https://doi.org/10.31896/k.27.2
Original scientific paper
Accepted 12. 10. 2023.

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A Miquel-Steiner Transformation

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ABSTRACT

Each complete quadrilateral has three Miquel-Steiner points. Any triangle together with an arbitrarily chosen point not on a triangle side also defines a complete quadrilateral, and thus, this pivot point defines three Miquel-Steiner points. These three Miquel points form a triangle which is perspective with the base triangle. The mapping that assigns to the pivot point the uniquely defined perspector is a quadratic and not involutive Cremona transformation and shall be called Miquel-Steiner transformation. We shall study the action of the Miquel-Steiner transformation and its inverse.

Key words: Miquel points, quadrilateral, triangle, quadratic Cremona transformation

MSC2020: 14A25, 51N15

Miquel-Steinerova transformacija

SAŽETAK

Svaki potpuni četverokut ima tri Miquel-Steinerove točke. Bilo koji trokut zajedno s po volji odabranom točkom koja ne leži na stranicama trokuta također definira potpuni četverokut, pa stoga ova točka određuje tri Miquel-Steinerove točke. Te tri Miquelove točke tvore trokut koji je perspektivan polaznom trokutu. Preslikavanje koje točki pridružuje jedinstveno definirano središte perspektiviteta je kvadratna neinvolutivna Cremonina transformacija koju ćemo zvati Miquel-Steinerova transformacija. Proučavat ćemo djelovanje Miquel-Steinerove transformacije i njen inverz.

Ključne riječi: Miquelove točke, četverokut, trokut, kvadratna Cremonina transformacija

1 Introduction

There are several theorems in geometry that are ascribed to the French geometer Auguste Miquel (1816-1851). The most common of his results (originally published in [10]) may be the following (see Figure 1):

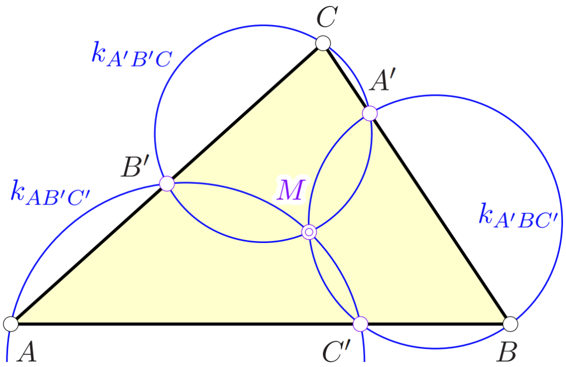

Figure 1: *Miquel's theorem as a theorem in triangle geometry.*

Let A, B, C be the vertices of a triangle and let A', B', C' be arbitrary points (different from A, B, C and not collinear) on the sides lines $[B,C]$, $[C,A]$, $[A,B]$. Then, the three circles $k_{AB'C'}$, $k_{A'BC'}$, $k_{A'B'C}$ have a common point M, the Miquel point. Here and in the following, k_{XYZ} denotes the circle on the (non-collinear) points X, Y, and Z. Sometimes, this theorem is also called the *Pivot Theorem* (see [5]).

There are other results on geometric configurations ascribed to Miquel:

(i) Miquel's *Five Circles Theorem* (cf. Figure 2, top) states that consecutive circumcircles of the spikes of a pentagonal star intersect in five concyclic points (see [3, pp. 151–152]).

(ii) Miquel's *Six Circles Theorem* (cf. Figure 2, bottom) states that if five circles meet four times in three points, then the remaining four common points are concyclic. This circle configuration can be viewed as an image of the stereographic projection of all circumcircles of the faces of a cube under a Möbius transformation. (cf. [1, 11]).

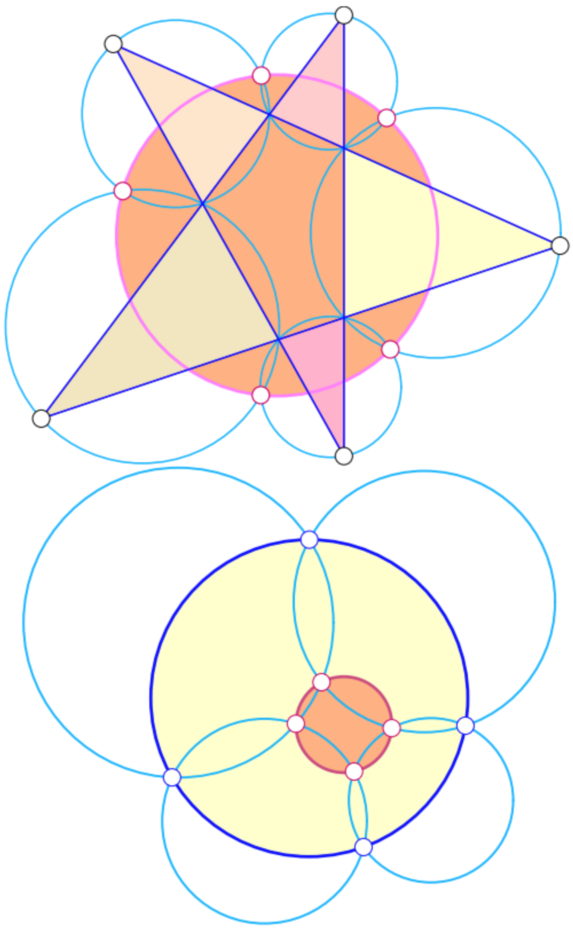

Figure 2: *Top:* MIQUEL*'s Five Circles Theorem. Bottom:* MIQUEL*'s Six Circles Theorem.*

In this article, we make use of the *Miquel-Steiner Quadrilateral Theorem*: We assume that $Q = ABCD$ is a quadrilateral, *i.e.*, no three of these points are collinear. The totality of the six lines $[A,B]$, ..., $[C,D]$ joining these points forms a complete quadrilateral. The points

$$\begin{aligned} D_1 &:= [A,B] \cap [C,D], \\ D_2 &:= [B,C] \cap [D,A], \\ D_3 &:= [C,A] \cap [B,D], \end{aligned}$$

are usually referred to as the diagonal points of Q. In the complete quadrilateral built on Q, we can find the following three quadruples of subtriangles

$$\begin{aligned} &ABD_3, \; CDD_3, \; ACD_1, \; BDD_1; \\ &ADD_1, \; BCD_1, \; ABD_2, \; CDD_2; \\ &ACD_2, \; BDD_2, \; ADD_3, \; BCD_3; \end{aligned}$$

each of which defining its own circumcircle. Then, the Miquel-Steiner Theorem reads:

Theorem 1 *The following quadruples of circumcircles of subtriangles in a complete quadrilateral share a single point:*

$$\begin{aligned} k_{ABD_3} \cap k_{CDD_3} \cap k_{ACD_1} \cap k_{BDD_1} &=: M_1, \\ k_{ADD_1} \cap k_{BCD_1} \cap k_{ABD_2} \cap k_{CDD_2} &=: M_2, \\ k_{ACD_2} \cap k_{BDD_2} \cap k_{ADD_3} \cap k_{BCD_3} &=: M_3. \end{aligned} \tag{1}$$

The *quadruple points*

$$M_1, \quad M_2, \quad M_3$$

are called the *Miquel-Steiner points* of Q, see [12, 13]. Figure 3 illustrates the contents of Thm. 1. (It is well-known, but nonetheless surprising that the four centers of the circles defining a Miquel point are concyclic, cf. [13].) As outlined in [12], the triangle $\Delta_M = M_1M_2M_3$ of Miquel points is perspective to the triangle $\Delta_D = D_1D_2D_3$ of diagonal points. Further, Δ_M is also perspective to $\Delta = ABC$ (see Figure 4). The perspector P of the triangles Δ and Δ_M shall be called *Miquel perspector*. Later, we shall replace the point D by an arbitrarily chosen point Z and consider the mapping $\mu : Z \mapsto P$ which shall be called the Miquel-Steiner transformation.

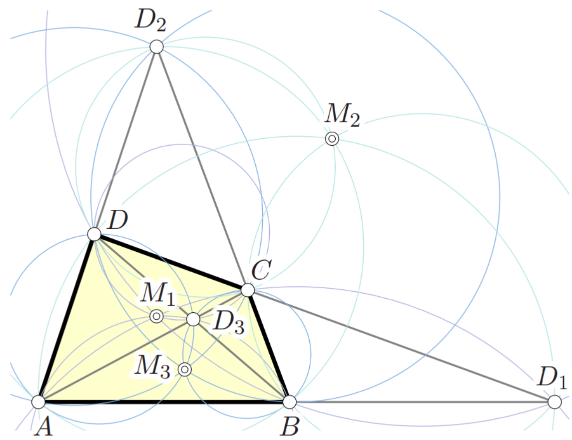

Figure 3: *The triple of Miquel points of a quadrilateral.*

In the following, we derive an analytical description of the Miquel-Steiner transformation μ. This will allow us to study its properties (cf. Section 2). Further, we derive the inverse which turns out to be different from the initial mapping. The Miquel-Steiner transformation is one of the rare examples of quadratic Cremona transformation that is not involutive as we shall see in Section 3. This is a good reason to have a closer look onto its properties and its action on objects which are occurring frequently in triangle geometry. In Section 3.3, we shall also investigate the six-parameter manifold of triangle cubics attached to the base triangle Δ which is left fixed as a whole under the Miquel-Steiner transformation. Besides that, we want to give a geometric meaning to at least some known triangle centers that show up in the inflationarily increasing *Encyclopedia of Triangle Centers* (cf. [9]).

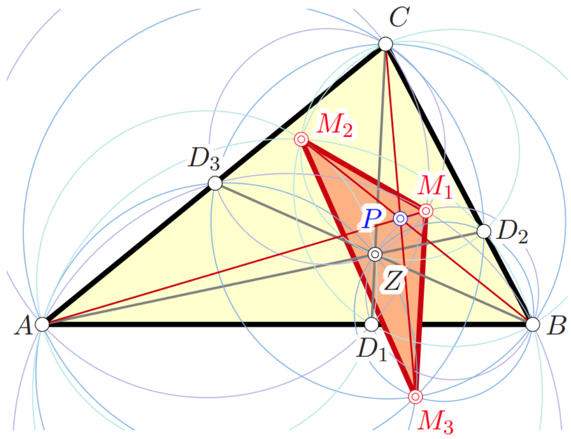

Figure 4: *Miquel points of a point Z with respect to a triangle $\Delta = ABC$ and the Miquel perspector P.*

2 A quadratic Miquel-Steiner transformation

Let us now assume that we are given a triangle $\Delta = ABC$ in the Euclidean plane. Any point Z that does not lie on a side line of Δ gives rise to a quadrilateral $Q = ABCZ$, *i.e.*, in comparison to Sec. 1, we have replaced D by Z, and the diagonal points are as defined above. Hence, the Miquel points are the quadruple points given in (1). Provided that Z is a triangle center (in the sense of [8, 9]), the Miquel perspector P is also a triangle center.

In order to study the mapping $\mu: \ Z \to P$, we shall derive an analytic description. For that purpose, we use homogeneous trilinear coordinates in the plane of Δ. The side lengths of Δ are

$$c := \overline{AB}, \quad a := \overline{BC}, \quad b := \overline{CA}.$$

We use the vertices of the triangle $\Delta = ABC$ as the base points of the projective frame and the incenter X_1 as the unit point. (Here and in the following, we use C. Kimberling's notation for triangle centers, cf. [8, 9]). Thus, we have

$$A = 1:0:0, \quad B = 0:1:0,$$
$$C = 0:0:1, \quad X_1 = 1:1:1.$$

With this coordinatization, the line at infinity (ideal line) ω is given by the homogeneous equation $a\xi + b\eta + c\zeta = 0$, or in terms of homogeneous trilinear line coordinates, as $a:b:c$.

We may assume that the fourth point Z has the homogeneous trilinear coordinates

$$\xi : \eta : \zeta \neq 0:0:0.$$

It is rather elementary to compute the three Miquel points M_1, M_2, M_3 as the intersections of the circumcircles mentioned in (1) and we find

$$\begin{aligned}
M_1 := {} & a(-a^2+b^2+c^2)\xi^2 - b(a^2-b^2)\xi\eta \\
& -abc\eta\zeta + c(c^2-a^2)\zeta\xi : \\
& : b(a\xi+b\eta)(a\xi+b\eta+c\zeta) : \\
& : c(c\zeta+a\xi)(a\xi+b\eta+c\zeta), \\
M_2 := {} & a(a\xi+b\eta)(a\xi+b\eta+c\zeta) : \\
& : a(a^2-b^2)\xi\eta + b(a^2-b^2+c^2)\eta^2 \\
& -abc\zeta\xi + c(c^2-b^2)\eta\zeta : \\
& : c(b\eta+c\zeta)(a\xi+b\eta+c\zeta), \\
M_3 := {} & a(c\zeta+a\xi)(a\xi+b\eta+c\zeta) : \\
& : b(b\eta+c\zeta)(a\xi+b\eta+c\zeta) : \\
& : c(a^2+b^2-c^2)\zeta^2 + a(a^2-c^2)\zeta\xi \\
& +b(b^2-c^2)\eta\zeta - abc\xi\eta.
\end{aligned}$$

With this it is easily verified that the triangles Δ and $\Delta_M = M_1M_2M_3$ are perspective. The Miquel perspector can be given in terms of trilinear coordinates

$$P = a(a\xi+b\eta)(a\xi+c\zeta) ::= \frac{1}{bc(b\eta+c\zeta)} ::, \qquad (2)$$

where the :: indicates that the subsequent coordinate functions are obtained by cyclically replacing all variables, *i.e.*, $a \to b \to c \to a$ and $\xi \to \eta \to \zeta \to \xi$.

The cyclic symmetry of the coordinate functions of the Miquel perspector indicates that the Miquel perspector assigned to a triangle center is also a triangle center (in the sense of C. Kimberling, see [8, 9]).

We can state:

Theorem 2 *The mapping $\mu: \ Z \mapsto P \notin \Delta_a$ that assigns to each point $Z = \xi:\eta:\zeta$ which does not lie on a side line of Δ's anticomplementary triangle Δ_a the Miquel perspector P as given in (2) is a quadratic Cremona transformation. The orthocenter X_4 of Δ is fixed under μ.*

Proof. The fact that μ from (2) is quadratic is obvious. We have to show that this mapping meets the requirements of a quadratic mapping to be invertible, *i.e.*, the (not necessarily regular) base conics defined by the three (homogeneous) quadratic coordinate functions (set equal to zero) share three points (cf. [4, 7]).

For that end, we look at the polynomial representation of μ given in (2) (in the middle). The two linear factors set equal to zero yield the equations of two straight lines: $a\xi + b\eta = 0$ is parallel to $[A,B]$ and passes through C, while $a\xi + c\zeta = 0$ is parallel to $[C,A]$ and passes through B. The latter lines meet in $A_a = -bc : ca : ab$. By virtue of the cyclic symmetry of μ's coordinate functions, we see that the exceptional set of μ consists of the lines

$$a\xi + b\eta = 0, \quad b\eta + c\zeta = 0, \quad c\zeta + a\xi = 0$$

which meet in the points

$$\begin{aligned} A_a &= -bc: \quad ca: \quad ab, \\ B_a &= \quad bc: -ca: \quad ab, \\ C_a &= \quad bc: \quad ca: -ab. \end{aligned}$$

The latter lines and points are the side lines and vertices of the anticomplementary triangle Δ_a of Δ. (Sometimes, Δ_a is called the *antimedial triangle*, see, *e.g.*, [6]).
The fact that $\mu(X_4) = X_4$ can easily be shown by inserting the orthocenters trilinear representation into (2). □

It is clear that no further point (different from X_4) can be fixed under μ. The Miquel-Steiner image of a point X can be found as the isogonal conjugate (with respect to Δ) of a collinear image of X (see Thm. 3). Under the isogonal conjugation ι, Δ's incenter X_1 is the only fixed point.
The base conics of the quadratic mapping μ are singular as is the case with the base conics in the isogonal and isotomic conjugation (cf. [7, 8]), and in the case of any inversion in a conic (see [7]).
According to Thm. 2, μ is a quadratic Cremona transformation, and as such, it is invertible. However, μ differs from the well-known quadratic Cremona transformations that occur frequently in triangle geometry. So, we state and prove:

Theorem 3 *The Miquel-Steiner transformation μ is not involutive. Its inverse is not defined on the side lines of Δ. The Miquel-Steiner transformation is the composition of the isogonal conjugation ι with respect to Δ and the central similarity α with Δ's centroid X_2 as the center and the scaling factor* 2, i.e., $\mu = \iota \circ \alpha$. *Δ's orthocenter is also fixed under μ^{-1}.*

Proof. The mapping μ is not involutive, since $\mu^2 \neq \mathrm{id}$ as can easily be verified.
By virtue of the right-hand side of (2), we set

$$\rho x = \frac{1}{bc(b\eta + c\zeta)}, \quad \rho y = \frac{1}{ca(c\zeta + a\xi)}, \quad \rho z = \frac{1}{ab(a\xi + b\eta)},$$

where $\rho \neq 0$ (is the complex but constant homogenizing factor). By applying the isogonal conjugation ι, we can rewrite the latter equations in the form

$$\begin{aligned} \rho^{-1}x^{-1} &= bc(b\eta + c\zeta), \\ \rho^{-1}y^{-1} &= ca(c\zeta + a\xi), \\ \rho^{-1}z^{-1} &= ab(a\xi + b\eta). \end{aligned}$$

Finally, we have to solve this system of three linear equations in the three unknowns ξ, η, ζ. By replacing x, y, z with ξ, η, ζ, we find

$$\mu^{-1}(\xi, \eta, \zeta) = bc(-a\eta\zeta + b\zeta\xi + c\xi\eta) :: . \quad (3)$$

The inverse of μ is not defined on the side lines of the base triangle. The coordinate functions of μ^{-1} describe three independent regular conics in the plane of Δ which share Δ's vertices.
The coordinate representation (2) of μ shows that μ can be considered as the composition of the isogonal transformation $\iota: (\xi, \eta, \zeta) \mapsto (\xi^{-1}, \eta^{-1}, \zeta^{-1})$ with respect to the base triangle Δ and a collineation α with the transformation matrix

$$\mathbf{T} = \begin{pmatrix} 0 & b^2c & bc^2 \\ a^2c & 0 & ac^2 \\ a^2b & ab^2 & 0 \end{pmatrix}.$$

The collineation α has Δ's centroid $X_2 = bc ::$ as fixed point and the ideal line $\omega = a ::$ of the projectively closed Euclidean plane of the initial triangle Δ is an axis of α. In order to show that α is a central similarity with the scaling factor -2, we compute the characteristic crossratio. For that end, we impose a projective frame on a fixed line (different from the axis, passing through the center X_2) and assign the coordinates $1:0$ to the center X_2 and $0:1$ to a generic point $Q \neq X_2$ and $Q \notin \omega$. We assume that the generic point Q has the homogeneous trilinear coordinates

$$m : n : o \neq 0 : 0 : 0$$

with respect to Δ. Then, the homogeneous coordinates of $\alpha(Q)$ and $U = [Q, \alpha(Q)] \cap \omega$ with respect to the frame on $[X_2, Q]$ are equal to

$$am + bn + co : -abc \text{ and } am + bn + co : -3abc.$$

Hence, we have

$$\mathrm{cr}(X_2, U, Q, \alpha(Q)) = -2.$$

The orthocenter of Δ is fixed under μ^{-1}. □

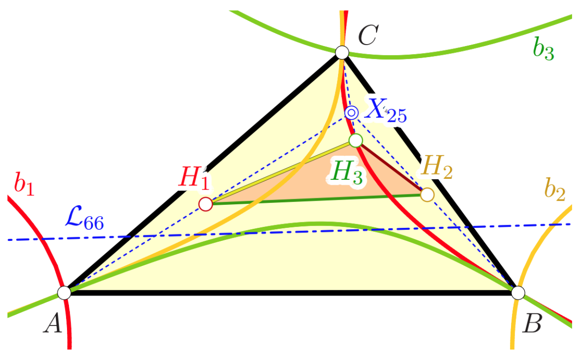

Figure 5: *The centers H_i of the three base conics b_i of μ^{-1} form a triangle perspective with Δ. X_{25} serves as the perspector, $\mathcal{L}_{66}$ is the perspectrix.*

Further, we can show what is illustrated in Figure 5:

Theorem 4 *The triangle of the centers of the three base conics of μ^{-1} is perspective with Δ. The perspector is the center triangle center X_{25} (of Δ).*

Proof. The centers H_1, H_2, H_3 of the conics given in (3) are found by multiplying the inverses of their coefficient matrices with a coordinate vector of the ideal line, *i.e.*, for example with (a,b,c). This yields

$$\begin{aligned} H_1 &= \sigma : 2b^2\cos C : 2c^2\cos B, \\ H_2 &= 2a^2\cos C : \sigma : 2c^2\cos A, \\ H_3 &= 2a^2\cos B : 2b^2\cos A : \sigma, \end{aligned}$$

where

$$\sigma := a^2+b^2+c^2$$

and

$$\cos A = \frac{b^2+c^2-a^2}{2bc} \quad \text{(cyclic)}$$

is the cosine of Δ's interior angle at A (cyclic). The perspector between Δ and $\Delta_H = H_1H_2H_3$ has the trilinear center function

$$a(-a^2+b^2+c^2)^{-1}$$

which belongs to the triangle center X_{25} in KIMBERLING's list (cf. [8, 9]). It is the homothetic center of the orthic triangle and the tangential triangle of Δ. □

The perspectrix p of Δ and Δ_H is the line carrying the triangle centers $X_{8673} \in \omega$ as well as the proper centers X_{2485}, X_{14396}, X_{52950}, *i.e.*, $p = [X_{2485}, X_{8673}] = \mathcal{L}_{66}$ (after canonical identification of line coordinates with point coordinates).

2.1 The square of μ

Since μ is not involutive, the square of the Miquel-Steiner transformation is a non-trivial and quartic Cremona transformation. It is obvious that μ^2 is a Cremona transformation, *i.e.*, it is invertible, since $(\mu^{-1})^2 \circ \mu^2 = \text{id}$. In terms of trilinear coordinates the *square* of μ reads

$$\mu^2(\xi,\eta,\zeta) = \big(bc(b\eta+c\zeta)(a(b^2+c^2)\xi+b^3\eta+c^3\zeta)\big)^{-1} :: .$$

The mapping μ^2 is not defined on the sides of the excentral triangle Δ_a of Δ and on the sides of further triangle Δ_f which is perspective with Δ. Here, X_4 (of Δ) serves as the perspector, while the perspectrix between Δ and Δ_f is the line with homogeneous coordinates $a^3 : b^3 : c^3$. The canonical identification of point and line coordinates assigns the perspectrix to the 3$^{\text{rd}}$ power point X_{32} (cf. [8, 9]).

3 Action of μ and μ^{-1}

Since μ is a quadratic mapping, it sends algebraic curves c of degree n to algebraic curves of degree $2n$. Degree reductions occur if c passes through a base point of the transformation. The same holds true for its inverse. In what follows, we shall have a look at the μ-images and μ^{-1}-images of some geometric objects related to the base triangle.
In order to increase the readability of equations, we shall write the coordinates ξ, η, ζ with bold characters.

3.1 Images of straight lines

We restrict ourselves to the μ-images and μ^{-1}-images of some very special lines related to a triangle. It is clear that images and pre-images of straight lines under the Miquel-Steiner transformations are conics in general, and straight lines only if the lines under consideration pass through at most one base point of the transformation.

3.1.1 Antiorthic axis

The antiorthic axis $\mathcal{L}_1 = 1:1:1$ is the harmonic conjugate of X_1 with respect to the base triangle Δ. Its μ-image is the central conic

$$\mu(\mathcal{L}_1) : \sum_{\text{cyclic}} c\,(bc+ca-ab)\,\xi\eta = 0$$

passing through the triangle centers X_i with

$$i \in \{100, 34071, 52923\}.$$

The center of the conic $\mu(\mathcal{L}_1)$ is the yet unnamed, and thus, unlabelled triangle center defined by the homogeneous trilinear center function

$$a(ab+ac-bc)\cdot \\ \cdot\big(a^3(b+c)-a^2bc-a(b+c)(b-c)^2-bc(b^2+c^2)\big).$$

The μ-pre-image of the antiorthic axis is again a conic and has the trilinear equation

$$\mu^{-1}(\mathcal{L}_1) : \sum_{\text{cyclic}} a^3\xi^2 + ab(a+b+c)\xi\eta = 0.$$

It is centered at the Gergonne point X_7 and houses the centers

$$i \in \{149, 4440, 20355, 20533, 21220, 21221, 30578, 37781\}.$$

Figure 6 shows a triangle with its antiorthic axis $\mathcal{L}_1$ the conics $\mu(\mathcal{L}_1)$ and $\mu^{-1}(\mathcal{L}_1)$.

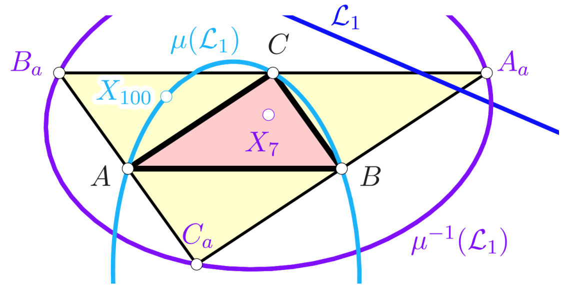

Figure 6: *Image and pre-image of the antiorthic axis $\mathcal{L}_1$.*

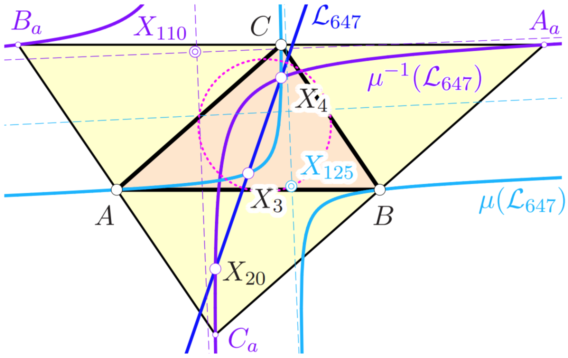

Figure 7: *The Euler line and its μ-image and μ^{-1}-image.*

3.1.2 Euler line

The μ-pre-image of the Euler line (cf. Figure 7)

$$\mathcal{L}_{647} = [X_2, X_3] = a(b^2-c^2)(a^2-b^2-c^2) ::$$

is the central conic with the trilinear equation

$$\mu^{-1}(\mathcal{L}_{647}) : \sum_{\text{cyclic}} a^4(b^2-c^2)(a^2-b^2-c^2)\xi^2 = \\ = 2\prod_{\text{cyclic}}(a^2-b^2)\sum_{\text{cyclic}} ab\xi\eta$$

centered at X_{110}, the focus of the Kiepert parabola.
The conic $\mu^{-1}(\mathcal{L}_{647})$ passes through the proper triangle centers X_i with the Kimberling indices

$$i \in \{4,20,69,146,193,2888,2889,2892,3868,3869, 5596,6193,6225,10340,11061,11271,11469, 12383,17220,18387,22647,32354,37889,39355\}$$

and carries also the centers X_{2574} and X_{2575} located on the line at infinity.
The μ-image of the Euler line is the conic

$$\mu(\mathcal{L}_{647}) : \sum_{\text{cyclic}} c(a^2-b^2)(a^2+b^2-c^2)\xi\eta = 0$$

centered at X_{125} which is the center of the Jeřabek hyperbola. The latter conic carries 272 known triangle center of which X_{2574} and X_{2575} are points at infinity while the proper points have the Kimberling indices

$$i \in \{3,4,6,54,64-74,248,265,290,695,879,895, 1173,1175-1177,1242-1246,1439,1798, 1903,1942,1987,2213,2435,3426,3431,3519, 3521,3527,3531,3532,3657,4846,5486,5504, 5505,5900,6145,6391,8044,8612,8795,8811, 8814,9399,9513,10097,10099,10100,10262, 10293,10378,10693,11270,11559,11564,11738, 11744,12023,13418,13452,13472,13603,13622, 13623,14220,14374,14375,14380,14457,14483, 14487,14490,14491,14498,14528,14542,14841, 14843,14861,15002,15077,15232,15316,15317, 15320,15321,15328,15453,15460,15461,15740, 15749,16000,16540,16620,16623,16665,16774, 16835,16867,17040,17505,17711,18123,18124, 18125,18296,18363,18368,18434,18532,18550, 19151,19222,20029,20421,21400,22334,22336, 22466,26861,28786-28788,30496,31366,31371, 32533,33565,34207,34221,34222,34259, 34435-34440,34483,34567,34800-34802,34817, 35364,35373,35512,35909,36214,37142,38005, 38006,38257,38260,38263,38264,38433,38436, 38439,38442,38443,38445,38447,38449,38534, 38535,38955,39372,39379,39665,39666,40048, 40441,41433,41435,41518,41519,42016,42021, 42059,42299,43689-43727,43834,43891,43908, 43918,43949,44207,44718,43892,44750,44835, 44836,45011,45088,45302,45733,45736,45788, 45835,45972,46765,46848,46851,47060,48362, 51223,51480,52222,52390,52391,52518,52559, 52560,52561,54124,54125\}.$$

The two conics $\mu(\mathcal{L}_{647})$ and $\mu^{-1}(\mathcal{L}_{647})$ are both passing through the circumcenter X_3 and the orthocenter X_4. Further, $\mu^{-1}(\mathcal{L}_{647})$ is a circumconic of Δ_a and contains the de Longchamp point X_{20} of Δ. Since X_{20} is at the same time the orthocenter of Δ_a, we can summarize and state:

Theorem 5 *The μ-image and the μ-pre-image of the Euler line are equilateral hyperbolae with the same ideal points (and hence, parallel asymptotes). The first is centered at X_{125}, the second is centered at X_{110}.*

3.1.3 Brocard axis

The Brocard axis $\mathcal{L}_{523} = [X_3, X_6]$ with trilinear coordinates $bc(b^2-c^2)$ :: is sent to the conic with the equation

$$\mu^{-1}(\mathcal{L}_{523}) : \sum_{\text{cyclic}} a^2(b^2-c^2)\xi^2 = 0$$

via the inverse of the Steiner-Miquel transformation. This conic is centered at X_{99} (Steiner point) and contains the triangle centers X_i with Kimberling indices

$$i \in \{1,2,20,63,147,194,487,488,616,617,627,628, 1670,1671,1764,2128,2582,2583,2896,\mathbf{3413},\mathbf{3414}, 6194,6462,6463,7616,8591,8782,9742,10336,11148, 13174,13678,13798,16552,16563,17147,18301,18596, 20371,21378,30562,30564,30579,33404,33405,33608, 33609,33610,33611,33612,33613,36857,41914,41923, 41930,44010,45029,46625,46717,46944,51860,51952, 51953,52025,52676,53856\},$$

where X_{3413} and X_{3414} are real points on the line at infinity.
Hence, $\mu^{-1}(\mathcal{L}_{523})$ is a hyperbola.
On the other hand, μ sends the Brocard axis to the central conic

$$\mu(\mathcal{L}_{523}) : \sum_{\text{cyclic}} c(a^2-b^2)\cdot\left(a^2(b^2+c^2)+c^2(b^2-c^2)\right)\xi\eta = 0$$

centered at the yet unnamed triangle center with the trilinear center function

$$a(b^2-c^2)^2(a^4-a^2b^2-a^2c^2-b^2c^2)\cdot \\ \cdot\big(a^6-2a^4(b^2+c^2)+a^2(b^4-b^2c^2+c^4)-b^2c^2(b^2+c^2)\big).$$

Finally, we shall note that the triangle centers X_i with Kimberling numbers

$$i \in \{54, 98, 251, 1078, 1179, 1342, 1343, 1629, 3453, \\ 5012, 5481, 10312, 11816, 38826, 39396, 42346\}$$

are located on the conic $\mu(\mathcal{L}_{523})$.

3.2 Images of conics

Again, the huge variety of conics makes it necessary to pick out some special representatives. It is clear that conics only map to conics if they are circumscribed to the triangle of base points, *i.e.*, the anticomplementary triangle Δ_a. The Miquel-Steiner transforms of circumconics of the initial triangle Δ are quartic curves in general.

3.2.1 Steiner circumellipse

Degeneracies of the image curves can only be expected if the circumconics of Δ touch the anticomplementary triangle. This happens escpecially in the following case:

Theorem 6 *The Miquel-Steiner pre-image of the Steiner circumellipse is the central line* $\mathcal{L}_{3051} = [X_{316}, X_{512}]$.

Proof. We insert (2) into the equation

$$s:\ bc\eta\zeta + ca\zeta\xi + ab\xi\eta = 0$$

of the Steiner ellipse and find

$$\mu^{-1}(s):\ \sum_{\text{cyclic}} a^3(b^2+c^2)\xi = 0.$$

Now, it is an elementary task to verify that $\mu^{-1}(s)$ is spanned by X_{316} (Droussent pivot) and $X_{512} \in \omega$. The canonical identification of the trilinear coordinates of $\mu^{-1}(s)$ with the coordinates results in a center with the trilinear center function $a^3(b^2+c^2)$ which is that of X_{3051} (cf. [8, 9]). □

Furthermore, the following triangle centers X_i with Kimberling indices

$$i \in \{316, 850, 3766, 3978, 11450, 14957, 14962, 17995, \\ 20022, 20295, 20352, 20556, 21282, 21301, 21302, \\ 21303, 33873, 44445, 47128, 52618, 53331, 53365, 54263\}$$

are located on $\mu^{-1}(s)$.

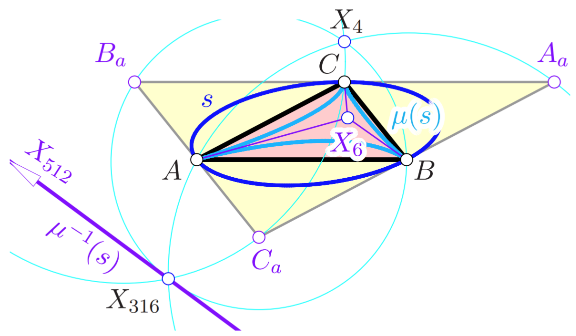

Figure 8: *Steiner-Miquel image and pre-image of the Steiner circumellipse.*

The μ-image of s is a quartic with three cusps at the vertices of Δ passing through the centers X_i with

$$i \in \{249, 1016, 1252, 1262, 2226, 6185, 10630, 23586, \\ 23592, 23964, 23984, 34536, 34537, 34538, 34539, 40384\}.$$

The tangents at the cusps concur in the Symmedian point $X_6 = a : b : c$. Figure 8 shows the Steiner-Miquel image and pre-image of the Steiner circumellipse s of Δ.

3.2.2 Circumcircle

The μ-pre-image of the circumcircle

$$u:\ a\eta\zeta + b\zeta\xi + c\xi\eta = 0$$

is the ideal line

$$\omega:\ a\xi + b\eta + c\zeta = 0.$$

The circumcircle is mapped under the Miquel-Steiner transformation to the quartic curve

$$\mu(u):\ \sum_{\text{cyclic}} a^2(-a^2+b^2+c^2)\eta^2\zeta^2 = \\ = 2abc\,\xi\eta\zeta(a\xi+b\eta+c\zeta) \tag{4}$$

housing the triangle centers X_i with

$$i \in \{59, 249, 250, 2065, 10419, 15378-15388, \\ 15395-15397, 15401-15407, 15460, \\ 15461, 41511, 44174\}.$$

The vertices of Δ are ordinary double points of $\mu(u)$. The tangents at the double points are the Cevians through the circumcenter X_3 and the Symmedian point X_6, cf. Figure 9. This can easily be verified by extracting the coefficients of ξ^2, η^2, and ζ^2 from (4) and showing that the resulting quadratic forms factor and split into two linear factors which (if set equal to zero) yield the equations of the tangents at the double points. For example, the coefficient of ξ^2 equals

$$(b\zeta - c\eta)\big(b(a^2-b^2+c^2)\zeta - c(a^2+b^2-c^2)\eta\big).$$

The first factor describes the Cevian through X_6, the second that through X_3.

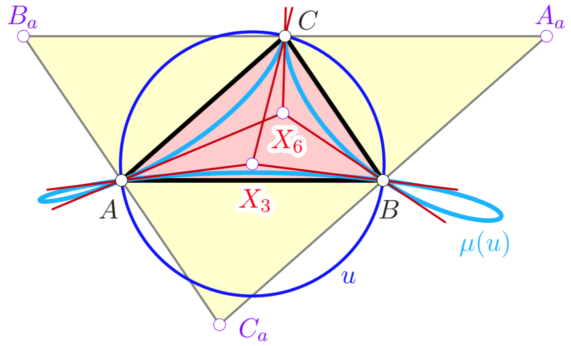

Figure 9: *The Miquel-Steiner transform of the circumcircle u is a quartic with three ordinary double points at the vertices of Δ. The tangents at the double points are the joins with the circumcenter X_3 and the Symmedian point X_6.*

By virtue of (4), it is clear that $\mu(u)$ degenerates if Δ is a right triangle. Let (for example) the right angle be at C. Then, $a^2+b^2=c^2$, the term $\xi^2\eta^2$ vanishes, and the right-hand side becomes

$$2a^2b^2\zeta^2(\xi^2+\eta^2).$$

Thus, the side $[A,B]$ (opposite to the vertex C) splits off from $\mu(u)$.
For an equilateral triangle Δ, *i.e.*, $a=b=c\neq 0$, the curve $\mu(u)$ becomes a Steiner hypocycloid.

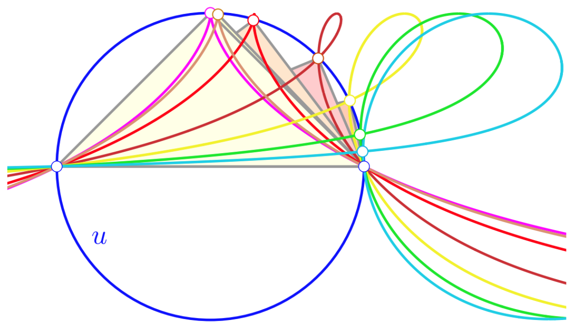

Figure 10: *A sequence of right triangles with ratios of cathetus's lengths 1:1, 20:21, 3:4, 5:12, 9:40, 19:180, 41:840 and the corresponding cubic curves as μ-images of the circumcircle u.*

3.2.3 Incircle

The μ-pre-image of the incircle

$$i:\sum_{\text{cyclic}} a^2(a-b-c)^2\xi^2=\sum_{\text{cyclic}} 2ab(a-b+c)(-a+b+c)\,\xi\eta$$

is the quartic curve

$$\begin{gathered}
\mu^{-1}(i):\sum_{\text{cyclic}} a^9bc(a+b+c)(a-b-c)^2\xi^4\\
-2a^5\big(b(b-c)a^6-(b^3-2b^2c-bc^2+c^3)a^5\\
-(2b^4-b^3c-2bc^3-c^4)a^4\\
+(2b^5-3b^4c-b^3c^2-7b^2c^3-bc^4+2c^5)a^3\\
+(b^6+2b^4c^2+2b^3c^3+3b^2c^4-2bc^5-2c^6)a^2\\
-(b-c)(b^6+2b^4c^2+b^3c^3+2b^2c^4-bc^5-c^6)a\\
c^3(b-c)^2(b+c)^3\big)\xi^3(b\eta+c\zeta)\\
-a^2b^2\big(b(4b-c)a^8-(b-c)(4b^2-bc-2c^2)a^7\\
-(8b^4-b^3c+6b^2c^2-3bc^3-2c^4)a^6\\
+(8b^5-9b^4c+7b^3c^2-5b^2c^3-bc^4+4c^5)a^5\\
+(4b^6-b^5c+10b^4c^2+2b^3c^3+8b^2c^4-5bc^5-4c^6)a^4\\
-(4b^7-5b^6c+15b^5c^2-6b^4c^3-5b^2c^5-bc^6+2c^7)a^3\\
c(b^5+2b^3c^2+10b^2c^3+7bc^4+2c^5)(b-c)^2a^2\\
-bc(b-c)^2(b-c)(b^4-2b^3c-2b^2c^2+c^4)a\\
+2b^3c^2(b+c)^2(b-c)^3\big)\xi^2\eta^2\\
-2a^2bc\big(b(3b-2c)a^8-(3b^3-5b^2c-2bc^2+3c^3)a^7\\
-(6b^4-2b^3c+2b^2c^2-5bc^3-3c^4)a^6\\
+(6b^5-8b^4c+2b^3c^2-18b^2c^3-2bc^4+6c^5)a^5\\
+(3b^6-b^5c+8b^4c^2+4b^3c^3+9b^2c^4-7bc^5-6c^6)a^4\\
-(3b^7-4b^6c+13b^5c^2-3b^4c^3-b^3c^4-6b^2c^5-bc^6-3c^7)a^3\\
c(b-c)^2(b^5+3b^3c^2+13b^2c^3+10bc^4+3c^5)a^2\\
-bc(b+c)(b-c)^2(b^4-2b^3c-2b^2c^2+c^4)a\\
+2b^3c^2(b+c)^2(b-c)^3\big)\xi^2\eta\zeta=0.
\end{gathered}$$

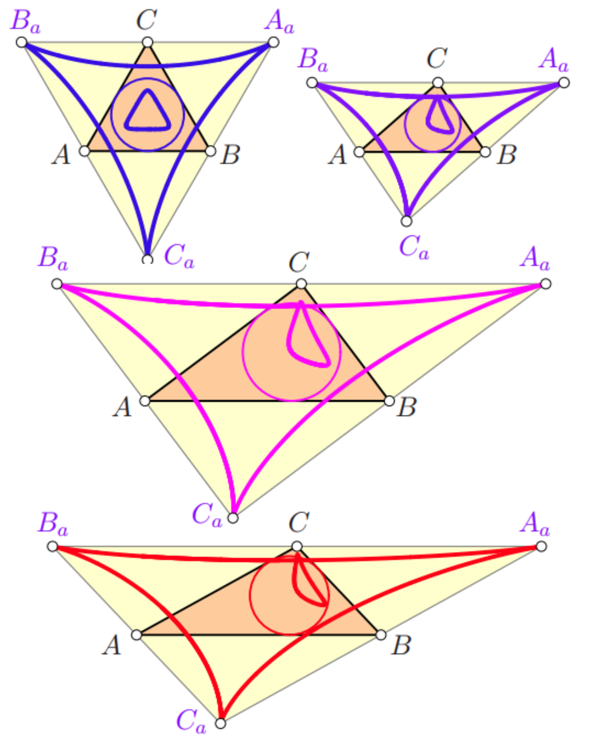

Figure 11: *Miquel-Steiner transforms and the (cusped) inverses of the incircle for an equilateral, an acute, a right, and an obtuse triangle.*

This quartic curve has three cusps at the vertices of the anticomplementary triangle Δ_a of Δ. Therefore, they map to

a conic that touches the three sides lines of the exceptional triangle of the mapping μ^{-1} (which is Δ). In the case of an equilateral triangle Δ (and thus also Δ_a), the curve $\mu^{-1}(i)$ is a Steiner hypocycloid (cf. Figure 11).
The μ-image of the incircle is the quartic

$$\begin{aligned}\mu(i):\ \sum_{\text{cyclic}} c^2\big((2a^6-4a^5(b-c)-3a^4(2b^2+bc-2c^2)\\ +a^3(8b^3-b^2c-bc^2-8c^3)\\ +a^2(6b^4-9b^3c+2b^2c^2+7bc^3+6c^4)\\ -a(4b^5-b^4c+b^3c^2+b^2c^3+7bc^4-4c^5)\\ -2(b^2+c^2)(b^2-c^2)^2\big)\,\xi^2\eta^2\\ +2b^2c\big(2a^5+a^4(2b-c)-a^3(b+c)(4b-7c)\\ -a^2(4b^3-7b^2c-2bc^2+c^3)\\ +a(b+c)(2b^3-b^2c-8bc^2+3c^3)\\ +2(b+c)^2(b-c)^3\big)\,\xi^2\eta\zeta=0.\end{aligned}$$

The vertices of Δ are isolated double points on the μ-images of i since the incircle of Δ always lies completely in the interior of the anticomplementary triangle Δ_a. Figure 11 shows the Miquel-Steiner image and pre-image of the incircle for four triangles (obtuse, right, acute, equilateral).

3.2.4 Nine-Point Circle

The nine-point circle n can be described by the homogeneous trilinear equation

$$n:\ \sum_{\text{cyclic}} a^2(-a^2+b^2+c^2)\,\xi^2 = 2abc(a\eta\zeta+b\zeta\xi+c\xi\eta).$$

The nine-point-circle is mapped under μ^{-1} to the quartic curve

$$\begin{aligned}\mu^{-1}(n):\ \sum_{\text{cyclic}} a^8(a^2-b^2-c^2)\,\xi^4+\\ 2a^5b\big(a^4-a^2(b^2+c^2)+2b^2c^2\big)\,\xi^3\eta\\ -2ab^5\big(a^2(b^2-2c^2)-b^2(b^2-c^2)\big)\,\xi\eta^3\\ +a^2b^2\big(a^6-a^4(b^2+c^2)-a^2(b^4-8b^2c^2+c^4)\\ +(b^2+c^2)(b^2-c^2)^2\big)\,\xi^2\eta^2=\\ =-2abc\,\xi\eta\zeta\sum_{\text{cyclic}} a(2a^6-2a^4(b^2+c^2)\\ -a^2(b^2+c^2)(b^2-c^2)^2)\,\xi.\end{aligned}$$

On it we can find the centers X_{35258}, X_{47785}, and X_{54280}.
The μ-image of n is also a quartic curve with the trilinear equation

$$\begin{aligned}\mu(n):\ \sum_{\text{cyclic}} c^2(c^2-3a^2-3b^2)\,\xi^2\eta^2=\\ =2\,\xi\eta\zeta\sum_{\text{cyclic}} bc(-5a^2+b^2+c^2)\,\xi.\end{aligned}$$

Surprisingly, there are only two labelled triangle centers on $\mu(n)$: X_{18771} (the Miquel-Steiner image of the Feuerbach point X_{11}) and X_{46426}. Depending on the shape of the triangle Δ, the curve $\mu(n)$ may have three cusps (equilateral triangle) or two double points and one cusp (isosceles triangle). For a right triangle Δ, $\mu(n)$ splits into a cubic and a straight line. If the right angle is at the vertex C, the linear component is given by $a\xi+b\eta=0$ and the cusp lies in the vertex C_a of the anticomplementary triangle Δ_a, *i.e.*, Δ_a's vertex opposite to C.

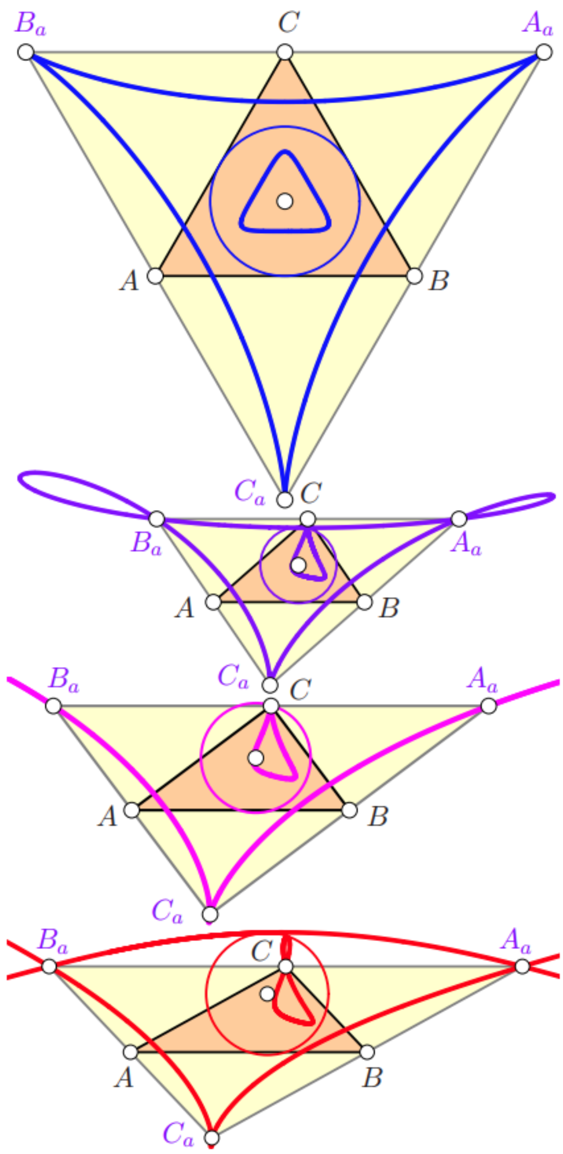

Figure 12: *Images and pre-images of the nine-point circle of an acute, an obtuse, a right, and an equilateral triangle.*

3.3 Triangle cubics

It is clear that the 6-parameter family of triangle cubics

$$
\begin{aligned}
\mathcal{C}^6: & \sum_{\text{cyclic}} \tfrac{a^2}{c^2} q_{102}\,\xi^3 - \sum_{\text{cyclic}} q_{210}\,\xi^2\eta - q_{111} \sum_{\text{cyclic}} \tfrac{a^2}{bc}\,\xi^3 \\
& + \tfrac{1}{a^2b^2c^2}\left(a^4c^2 q_{120}\,\xi^3 + b^4a^2 q_{012}\,\eta^3 + c^4b^2 q_{201}\,\zeta^3\right) \\
& -(q_{120}\,\xi\eta^2 + q_{012}\,\eta\zeta^2 + q_{201}\,\zeta\xi^2) + q_{111}\,\xi\eta\zeta = 0
\end{aligned} \tag{5}
$$

that pass through the vertices of Δ_a are mapped to cubics under μ (since the side lines of Δ_a split off from the image curve). According to Thms. 2 and 3, the orthocenter X_4 of Δ is fixed under μ and μ^{-1}. If a triangle cubic $\mathcal{C}$ contains X_4, then $X_4 \in \mu(\mathcal{C})$ and $X_4 \in \mu^{-1}(\mathcal{C})$.
Among the triangle cubics listed in B. GIBERT's *Catalogue of Triangle Cubics* (see [6]), we find the following cubics $\mathcal{K}_i$ with indices

$$
\begin{aligned}
i \in \{&7,8,45,80,92,133,141,142,144,146,154,\\
&170,211,240,242,254,279,311,347,355,371,\\
&380,449,455,548,605,611,617,659,753,860,\\
&985,1000,1002,1004,1053a,b,1078,1131\}
\end{aligned}
$$

which are also contained in the 6-parameter family (5).
For some of the cubics in B. GIBERT's list, their μ-images are also contained in the catalogue of cubics, see Tab. 1.

$\mathcal{K}_i$	7	8	80	141	170
$\mathcal{K}_j$	2	273	361	644	233
$\mathcal{K}_i$	254	311	355	449	611
$\mathcal{K}_j$	379	454	380	447	1172
$\mathcal{K}_i$	617	753	1000	1002	1037
$\mathcal{K}_j$	28	73	354	135	1013
$\mathcal{K}_i$		1053a,b		1131	
$\mathcal{K}_j$		1145a,b		1134	

Table 1: *Triangle cubics $\mathcal{K}_i$ with μ-images $\mathcal{K}_j$ both contained in* B. GIBERT*'s catalogue* [6].

$\mathcal{K}_i$	$X_j \in \mu(\mathcal{K}_i)$
45	2, 4, 6, 54, 275, 1993, 8882, 34756
240	6, 69, 316, 512, 3448, 14360, 53365
242	6, 69, 316, 3448, 14360, 53365
279	2, 4, 6, 30, 323, 2986, 5504, 10419, 15262
380	4, 6, 251, 1976, 2065
455	1, 6, 35, 37, 1126, 1171, 1255, 21353, 33635
605	6, 58, 63, 81, 284, 2287, 7123, 40403
659	6, 32, 83, 251, 51951
860	6, 15, 16, 74, 40384
985	6, 58, 81, 291, 1922, 2311, 7132, 24479, 38810, 38813
1078	1, 6, 56, 57, 266, 289, 1743

Table 2: *Triangle cubics $\mathcal{K}_i$ (from* GIBERT*'s) catalogue whose images are defined by triangle centers X_j (from* KIMBERLING*'s encyclopedia).*

The images of some other cubics are not contained in GIBERT's catalogue, but nevertheless, well defined solely by the triangle centers contained in them, see Tab. 2.
As can be seen in Tab. 1, the Lucas cubic $\mathcal{K}_{007}$ is mapped to the Thomson cubic $\mathcal{K}_{002}$. The image of the Droussent cubic $\mathcal{K}_{008}$ is the pivotal isocubic $\mathcal{K}_{273}$. Figure 13 shows the cubic $\mathcal{K}_{254}$ with some triangle centers on it. The μ-image $\mathcal{K}_{379}$ as well as the μ^{-1}-image of $\mathcal{K}_{254}$ is shown.

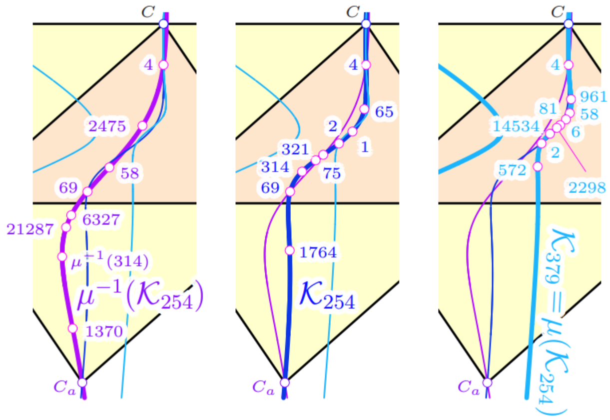

Figure 13: *The cubic $\mathcal{K}_{254}$, its μ-image $\mathcal{K}_{379}$, and its μ^{-1} image together with the centers on it.*

4 Final remarks

As mentioned earlier (and also in [12]), there exists a perspector R for the triangles Δ_M and Δ_D. In terms of trilinear coordinates and depending on $Z = \xi : \eta : \zeta$, the perspector R reads

$$
\begin{aligned}
R = & \left(a(a^2-b^2-c^2)\xi^2 + b(a^2-b^2)\xi\eta + c(a^2-c^2)\zeta\xi + abc\eta\zeta\right) \\
& \left(a(b\eta + c\zeta)\eta\zeta - bc\xi(\eta^2+\zeta^2) + (2a^2-b^2-c^2)\xi\eta\zeta\right) :: .
\end{aligned}
$$

The mapping $Z \mapsto R$ is quintic and by no means involutive. The Miquel-Steiner transformation is not involutive. We can give some chains of triangles centers, where each triangle center in the chain is the Miquel-Steiner transformation of its predecessor (see Tab. 3.).
It is possible to define some more algebraic transformations based on Miquel's theorem (the triangle related theorem illustrated in Figure 1). For example, the assumption that the three points A', B', C' be collinear yields a quartic transformation that sends lines to to points. Unfortunately, this transformation is not invertible. If the points A', B', C' are the vertices of the Cevian triangle of a point P, then the mapping that sends P to the respective Miquel point (as illustrated in Figure 1) is sextic. In this case it has to be clarified under which circumstances this mapping is invertible.

μ^{-3}	μ^{-2}	μ^{-1}	μ^{0}	μ^{1}	μ^{2}
	3436	8	**1**	58	3453
	1370	69	**2**	6	251
	6225	20	**3**	54	1166
			4		
		2888	**5**	1173	
			6 ↑ 2		
		3434	**7**	57	3451
			8 ↑ 1		
		329	**9**	1174	
		1330	**10**	1126	
		17035	**51**		
			52	1179	
			54 ↑ 3		
		144	**55**	3449	
		42020	**56**	3450	
			57 ↑ 7		
			58 ↑ 1		
149	513	100	**59**		
		2975	**60**		
		146	**30**	74	10419
		146	**66**	18018	40404
		6327	**75**	81	1169
	147	511	**98**	2065	
	148	512	**99**	249	
			100 ↑ 59		
	150	514	**101**	15378	
	151	515	**102**	15379	
	152	516	**103**	15380	
	153	517	**104**	15381	
	20344	518	**105**	15382	
	21290	519	**106**	15383	
	34186	520	**107**	15384	
	34188	521	**108**	15385	
	33650	522	**109**	15386	
	3448	523	**110**	250	
	14360	524	**111**	15387	
	13219	525	**112**	15388	

Table 3: *Some centers and the repeated μ-images.* **6** ↑ 2 *indicates that the center with Kimberling index* **6** *already shows up in the chain defined by center with Kimberling index* **2**.

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