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On Symmetric Designs with Parameters (101, 25, 6)

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ABSTRACT

There is known only one symmetric design with parameters (101, 25, 6) which has a Singer group (see [3]). Consequently, it is of interest to try to construct such a design without a Singer group.

Key words: symmetric design, automorphism, Frobenius group, orbit structure

O simetričnim dizajnima s parametrima (101, 25, 6)

SAŽETAK

Poznat je samo jedan simetrični dizajn s parametrima (101, 25, 6) koji ima Singerovu grupu (vidi [3]). U ovom radu pokušavamo konstruirati takav dizajn bez Singerove grupe.

Ključne riječi: simetrični dizajn, automorfizam, Frobeniusova grupa, orbitna struktura

The symmetric design with parameters (v, k, λ) is the finite incidence structure $\boldsymbol{D}$ which has v points and v lines (blocks) so that every line of $\boldsymbol{D}$ is incident with k points of $\boldsymbol{D}$, every point of $\boldsymbol{D}$ lies on k lines of $\boldsymbol{D}$, every two lines of $\boldsymbol{D}$ intersect in λ points of $\boldsymbol{D}$, every two points of $\boldsymbol{D}$ are incident with λ lines of $\boldsymbol{D}$. Much more about symmetric designs see [1].

In this paper we assume that the Frobenius group $\boldsymbol{E}_{25} \cdot \boldsymbol{Z}_3$ of order 75 acts on such a design in five orbits of lengths $1, 25, 25, 25, 25$. Thus, we assume that $\boldsymbol{Z}_3$ has five fixed points. However we prove the following:

Theorem.

There is no symmetric design with parameters (101, 25, 6) *acted upon by the Frobenius group* $\boldsymbol{G} = \boldsymbol{E}_{25} \cdot \boldsymbol{Z}_3$ *(a faithful extension of an elementary abelian group* $\boldsymbol{E}_{25}$ *of order 25 by a cyclic group* $\boldsymbol{Z}_3$ *of order 3) so that* $\boldsymbol{Z}_3$ *has exactly five fixed points.*

Proof. Let $\boldsymbol{D}$ be a symmetric design with parameters $(101, 25, 6)$ on which the Frobenius group $\boldsymbol{G} = \boldsymbol{E}_{25} \cdot \boldsymbol{Z}_3$ operates, where $\boldsymbol{G}$ is given (without loss of generality) by:

$$\boldsymbol{G} = \langle a, b, c / a^5 = 1,\ b^5 = 1,\ c^3 = 1,\\ aba^4b^4 = 1,\ c^2acb^4 = 1,\ c^2bcab = 1 \rangle.$$

For a reduction of a number of cases we will use the non-abelian group $\boldsymbol{G}_{16}$ of order 16, where

$$\boldsymbol{G}_{16} = \langle d, e / d^8 = 1, e^2 = 1, eded^3 = 1 \rangle,$$

which normalizes the Frobenius group $\boldsymbol{G} = \langle a, b, c \rangle$ so that the following relations

$$dcd^7c^2 = 1; \quad d^7adab^2 = 1; \quad (ec)^2 = 1; \quad eaea^4 = 1$$

are satisfied.

The normalizer $\boldsymbol{G}_{16}$ of the group $\boldsymbol{G}$ is counted in a full automorphism group $(Aut\,\boldsymbol{G})$ of $\boldsymbol{G}$.

We see that there is a unique orbit structure $\boldsymbol{M}$ for $\boldsymbol{E}_{25}$ (in the sense of [2]), which admits the action of $\boldsymbol{Z}_3$, i. e. where all coefficients are $\equiv 0$ or $1 \pmod 3$. We got it "easily" and we checked the result with the help of a computer. So we have:

$$\boldsymbol{M} = \begin{bmatrix} 0 & 25 & 0 & 0 & 0 \\ 1 & 6 & 6 & 6 & 6 \\ 0 & 6 & 9 & 6 & 4 \\ 0 & 6 & 6 & 4 & 9 \\ 0 & 6 & 4 & 9 & 6 \end{bmatrix}.$$

This orbit structure $\boldsymbol{M}$ has an automorphism (a symmetry) ξ of order 3, which permutes cyclically the last three columns and rows in $\boldsymbol{M}$. We also use this symmetry ξ for a reduction.

The complete group $\boldsymbol{G} = \boldsymbol{E}_{25} \cdot \boldsymbol{Z}_3$ has one fixed point, which we will denote with ∞, and the other point–orbits of length 25 will be denoted by $\mathbf{1}, \mathbf{2}, \mathbf{3}, \mathbf{4}$.

We shall denote the points of our design $\boldsymbol{D}$ with $\infty, \boldsymbol{I}_1, \boldsymbol{I}_2, \ldots, \boldsymbol{I}_{25}$, $\boldsymbol{I} \in \{\mathbf{1,2,3,4}\}$ and the automorphisms will be:

$$a = (\infty)(I_1,I_2,I_3,I_4,I_5)(I_6,I_{10},I_{11},I_{12},I_{13})(I_7,I_{16},I_{22},I_{25},I_{20})\\ (I_8,I_{17},I_{23},I_{24},I_{15})(I_9,I_{18},I_{19},I_{21},I_{14}),$$

$$b = (\infty)(I_1,I_6,I_7,I_8,I_9)(I_2,I_{10},I_{16},I_{17},I_{18})(I_3,I_{11},I_{22},I_{23},I_{19})\\ (I_4,I_{12},I_{25},I_{24},I_{21})(I_5,I_{13},I_{20},I_{15},I_{14}),$$

$$c = (\infty)(I_1)(I_2,I_6,I_{14})(I_3,I_7,I_{24})(I_4,I_8,I_{22})(I_5,I_9,I_{10})(I_{11},I_{13},I_{15})\\ (I_{12},I_{20},I_{25})(I_{16},I_{21},I_{18})(I_{17},I_{23},I_{19}), \text{ where } I \in \{1,2,3,4\}.$$

For a reduction we use the following collineations :

$d = (\infty)(I_1)(I_2,I_{15},I_3,I_{12},I_5,I_{16},I_4,I_{19})(I_6,I_{11},I_7,I_{20},I_9,I_{21},I_8,I_{17})$
$(I_{10},I_{18},I_{22},I_{23},I_{14},I_{13},I_{24},I_{25})$,
$e = (\infty)(I_1)(I_2)(I_3)(I_4)(I_5)(I_6,I_{14})(I_7,I_{24})(I_8,I_{22})(I_9,I_{10})(I_{11},I_{18})$
$(I_{12},I_{19})(I_{13},I_{21})(I_{15},I_{16})(I_{17},I_{25})(I_{20},I_{23})$,
where $I \in \{1,2,3,4\}$.

We got the automorphisms a, b, c and d, e in the explicit form with the help of Hrabe de Angelis's programme for "coset enumeration".

The block ℓ_0 is $G = \langle a, b, c\rangle$–invariant and is uniquelly determined:

$\ell_0 = \mathbf{1}_1\ \mathbf{1}_2\ \mathbf{1}_3\ \ldots\ \mathbf{1}_{23}\ \mathbf{1}_{24}\ \mathbf{1}_{25}$.

In the next construction we denote with $\ell_1, \ell_2, \ell_3, \ell_4$ the $\langle c\rangle$–invariant representantives of E_{25}–orbits of blocks. The block ℓ_1 contains the point ∞, and six points from each of the orbits $\mathbf{1}, \mathbf{2}, \mathbf{3}, \mathbf{4}$.

With the help of a computer we got the following 28 possibilities for the choice of the first six points of orbit $\mathbf{1}$:

$\ell_1 = \infty \ldots \{(2,6,14,3,7,24)^*, (2,6,14,4,8,22), (2,6,14,5,9,10)^*,$
$(2,6,14,11,13,15)^*, (2,6,14,12,20,25), (2,6,14,16,21,18),$
$(2,6,14,17,23,19), (3,7,24,4,8,22), (3,7,24,5,9,10),$
$(3,7,24,11,13,15), (3,7,24,12,20,25), (3,7,24,16,21,18),$
$(3,7,24,17,23,19), (4,8,22,5,9,10), (4,8,22,11,13,15),$
$(4,8,22,12,20,25), (4,8,22,16,21,18), (4,8,22,17,23,19),$
$(5,9,10,11,13,15), (5,9,10,12,20,25),$
$(5,9,10,16,21,18), (5,9,10,17,23,19),$
$(11,13,15,12,20,25), (11,13,15,16,21,18),$
$(11,13,15,17,23,19), (12,20,25,16,21,18),$
$(12,20,25,17,23,19), (16,21,18,17,23,19)\}$.

After the reduction with the help of group $G_{16} = \langle d, e\rangle$ only three possibilities remain (signed with *). On the orbits $\mathbf{2}, \mathbf{3}, \mathbf{4}$ the symmetry ξ is used for a reduction. Thus, with the help of a computer, we get 444 solutions for the block ℓ_1.

The block ℓ_2 has six points from each of the orbits $\mathbf{1}$ and $\mathbf{3}$, nine points from the orbit $\mathbf{2}$, and four points from the orbit $\mathbf{4}$. With the help of a computer we get 58 solutions for ℓ_2 which are compatible with ℓ_1.

The block ℓ_3 has six points from each of the orbits $\mathbf{1}$ and $\mathbf{2}$, four points from the orbit $\mathbf{3}$, and nine points from the orbit $\mathbf{4}$. Again with the help of a computer we see that there is no solution for ℓ_3, which is compatible with orbits containing 25 blocks, whose representantives are ℓ_1 and ℓ_2.

This proves our Theorem.

Remark. It remains to investigate the more complicated problem of a construction of this design with the help of the group $G = E_{25} \cdot Z_3$, where Z_3 has only two fixed points. In this case the group G acts on such a design in three orbits of lengths $1, 25, 75$. However, presently this cannot be done with a computer.

References

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