidija Pletenac. Jelena Beban-Brkić izložila je svoj rad *Isometric Invariants of Conics in Isotropic Plane - Classification of Conics*, a Sonja Gorjanc *Classification of Pedal Surfaces of (1,2) Congruences*.

Sonja Gorjanc



## DAN HDKGIKG-A ZAGREB, 25. LIPNJA 2001.

U okviru redovitog lipanjskog druženja članova HDKGIKG-a, osim uobičajenih dogovora i diskusija o problemima u struci održana su dva zanimljiva stručna izlaganja.

Sonja Gorjanc je demonstrirala kako se jednostavno može oblikovati web stranica u Wordu, te pokazala kako ona izgleda za geometrijske predmete na Građevinskom fakultetu u Zagrebu.

[www.grad.hr/nastava/geometrija](http://www.grad.hr/nastava/geometrija)

Pod naslovom "Geometrijsko 3D modeliranje u programu DesignCAD 3000" Lidija Pletenac prezentirala je mogućnosti rada u programu DesignCAD 3000 koji se, od svih do sada viđenih, čini najprikladnijim za našu nastavnu struku. Lidija Pletenac dala je i opširan opis svog rada sa studentima u onom dijelu nastave u kojem koristi računala. Uz puno uloženog truda i vremena ona je uspjela, bez obzira na prosječnu opremljenost računalima na Građevinskom fakultetu u Rijeci, svakog svojeg studenta dovesti do situacije da samostalno radi barem jednu vježbu na računalu.

## PrimMath [2001]

ZAGREB, 27-28. RUJNA 2001.

U Zagrebu će se od 27. do 28. rujna 2001. godine na Fakultetu elektrotehnike i računarstva održati 1. znanstveno-stručni skup Programski sustav *Mathematica* u znanosti, tehnologiji i obrazovanju - PrimMath[2001].

Cilj je ovoga skupa okupiti znanstvenike i druge korisnike programskog sustava *Mathematica* te im omogućiti da prikažu, kritički sagledaju i objave postignute rezultate. Kroz pozivna predavanja sudionici skupa upoznat će se s trendovima primjene programskog sustava *Mathematica* u svijetu.

Područja kojima će se skup baviti obuhvaćaju primjenu programskog sustava *Mathematica* u matematici, računarstvu, prirodnim i tehničkim znanostima, biotehnici i medicini, ekonomiji, humanističkim znanostima, obrazovanju, te umjetnosti i dizajnu.

Detaljne informacije mogu se naći na Internet adresi <http://mrc.math.hr/PrimMath/>.

Sonja Gorjanc



## Javni pozivi MTZ-a

Vlada Republike Hrvatske prihvatila je 5. travnja 2001. godine *Program hrvatskog inovacijskog tehnološkog razvika HITRA*, te donijela *Smjernice za provedbu Programa hrvatskog inovacijskog tehnološkog razvika uključivanjem domaćeg znanstvenoistraživačkog potencijala*, kao i *Uredbu o postupku provedbe programa Razvoj na znanju utemeljenih poduzeća, te uvjetima i načinu financiranja programa* (NN 33/2001).

*Program hrvatskog inovacijskog tehnološkog razvika (HITRA)* slijedi moderna razvojna razmatranja prema kojima je, uvjetno govoreći, industrijska era završena, te se nalazimo na pragu gospodarstva temeljenog na znanju, tj. razvoju i istraživanju kao glavnim proizvodnim resursima. Temeljna pretpostavka gospodarstva znanja je uključivanje znanstvenoistraživačkih potencijala u gospodarski razvoj (suradnja znanosti i industrije, gospodarstva) kao i podupiranje strateških istraživanja i transfernih znanosti kojima se stvaraju novi generički proizvodi, te nove gospodarske grane i sektori.

Program *Razvoj na znanju utemeljenih poduzeća* namijenjen je svim poduzetnicima, istraživačima i znans-

tvenicima te malim ili srednjim na znanju utemeljenim poduzećima koji nude proizvod ili uslugu čija se tržišna vrijednost temelji na vlastitom istraživanju i razvoju, tehnološki je napredan i razvija se za prepoznato tržište. Program pruža povoljno financiranje projekata, stručnu potporu u svim fazama razvoja projekta uključujući suradnju sa znanstvenoistraživačkim i razvojnim institucijama, razvoj postojećih i novih tržišta, razvoj tehnologija i poslovnih partnerstava.

Temeljem Programa HITRA Ministarstvo znanosti i tehnologije Republike Hrvatske objavilo je dva **javna poziva**. Jedan poziv je za podnošenje prijedloga **tehnoloških istraživačko-razvojnih** projekata, a drugi za prijavljivanje **poduzetničkih** projekata u sklopu programa "Razvoj na znanju utemeljenih poduzeća".

Navedeni dokumenti dostupni su na Internet stranici Ministarstva znanosti i tehnologije.

[www.mzt.hr/tehnoprojekti/](http://www.mzt.hr/tehnoprojekti/)

[www.mzt.hr/tehnopoduzeca/](http://www.mzt.hr/tehnopoduzeca/)

Prijave se primaju do 15. srpnja 2001. godine pa valja požuriti.



## Kako nabaviti KoG?

KoG je najjednostavnije nabaviti u uredništvu časopisa:

Vlasta Szirovicza  
(E-mail: [szvlasta@juraj.gradnz.grad.hr](mailto:szvlasta@juraj.gradnz.grad.hr))

Građevinski fakultet  
Av. V. Holjevca 15, 10 010 Zagreb  
Tel: 01 66 70 509  
Fax: 01 66 00 642

Za Hrvatsku je cijena primjerka 60 KN + 10 KN za poštarinu.

Nakon uplate za:  
HDKGIKG (za KoG), Kačićeva 26, 10000 Zagreb  
žiro račun broj 30105-678-94594  
poslati ćemo časopis na Vašu adresu.

Ako Vas zanima tematika časopisa i rad našega društva, preporučamo Vam da postanete članom HDKGIKG (godišnja članarina iznosi 100 KN). Za članove društva časopis je besplatan.

## How to get KoG?

The easiest way to get your copy of KoG is by contacting the editor's office:

Vlasta Szirovicza  
(E-mail: [szvlasta@juraj.gradnz.grad.hr](mailto:szvlasta@juraj.gradnz.grad.hr))  
Faculty of Civil Engineering  
Av. V. Holjevca 15, 10 010 Zagreb, Croatia  
Tel: (385 1) 66 70 509  
Fax: (385 1) 66 00 642

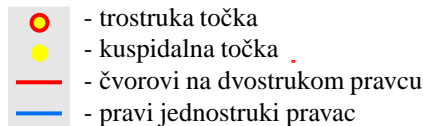
The price of the issue is \$12 + mailing expenses \$5 for European countries and \$10 for other parts of the world.

The amount is payable to the account of Croatian Society for Constructive Geometry and Computer Graphics HDKGIKG (for KoG), Kačićeva 26, 10000 Zagreb, Croatia  
No. 30101-620-16-2421809001, Zagrebačka banka

## OVITAK

10 primjera nožišnih ploha (1,2) kongruencija numeriranih prema klasifikaciji: S. Gorjanc, Klasifikacija nožišnih ploha (1,2) kongruencija, disertacija, PMF, Matematički odjel, Zagreb, 2000.

### Oznake



### Naslovna stranica

tip I<sub>1</sub> - 2 trostruke točke

tip I<sub>2</sub> - 1 trostruka točka

tip I<sub>3</sub> - 1 trostruka točka, 2 kuspidane

tip I<sub>4</sub> - 1 trostruka točka, 1 kuspidalna

### Posljednja stranica

tip I - nema realnih jednostrukih pravaca

tip II - 2 realna jednostruka pravca (1 pravi, 1 nepravi)

tip II - 3 realna jednostruka pravca (2 prava, 1 nepravi)

tip IV - 4 realna jednostruka pravca (2 prava, 2 neprava)

tip V - 5 realnih jednostrukih pravaca (3 prava, 2 neprava)

**Jednadžbe** pojedinih primjera u homogenim Kartezijevim koordinatama ( $x : y : z : w$ ).

tip I<sub>1</sub>

$$(x^2 + y^2 + z^2)(x^2 + 3y^2) - 2[(x^2 + y^2)(x - y) + (x^2 + 3y^2)z]w = 0$$

tip I<sub>2</sub>

$$(x^2 + y^2 + z^2)(x^2 + 0.5y^2) - 2(x^2 + y^2)(x - y)w = 0$$

tip I<sub>3,1</sub>

$$(x^2 + y^2 + z^2)(x^2 + y^2) + 2[(x^2 + y^2 - xz + yz)(x + 2y) - 4(x^2 + y^2)z]w + 2(x + 2y)(4x - 4y)w^2 = 0$$

tip I<sub>3,2</sub>

$$(x^2 + y^2 + z^2)(x^2 + y^2) - (x^2 + y^2 - xz - yz)(x - y)w = 0$$

tip I<sub>4</sub>

$$(x^2 + y^2 + z^2)(x^2 + y^2) - [(x^2 + y^2 - xz - yz)(x - y) + (x^2 + y^2)(x + y + z)]w = 0$$

tip I<sub>5</sub>

$$(x^2 + y^2 + z^2)(x^2 + 0.5y^2) + [(x^2 + y^2 - xz - yz)(x + y) - (x^2 + 0.5y^2)(x + y + 5z)]w + 4(x + y)^2w^2 = 0$$

tip II<sub>6,1</sub>

$$(x^2 + y^2 + z^2)x^2 - [x^2(x + y + 3.5z) + (x^2 + y^2)(x + 4y)]w + (x + 4y)(x + y)w^2 = 0$$

tip III<sub>5</sub>

$$(x^2 + y^2 + z^2)x^2 - [(x^2 + y^2)x + x^2(3x - z) + y^2z]w + (3x^2 - y^2)w^2 = 0$$

tip IV<sub>5</sub>

$$(x^2 + y^2 + z^2)(x^2 - y^2) - [(x^2 + y^2 - yz)(x + 3y) + (x^2 - y^2)(x + y + 3.5z)]w - (x + 3y)(-x + 2.5y)w^2 = 0$$

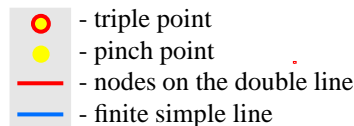
tip V<sub>6</sub>

$$(x^2 + y^2 + z^2)xy - (y^3 + 3x^2y + 3xy^2 + x^2z + 2xyz)w + (2x^2 + 2xy + 3y^2)w^2 = 0$$

## COVER PAGE

10 examples of the pedal surfaces of (1,2) congruences numerated according to the following classification: S. Gorjanc, The Classification of the Pedal Surfaces of (1,2) Congruences, dissertation, Department of Mathematics, University of Zagreb, 2000.

### Marks



### Front cover page

type I<sub>1</sub> - 2 triple points

type I<sub>2</sub> - 1 triple point

type I<sub>3</sub> - 1 triple point, 2 pinch points

type I<sub>4</sub> - 1 triple point, 1 pinch point

### Back cover page

type I - no real simple lines

type II - 2 real simple lines (1 finite, 1 infinite)

type II - 3 real simple lines (2 finite, 1 infinite)

type IV - 4 real simple lines (2 finite, 2 infinite)

type V - 5 real simple lines (3 finite, 2 infinite)

The **equations** of the examples in homogeneous Cartesian coordinates ( $x : y : z : w$ ).

type I<sub>1</sub>

$$(x^2 + y^2 + z^2)(x^2 + 3y^2) - 2[(x^2 + y^2)(x - y) + (x^2 + 3y^2)z]w = 0$$

type I<sub>2</sub>

$$(x^2 + y^2 + z^2)(x^2 + 0.5y^2) - 2(x^2 + y^2)(x - y)w = 0$$

type I<sub>3,1</sub>

$$(x^2 + y^2 + z^2)(x^2 + y^2) + 2[(x^2 + y^2 - xz + yz)(x + 2y) - 4(x^2 + y^2)z]w + 2(x + 2y)(4x - 4y)w^2 = 0$$

type I<sub>3,2</sub>

$$(x^2 + y^2 + z^2)(x^2 + y^2) - (x^2 + y^2 - xz - yz)(x - y)w = 0$$

type I<sub>4</sub>

$$(x^2 + y^2 + z^2)(x^2 + y^2) - [(x^2 + y^2 - xz - yz)(x - y) + (x^2 + y^2)(x + y + z)]w = 0$$

type I<sub>5</sub>

$$(x^2 + y^2 + z^2)(x^2 + 0.5y^2) + [(x^2 + y^2 - xz - yz)(x + y) - (x^2 + 0.5y^2)(x + y + 5z)]w + 4(x + y)^2w^2 = 0$$

type II<sub>6,1</sub>

$$(x^2 + y^2 + z^2)x^2 - [x^2(x + y + 3.5z) + (x^2 + y^2)(x + 4y)]w + (x + 4y)(x + y)w^2 = 0$$

type III<sub>5</sub>

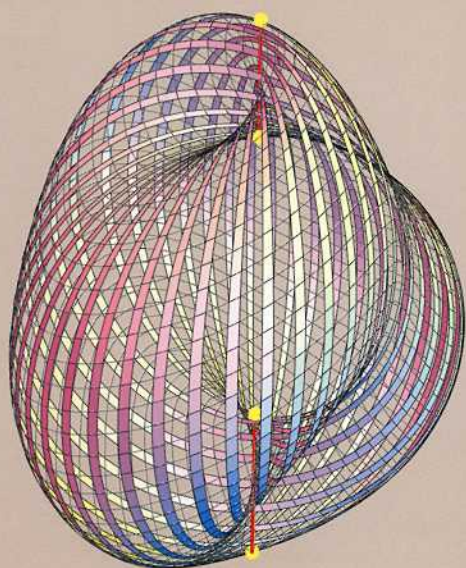
$$(x^2 + y^2 + z^2)x^2 - [(x^2 + y^2)x + x^2(3x - z) + y^2z]w + (3x^2 - y^2)w^2 = 0$$

type IV<sub>5</sub>

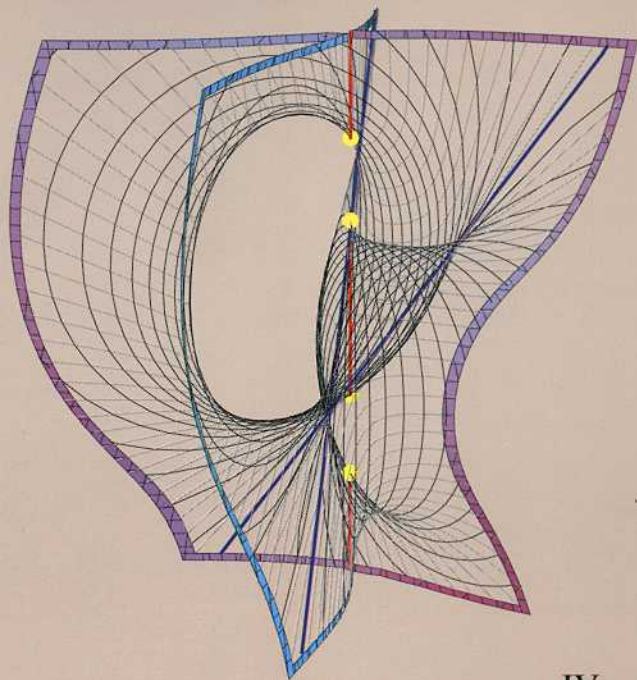
$$(x^2 + y^2 + z^2)(x^2 - y^2) - [(x^2 + y^2 - yz)(x + 3y) + (x^2 - y^2)(x + y + 3.5z)]w - (x + 3y)(-x + 2.5y)w^2 = 0$$

type V<sub>6</sub>

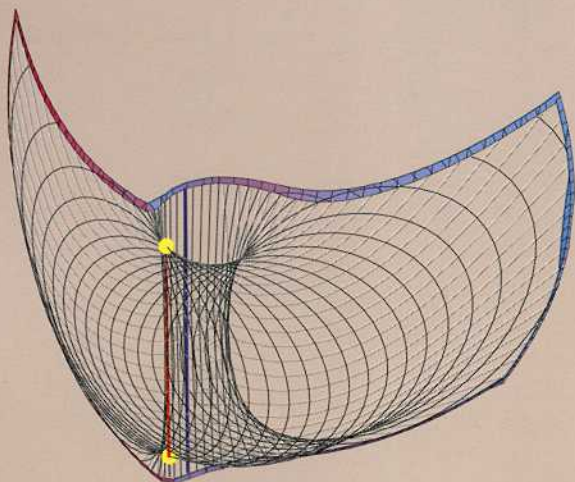
$$(x^2 + y^2 + z^2)xy - (y^3 + 3x^2y + 3xy^2 + x^2z + 2xyz)w + (2x^2 + 2xy + 3y^2)w^2 = 0$$



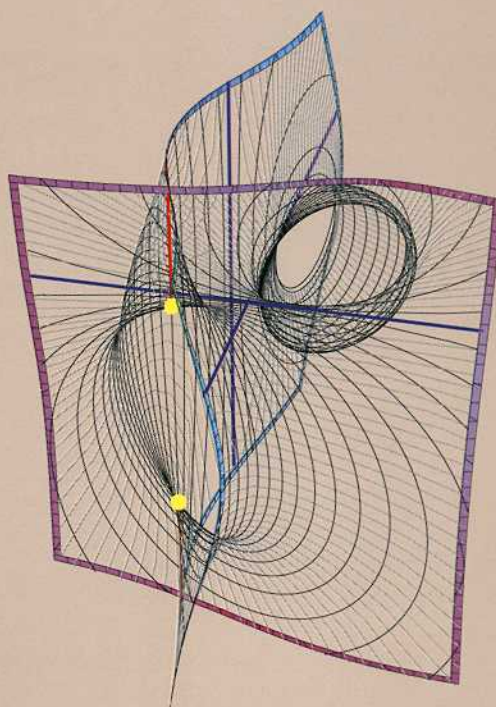
$I_5$



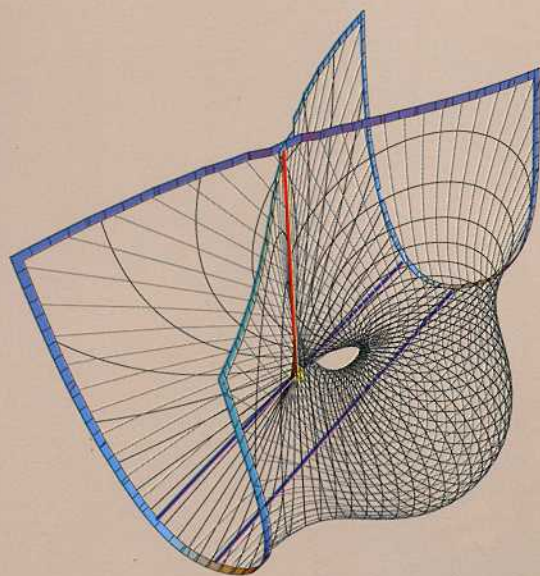
$IV_5$



$II_{6,1}$



$V_6$



$III_5$

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