No. 22. (2018)
ISSN 1331-1611

KoG

SCIENTIFIC - PROFESSIONAL JOURNAL
OF CROATIAN SOCIETY FOR GEOMETRY AND GRAPHICS

Official publication of the Croatian Society for Geometry and Graphics publishes scientific and professional papers from the fields of geometry, applied geometry and computer graphics.

Founder and Publisher

Croatian Society for Geometry and Graphics

Design

Miroslav Ambruš-Kiš

Layout

Ema Jurkin

Cover Illustration

Georg Glaeser, photography

Print

SAND d.o.o., Zagreb

URL address

http://www.hdgg.hr/kog
http://hrcak.srce.hr

Edition

150

Published annually

Guide for authors
Please, see the last page

KoG is reviewed by Mathematical Reviews (cover to cover) and zbMATH.

This issue has been financially supported by the the Ministry of Science and Education.

ISSN 1331–1611

KoG

No. 22
Zagreb, 2018

SCIENTIFIC AND PROFESSIONAL JOURNAL OF
CROATIAN SOCIETY FOR GEOMETRY AND GRAPHICS

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ISSN 1331–1611

BROJ 22
Zagreb, 2018

ZNANSTVENO-STRUČNI ČASOPIS
HRVATSKOG DRUŠTVA ZA GEOMETRIJU I GRAFIKU

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https://doi.org/10.31896/k.22.1
Original scientific paper
Accepted 27. 10. 2017.

Boris Odehnal

Two Convergent Triangle Tunnels

Two Convergent Triangle Tunnels

ABSTRACT

A semi-orthogonal path is a polygon inscribed into a given polygon such that the i-th side of the path is orthogonal to the i-th side of the given polygon. Especially in the case of triangles, the closed semi-orthogonal paths are triangles which turn out to be similar to the given triangle. The iteration of the construction of semi-orthogonal paths in triangles yields infinite sequences of nested and similar triangles. We show that these two different sequences converge towards the bicentric pair of the triangle's Brocard points. Furthermore, the relation to discrete logarithmic spirals allows us to give a very simple, elementary, and new constructions of the sequences' limits, the Brocard points. We also add some remarks on semi-orthogonal paths in non-Euclidean geometries and in n-gons.

Key words: triangle, semi-orthogonal path, Brocard points, symmedian point, discrete logarithmic spiral, Tucker-Brocard cubic

MSC2010: 51A05, 51A20

Dva konvergentna niza trokuta

SAŽETAK

Poluortogonalan put je poligonalna linija upisana u dani mnogokut takva da je i-ta stranica poligonalne linije okomita na i-tu stranicu danog mnogokuta. U slučaju trokuta, zatvoreni poluortogonalni putovi su trokuti slični danom trokutu. Iteracijom konstrukcije poluortogonalnih putova u trokutima dobivaju se beskonačni nizovi upisanih sličnih trokuta. Pokazujemo da ova dva različita niza konvergiraju prema bicentričnom paru Brocardovih točaka trokuta. Nadalje, veza s diskretnim logaritamskim spiralama omogućuje vrlo jednostavnu, elementarnu i novu konstrukciju limesa ovih nizova, Brocardovih točaka. Iznosimo i neke napomene o poluortogonalnim putovima kako u neeuklidskim geometrijama i tako i za n-kute.

Ključne riječi: trokut, poluortogonalan put, Brocardove točke, sjecište simedijana, diskretna logaritamska spirala, Tucker-Brocardova kubika

1 Introduction

1.1 Sequences of triangles

Little is known about sequences of Cevian triangles within a given triangle. Sequences of medial triangles and Routh triangles are studied in [3]. There, triangles are considered as triplets of points in the complex plane and a shape function which is actually a complex affine ratio is defined and describes how the shape of a triangle changes during the iteration process. It turns out that the above mentioned classes of triangles converge in shape, in most cases to equilateral triangles.

It is well-known (and rather trivial) that the sequence of Cevian triangles of a triangle's centroid converges towards the centroid. The intouch triangle (contact points of the incircle and the triangle sides) is always in the interior of the initial triangle (cf. Fig. 1). So, it is nearby to expect that the sequence of nested intouch triangles has a point shaped limit which is yet undiscovered. On the contrary, the orthic triangle is an interior triangle only if the base triangle is acute and, unfortunately, acute triangles may have an obtuse orthic triangle and convergence cannot be expected in the generic case.

We try to leave the beaten tracks by starting the construction of the triangles of the sequence in a different way. The edges of the triangles in the sequences in question shall form a semi-orthogonal path, *i.e.*, the i-th edge of the new triangle shall be orthogonal to the i-th side of the given triangle. Depending on the ordering of the sides of the base triangle, we find two closed semi-orthogonal paths which shall be constructed and discussed in Sec. 2. Further, the case of generic n-laterals (n straight lines in generic position such that no two lines enclose a right angle) shall be addressed in Sec. 2, besides some comments on closed semi-orthogonal paths in non-degenerate Cayley-Klein geometries, *i.e.*, the elliptic and the hyperbolic plane. Sec. 3 is dedicated to the computation of the limits of the triangle sequences. We show that the triangles in one sequence shrink to one Brocard point, while the others converge to the other Brocard point, and thus, these two limits are located on the Tucker-Brocard cubic. Finally, in Sec. 4, we conclude and address some open problems. The remaining part of this section (Sec. 1) collects some prerequisites.

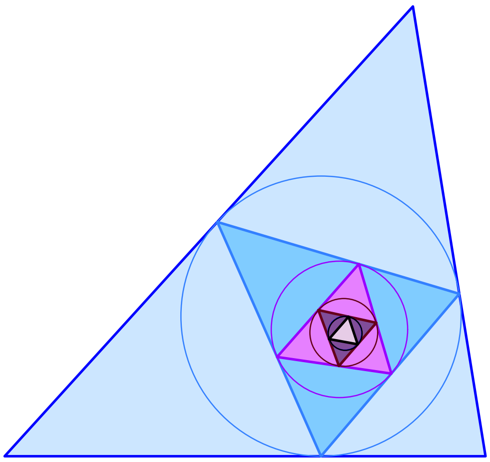

Figure 1: *Where and what is the limit of the sequence of intouch triangles?*

1.2 Prerequisites and conventions

Since we deal with triangles in the Euclidean plane $\mathbb{R}^2$, we use Cartesian coordinates in order to describe points. It will turn out useful to perform the projective closure of the Euclidean plane by adding the ideal line ω to $\mathbb{R}^2$. Whenever, we deal with points and lines in the projectively extended plane, we can switch between Cartesian and homogeneous coordinates of points by

$$(1,x,y) \longleftrightarrow (x_0 : x_1 : x_2)$$

as long as $x_0 \neq 0$. Lines $l: \; a_0 + a_1x + a_2y = 0$ can also be described by homogeneous coordinates $(a_0 : a_1 : a_2)$. Especially, the ideal line (or line at infinity) is simply given by $\omega = (1:0:0)$.

For the moment, it is sufficient to assume that the Cartesian coordinates of the vertices of the base triangle Δ_0 are

$$A_0 = (0,0), \;\; B_0 = (c,0), \;\; C_0 = (u,v). \tag{1}$$

We assume that $c, v \neq 0$ so that $A_0 \neq B_0$ and $C_0 \notin [A_0,B_0]$. Further, $u^2+v^2 \neq 0$ which implies $A_0 \neq C_0$. In the following, no interior angle of Δ_0 shall be a right one. This is expressed algebraically by $u \neq c$, $u \neq 0$, and $u^2 - uc + v^2 \neq 0$.

We shall agree that the side lengths of Δ_0 are

$$a := \overline{B_0C_0}, \;\; b := \overline{C_0A_0}, \;\; c := \overline{A_0B_0}.$$

Later, when we try to express especially metric properties of the triangle in terms of Δ_0's side lengths a, b, c, we should be able to replace u and v from (1) by functions depending on a, b, c. For that purpose, we compute $C_0 = (u,v)$ as the intersection of two circles: one centered at A_0 with radius b; the other one centered at B_0 with radius a such that $v > 0$. This results in

$$u = \frac{b^2+c^2-a^2}{2c} \quad \text{and} \quad v = \frac{2F}{c} \tag{2}$$

with F being the area of Δ_0. Note that F can be expressed in terms of Δ_0's side lengths using Heron's formula or, equivalently, with help of the Cayley-Menger determinant.

2 Closed semi-orthogonal paths

2.1 Triangles in the Euclidean plane

Let $\Delta_0 = A_0B_0C_0$ be a triangle in the Euclidean plane. Let further P_0 be a point on the side line $[A_0,B_0]$. We construct a sequence P_0, P_1, P_2, P_3 of points on the lines $[A_0,B_0]$, $[B_0,C_0]$, $[C_0,A_0]$, $[A_0,B_0]$ in the following way (cf. Fig. 2):

$$\begin{gathered} P_0 \in [A_0,B_0], \; [P_0,P_1] \perp [A_0,B_0], \; P_1 \in [B_0,C_0] \\ \text{with } P_0 \to P_1 \to P_2 \to P_3, \;\; A_0 \to B_0 \to C_0 \to A_0, \\ \text{cyclical replacement.} \end{gathered}$$

Henceforth, we shall refer to such paths as *semi-orthogonal paths*. It doesn't make a difference if we start at $[A_0,B_0]$ or at any other side line of Δ_0.

Obviously, the mapping $\pi: \; P_0 \mapsto P_3$ is a projective mapping $[A_0,B_0] \to [A_0,B_0]$, since it is a chain of three perspectivities:

$$\underbrace{[A_0,B_0] \overset{R^\perp}{\barwedge} [B_0,C_0] \overset{S^\perp}{\barwedge} [C_0,A_0] \overset{T^\perp}{\barwedge} [A_0,B_0]}_{\barwedge}$$

where $R := [A_0,B_0] \cap \omega$, $S := [B_0,C_0] \cap \omega$, and $T := [C_0,A_0] \cap \omega$ are the ideal points of Δ_0's side lines and $R^\perp$, $S^\perp$, and $T^\perp$ are the ideal points of the respective orthogonal directions. Note, that the three perspectors are collinear as indicated in Fig. 2.

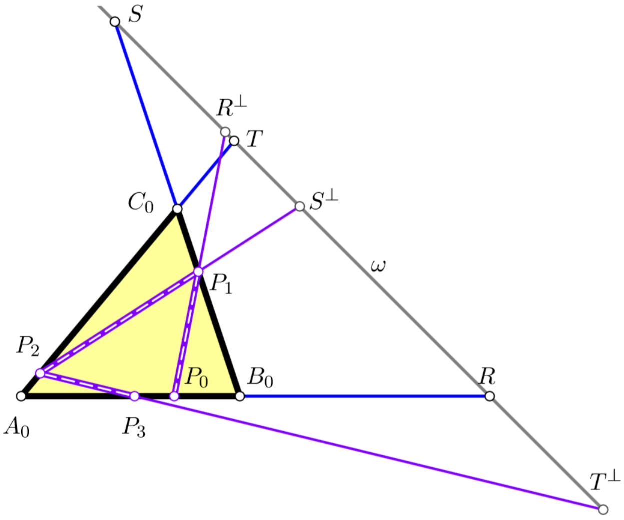

Figure 2: *The projective mapping $\pi : [A_0, B_0] \to [A_0, B_0]$ is the product of three perspectivities. The perspectors themselves are assigned to the ideal points of Δ_0's side lines in a projective way: They are joined by the absolute polarity acting on ω.*

Remark 1 *Indeed, the projective mapping $\perp : \omega \to \omega$ that assigns the ideal point of the orthogonal direction to any ideal point can be replaced by any other elliptic projective mapping acting on ω. The pseudo-Euclidean case would be covered if $\perp$ is hyperbolic.*

The ideal point R of $[A_0, B_0]$ is self-assigned in π, since $\pi(R) = R$, and therefore, we can expect only one further fixed point A_1 of π.

Let now, $P_0 = (t, 0)$ with $t \in \mathbb{R}$ which is a parametrization of the line $[A_0, B_0]$ and we find

$$P_1 = \left(t, \tfrac{v(c-t)}{c-u}\right), \quad P_2 = \frac{\alpha(t)}{\beta}(u, v), \quad P_3 = \frac{\alpha(t)(u^2+v^2)}{u\beta}(1, 0) \tag{3}$$

where

$$\alpha(t) = ((c-u)^2+v^2)t - cv^2,$$
$$\beta = ((c-u)^2+v^2)u - cv^2.$$

Note that $c - u$ is a divisor of β that cannot vanish due to assumptions made earlier. The path $P_0P_1P_2P_3$ is closed if the points P_0 and P_3 coincide. This is equivalent to $t = \frac{\alpha(t)}{u\beta}(u^2+v^2)$, and thus,

$$t = \frac{c(u^2+v^2)}{c^2 - cu + u^2 + v^2}. \tag{4}$$

We shall make explicit the fact that the denominator of t in (4) cannot vanish: Substituting (2) into (4), we find

$$t = 2b^2 c\sigma^{-1} \tag{5}$$

where

$$\sigma := a^2 + b^2 + c^2 \tag{6}$$

which cannot vanish for $a, b, c \in \mathbb{R}^\star$.

If we insert (4) into (3) and relabel the points by letting $A_1 = P_0 = P_3$, $B_1 = P_1$, and $C_1 = P_2$, we arrive at

$$A_1 = \tfrac{c}{\gamma}\left(c(u^2+v^2), 0\right), \quad B_1 = \tfrac{c}{\gamma}\left(u^2+v^2, cv\right), \quad C_1 = \tfrac{cu}{\gamma}(u, v) \tag{7}$$

with $\gamma = c^2 - cu + u^2 + v^2$. Such a closed triangular path $A_1B_1C_1$ is shown in Fig. 3.

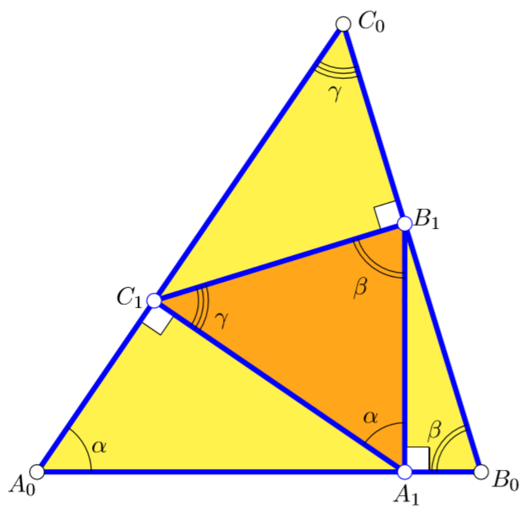

Figure 3: *The first triangle $\Delta_1 = A_1B_1C_1$ inscribed into $\Delta_0 = A_0B_0C_0$.*

As can be seen in Fig. 3,

$$\sphericalangle C_1A_1B_1 = \sphericalangle C_0A_0B_0,$$
$$\sphericalangle A_1B_1C_1 = \sphericalangle A_0B_0C_0,$$
$$\sphericalangle B_1C_1A_1 = \sphericalangle B_0C_0A_0.$$

Thus, we have

Lemma 1 *The triangles Δ_0 and Δ_1 are similar.*

In order to construct the path $A_1B_1C_1$, we started at P_0 leaving $[A_0, B_0]$ in the orthogonal direction until we meet $[B_0, C_0]$. We could also look for a path $Q_0Q_1Q_2Q_3$ with

$Q_0 = P_0$ and $Q_1 \in [C_0, A_0]$, *i.e.*, leaving $[A_0B_0]$ in the orthogonal direction until we meet $[C_0A_0]$, and so forth. This yields a second closed triangular path if we achieve $Q_0 = Q_3$. Then, $L_1 = Q_0$, $K_1 = Q_1$, and $M_1 = Q_2$ form a triangle ∇_1, see Fig. 4. Similar to Lem. 1 and due to the same reasoning, we have

Lemma 2 *The triangles Δ_0 and ∇_1 are similar.*

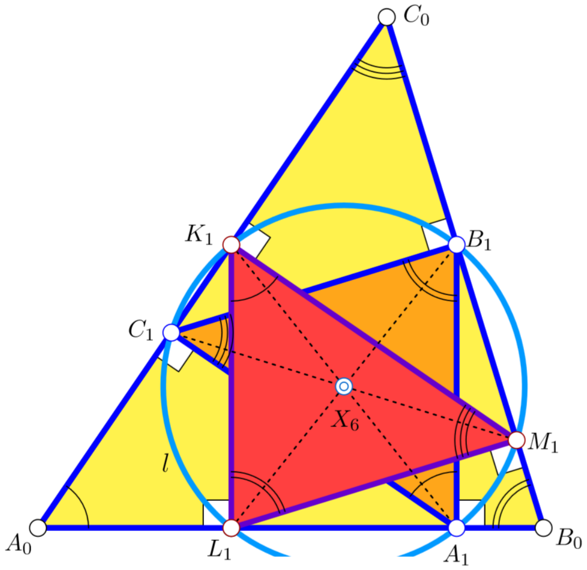

Figure 4: *The two perspective triangles Δ_1 and ∇_1 with their common circumcircle (centered at the symmedian point X_6).*

Moreover, we can show the following

Theorem 1 *The triangles Δ_1 and ∇_1 are congruent, share the circumcircle l, and therefore, the circumcenter which is the symmedian point X_6 of Δ_0.*[1]

Proof. The similarity of Δ_0 and Δ_1 needs no further confirmation, since this is done right before Lem. 1. The similarity of Δ_0 and ∇_1 can be shown in the same way. From $\Delta_1 \sim \Delta_0$ and $\nabla_1 \sim \Delta_0$ we can infer $\Delta_1 \sim \nabla_1$.

Since C_1M_1 is seen from B_1 and K_1 at right angles, B_1, C_1, K_1, and M_1 are concyclic. Further, B_1L_1 is seen from A_1 and M_1 at right angles. Thus, the circumcircle of B_1, C_1, K_1, and M_1 equals that of A_1, B_1, L_1, and M_1.

Two similar triangles can only share the circumcircle if they are congruent (which can be confirmed with help of the Law of sines).

Finally, we have to show that the circumcenter of the six points $A_1, B_1, C_1, K_1, L_1, M_1$ is the symmedian point X_6 of Δ_0. We compute the actual distances of the midpoint M of the segment A_1K_1 to Δ_0's side lines and find

$$\overline{M[A_0,B_0]} = 2cF\sigma^{-1},$$
$$\overline{M[B_0,C_0]} = 2aF\sigma^{-1},$$
$$\overline{M[C_0,A_0]} = 2bF\sigma^{-1},$$

and thus, the homogeneous trilinear coordinates are

$$M = (a : b : c)$$

which confirms that M equals the symmedian point X_6 of Δ_0. □

From Lem. 1, we can deduce a simple linear construction of Δ_1 (and ∇_1) which is shown in Fig. 5: An arbitrary triangle $\Delta = A'B'C'$ with

$$A' \in [A_0, B_0],\ B' \in [B_0, C_0],$$
$$[A', B'] \perp [A_0, B_0]$$

similar to Δ_0 is drawn, *i.e.*,

$$\sphericalangle C'A'B' = \sphericalangle C_0A_0B_0,$$
$$\sphericalangle A'B'C' = \sphericalangle A_0B_0C_0.$$

The central similarity with center B_0 sends C' to $C_1 \in [C_0, A_0]$, and thus, it maps $\Delta' \to \Delta_1$.

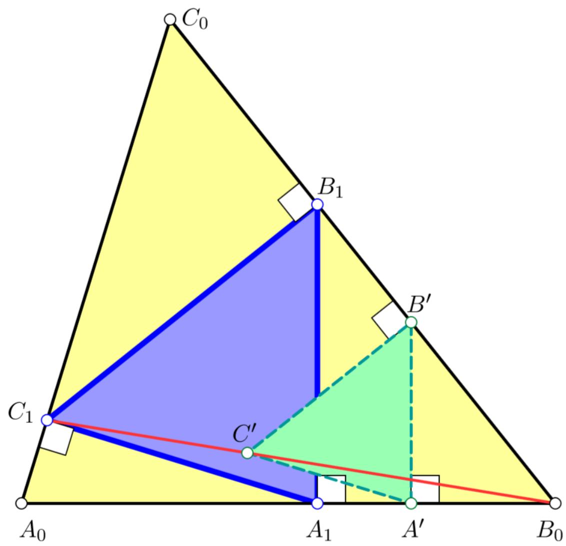

Figure 5: *The construction of Δ_1 is linear and uses the central similarity from B_0.*

From this construction it is clear that there is a second triangle $\nabla_1 = K_1L_1M_1$ similar to Δ_0 and inscribed into Δ_0,

[1] Here, and in the following X_i means the i-th point in Kimberlings's encyclopedia of triangle centers [4, 5].

but different from Δ_1: We start with a triangle $\nabla = K'L'M'$ and let

$$L' \in [A_0, B_0], K' \in [C_0, A_0],$$
$$[K', L'] \perp [A_0, B_0]$$

with

$$\sphericalangle K'L'M' = \sphericalangle A_0B_0C_0,$$
$$\sphericalangle M'K'L' = \sphericalangle C_0A_0B_0.$$

Now, the similarity with center A_0 sends $K'L'M'$ to $K_1L_1M_1$.

2.2 Non-Euclidean planes

The case of the pseudo-Euclidean plane was the subject of Rem. 1 since its only difference is the hyperbolic projectivity on the ideal line (in the plane's projective extension).

In hyperbolic and elliptic geometry, there is still a projective mapping $\pi : [A_0, B_0] \to [A_0, B_0]$. However, since there is no ideal line, but rather an ideal conic, we miss a self-assigned ideal point on $[A_0, B_0]$. Thus, π has up to two real fixed points:

Theorem 2 *In each generic triangle in the hyperbolic or elliptic plane, there exist two closed semi-orthogonal paths for a particular chosen ordering of side lines.*

The reality of the fixed points mentioned in Thm. 2 is clear in the hyperbolic and in the elliptic case: Let ω denote the absolute conic. Each pair (V_i, V_j) of proper vertices of the triangle and the pair of absolute points $(A_1, A_2) = [V_i, V_j] \cap \omega$ of $[V_i, V_j]$ are not entangled. Therefore, there are two real fixed points on $[V_i, V_j]$, see [2, p. 254].

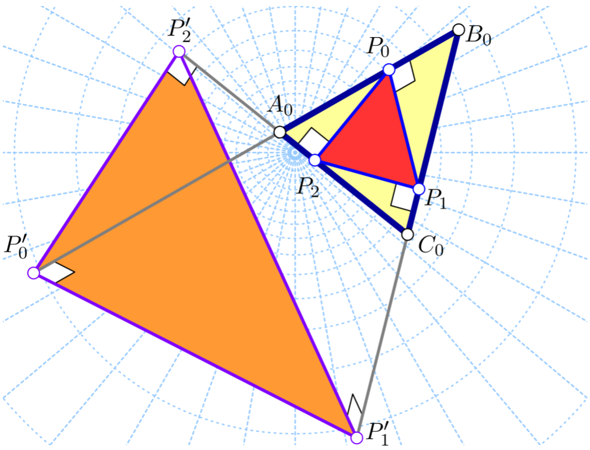

Figure 6: *Two closed semi-orthogonal paths in a triangle $\Delta_0 = A_0B_0C_0$ of the elliptic plane. The paths are starting with $[P_0, P_1] \perp_e [A_0, B_0]$ and $[P'_0, P'_1] \perp_e [A_0, B_0]$, respectively.*

Fig. 6 shows a triangle $\Delta_0 = A_0B_0C_0$ in the projective model of the elliptic plane together with the two closed semi-orthogonal paths to a chosen ordering of Δ_0's side lines.

There are two different closed semi-orthogonal paths in a triangle in the elliptic or in the hyperbolic plane. Since the projective mapping $\pi : [A_0, B_0] \to [A_0, B_0]$ is hyperbolic in elliptic as well as hyperbolic geometry, there are four closed semi-orthogonal (triangular) paths in a generic triangle. Fig. 7 shows these four closed semi-orthogonal paths in a triangle in the elliptic plane. This four-fold symmetry resembles the four-fold symmetry in Universal Hyperbolic Geometry (cf. [8, 9]) and shows up when classical triangle geometry is also studied from the viewpoint of projective geometry, see [7].

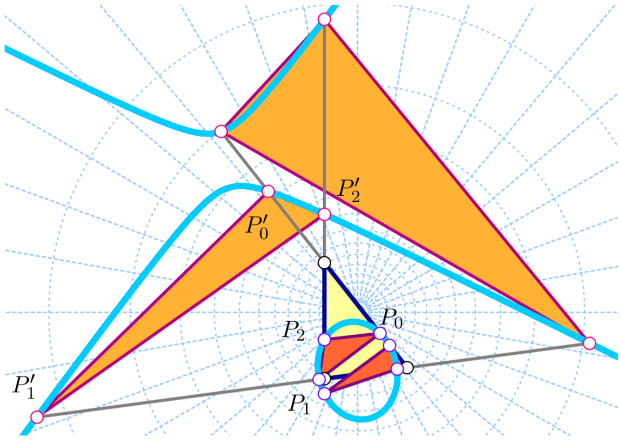

Figure 7: *There are four closed semi-orthogonal paths in a triangle in the elliptic plane. Two by three vertices gather on a conic: three points from an $[A_0, B_0] - [B_0, C_0] - [C_0, A_0]$ path and three vertices from an $[A_0, B_0] - [C_0, A_0] - [B_0, C_0]$ path.*

Neither in the elliptic nor in the hyperbolic plane we can use Thales's theorem in order to show that two of the triangular paths share a circumconic as illustrated in Fig. 7.

3 Infinite sequences of inscribed triangles

In this section, we return to Euclidean geometry in order to attack the main problem.

We have seen that Δ_0 and Δ_1 are similar triangles. Consequently, the triangle Δ_2 inscribed into Δ_1 whose vertices A_2, B_2, C_2 are obtained in the same way as A_1, B_1, C_1 is also similar to Δ_1, and thus, to Δ_0. This procedure can be repeated arbitrarily often which yields a sequence of similar triangles $\Delta_0, \Delta_1, \Delta_2, \Delta_3, \Delta_4, \ldots$ for a particular triangle Δ_0.

Due to the construction of Δ_1, subsequent edges A_iA_{i+1} and $A_{i+1}A_{i+2}$ of the polygon $A_0A_1A_2A_3A_4\ldots$ are orthogonal. Further, the edges A_iA_{i+1} and $A_{i+2}A_{i+3}$ are antiparallel. The same holds true for the polygons $B_0B_1B_2\ldots$ and $C_0C_1C_2\ldots$.

Fig. 8 shows the polygon $A_0A_1A_2A_3\ldots$, while Fig. 9 shows the six discrete logarithmic spirals encircling two different limits.

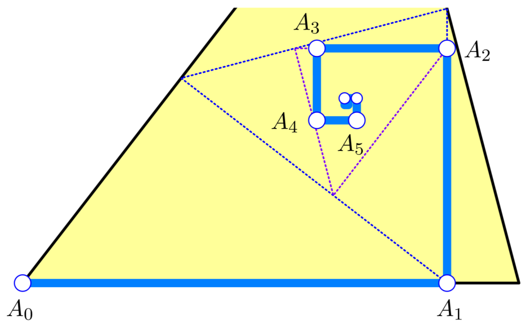

Figure 8: *The discrete logarithmic spiral formed by A_0, A_1, ... winding towards a limit.*

Now, we want to show that the triangles Δ_i and ∇_i converge to a point as $i \to \infty$. In this case, it is not necessary to apply shape functions like in [3]. We need some other prerequisites. With (7) it is easy to verify that the side lengths $a_1 = \overline{B_1C_1}$, $b_1 = \overline{C_1A_1}$, $c_1 = \overline{A_1B_1}$ of Δ_1 are

$$a_1 = a\lambda, \quad b_1 = b\lambda, \quad c_1 = c\lambda$$

where λ is the scaling factor of the similarity $\Delta_0 \to \Delta_1$ which can be computed from a, b, and c via

$$\lambda = 4F\sigma^{-1} \tag{8}$$

where F equals the area of Δ_0. This allows us to express the radius of the circle l (cf. Thm. 1) in terms of Δ_0's side lengths:

Corollary 1 *The radius of l equals*

$$R_1 = \frac{abc}{a^2+b^2+c^2}.$$

Proof. Lengths are scaled with the factor λ when applying the similarity $\Delta_0 \to \Delta_1$. Thus, the circumradius R_0 changes to $R_1 = R_0\lambda$ with λ given in (8). According to the Law of sines, $R_0 = \frac{abc}{4F}$, and thus, $R_1 = abc\sigma^{-1}$. □

The fact that the sequence of triangles Δ_0, Δ_1, Δ_2, ... (as well as the sequence of all ∇_i) consists of scaled versions of the initial triangle Δ_0 together with the fact that the scaling factor depends on the side lengths of each triangle in the same way (cf. Eq. (8)) makes the traces of Δ_0's vertices a special polygon. For example, the segment A_1A_2 is orthogonal to A_0A_1 and $\overline{A_1A_2} = \lambda \cdot \overline{A_0A_1}$. This holds true for any pair $(A_iA_{i+1}, A_{i+1}A_{i+2})$ of subsequent segments. Thereby, a sequence of discrete equiform motions (consisting of a quarter turn and a constant scaling) *moves* the polygon $A_0A_1A_2\ldots$ into itself and we can say that $A_0A_1A_2\ldots$ is invariant under the sequence of discrete equiform motions. In the smoth case, the trace of a point undergoing one-parameter equiform motion with constant parameter is a logarithmic spiral. Therefore, we can call $A_0A_1A_2\ldots$ a *discrete logarithmic spiral*. Fig. 9 shows the six discrete logarithmic spirals traced by the three vertices of Δ_0 undergoing the two independent sequences of discrete equiform motions.

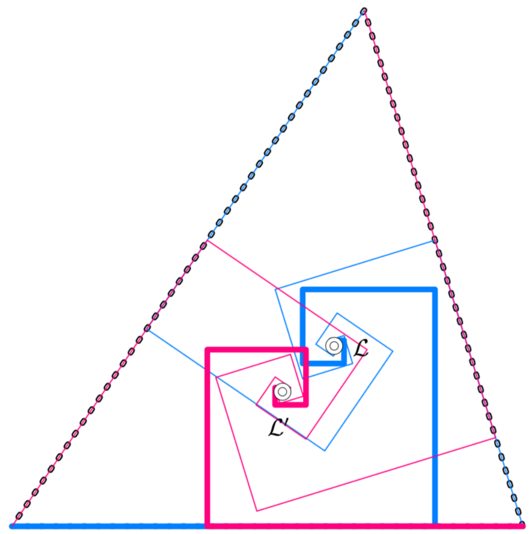

Figure 9: *The six discrete logarithmic spirals orbiting the limit points $\mathcal{L}$ and $\mathcal{L}'$.*

Now, we are able to state and prove:

Theorem 3 *The limit position $\mathcal{L}$ of all points A_i, B_i, or C_i is the first Brocard point of Δ_0 with homogeneous trilinear coordinates*

$$\mathcal{L} = (ac^2 : ba^2 : cb^2) \tag{9}$$

while the limit $\mathcal{L}'$ of all points K_i, L_i, or M_i is the second Brocard point of Δ_0, i.e., *in terms of homogeneous trilinear coordinates*

$$\mathcal{L}' = (ab^2 : bc^2 : ca^2). \tag{10}$$

Proof. With the initial choice of the coordinate frame, we can find the limit position

$$\mathcal{L} = \lim_{i\to\infty} A_i.$$

The x- and y-coordinate $x_{\mathcal{L}}$ and $y_{\mathcal{L}}$ are

$$x_{\mathcal{L}} = \overline{A_0A_1} - \overline{A_2A_3} + \overline{A_4A_5} - \overline{A_6A_7} \pm \ldots,$$
$$y_{\mathcal{L}} = \overline{A_1A_2} - \overline{A_3A_4} + \overline{A_5A_6} - \overline{A_7A_8} \pm \ldots. \tag{11}$$

The similarity $\Delta_i \to \Delta_{i+1}$ changes lengths by scaling them with the factor λ, *i.e.*,

$$\overline{A_iA_{i+1}} = \lambda^k \cdot \overline{A_{i+k}A_{i+k+1}}$$

for $i,k \in \{0,\ldots,n\}$. Consequently, the coordinates $x_{\mathcal{L}}$ and $y_{\mathcal{L}}$ from (11) change to

$$\begin{aligned} x_{\mathcal{L}} &= \overline{A_0A_1}\ (1-\lambda^2+\lambda^4 \mp \ldots), \\ y_{\mathcal{L}} &= \overline{A_0A_1}\lambda(1-\lambda^2+\lambda^4 \mp \ldots). \end{aligned} \tag{12}$$

The length of the segment $\overline{A_0A_1}$ follows from (1), (7) with (2) and is already given in (4):

$$\overline{A_0A_1} = t = 2b^2c\sigma^{-1}.$$

We let

$$\tau := a^2b^2 + b^2c^2 + c^2a^2 \tag{13}$$

and with (8), we can infer

$$\lambda = \sqrt{\frac{2\tau - a^4 - b^4 - c^4}{2\tau + a^4 + b^4 + c^4}} < 1$$

for any admissible choice of a, b, and c. Consequently, the infinite alternating sum of even powers of λ attains the value

$$\frac{1}{1+\lambda^2}$$

which gives

$$\mathcal{L} = \frac{b^2c}{2\tau}(\sigma, 4F). \tag{14}$$

Note that $\mathcal{L}$ given in (14) is at the same time the limit of B_i and C_i too.

The y-coordinate of $\mathcal{L}$ in (14) equals the distance of $\mathcal{L}$ to the line $[A_0, B_0]$. Therefore, it is the third actual trilinear coordinate of $\mathcal{L}$. It is elementary to compute the first and second actual trilinear coordinate of $\mathcal{L}$ and it turns out that they can be obtained by cyclically replacing a, b, c once and twice in $2b^2cF\tau^{-1}$. We observe that both F and τ are cyclic symmetric in a, b, c, and therefore, they do not change. For the sake of simplicity, we aim at homogeneous trilinear coordinates of $\mathcal{L}$ which allows us to cancel cyclic symmetric factors as long as they are common to all coordinate functions. So, we obtain (9).

A comparison of (9) with the trilinear representation of the first Brocard point given in [6] confirms that $\mathcal{L}$ is indeed the first Brocard point.

The calculations do not really differ if we compute the limit position $\lim\limits_{i\to\infty} K_i = \mathcal{L}'$. In this case, it is beneficial to start at $B_0 = (c, 0)$ and determine

$$\mathcal{L}' = \begin{pmatrix} c \\ 0 \end{pmatrix} + \overline{B_0L_1} \begin{pmatrix} -1+\lambda^2-\lambda^4 \pm \ldots \\ \lambda(1-\lambda^2+\lambda^4 \mp \ldots) \end{pmatrix}$$

where $\overline{B_0L_1} = 2a^2c\sigma^{-1}$ which results in

$$\mathcal{L}' = \frac{c}{2\tau}\left(b^2c^2 - a^4 + \tau, 4a^2F\right). \tag{15}$$

In the same way as above, we end with

$$\mathcal{L}' = (ab^2 : bc^2 : a^2c). \qquad \square$$

Remark 2 *The two limit positions $\mathcal{L}$ and $\mathcal{L}'$ are no triangle centers: Of course, the trilinear representation is apparently cyclic symmetric in the side lengths a, b, c of Δ_0, but, they do not satisfy the norming condition: If $f(a,b,c)$ is a center function, then it also has to satisfy $|f(a,b,c)| = |f(a,c,b)|$ in order to make $(f(a,b,c) : f(b,c,a) : f(c,a,b))$ a center.*

Now, we show

Theorem 4 *The point $\mathcal{L}$ is the only (real and proper) common point of the three Thaloids of the segments A_0A_1, B_0B_1, C_0C_1.*
The point $\mathcal{L}'$ is the only (real and proper) common point of the three Thaloids of the segments A_0K_1, B_0L_1, C_0M_1.

Proof. From the asymptotic point $\mathcal{L}$ of the discrete logarithmic spiral $A_0A_1A_2\ldots$, any segment A_iA_{i+1} can be seen at a right angle, since each subsequent segment of the discrete logarithmic spiral corresponds to a quarter turn plus a scaling with the factor λ. This is also the case for the segments B_iB_{i+1} and C_iC_{i+1}.

For the same reasons, the limit $\mathcal{L}'$ of the second sequence is simultaneously located on three Thaloids. $\square$

Thm. 4 provides a very elegant and elementary construction of the two Brocard points $\mathcal{L}$ and $\mathcal{L}'$. This construction seems not to be mentioned in the literature.

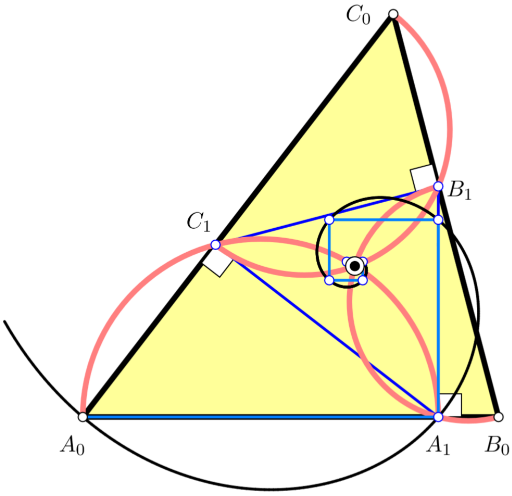

Figure 10: *Three Thaloids concur in $\mathcal{L}$.*

Fig. 10 shows the three Thaloids through $\mathcal{L}$. Clearly, there are also three Thaloids passing through $\mathcal{L}'$.

As a consequence of Thm. 3, the two limits of the triangle tunnels are located on a special self-isotomic pivotal cubic K012 (cf. [1]), better known as the Tucker-Brocard cubic $\mathcal{T}$. Only a few centers from Kimberling's list [5] are known to lie on $\mathcal{T}$: the symmedian point (also Lemoine point or Grebe point) X_6, its isotomic conjugate X_{76} (the third Brocard point), and further the two centers X_{880} and X_{882}. (X_{882} lies on the Brocard axis $[\mathcal{L},\mathcal{L}']$.) Fig. 11 shows the tunnel limits (Brocard points) together with the Tucker-Brocard cubic $\mathcal{T}$ and the centers X_6, X_{76}, X_{880}, X_{882}.

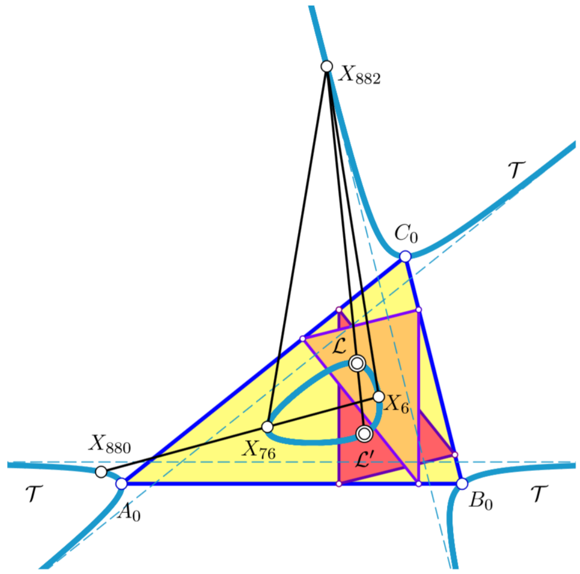

Figure 11: *The Tucker-Brocard cubic contains the first and second Brocard point $\mathcal{L}$ and $\mathcal{L}'$, besides X_6, X_{76}, X_{880}, and X_{882}.*

4 Conclusion and open problems

The semi-orthogonal paths in quadrilaterals will, in general, not be similar to the initial quadrilateral, for there exists no equiform transformation that maps two quadrilaterals onto each other even when they agree in their interior angles. Nevertheless, the iteration of the computation of semi-orthogonal paths in a quadrilateral produces sequences of shrinking and nested quadrilaterals which preserve their interior angles. Fig. 12 shows an example. In fact, there exist up to six such sequences in a generic quadrilateral. Apparently, there exists an attractor for the six different semi-orthogonal paths in a quadrilateral. It would be interesting to find a tool to compute the limits, if they exist. Shape functions for quadrilaterals as used for triangles in [3] or for generic n-gons defined in [10] could help.

Only the case of cyclic quadrilaterals seems hopeful. Since the measures of interior angles are preserved during the transition from one quadrilateral to the next in the sequence of semi-orthogonal paths, so is there sum or difference. In a cyclic quadrilateral opposite interior angels sum up to π, *i.e.*, $\sphericalangle ABC + \sphericalangle CDA = \pi$ and $\sphericalangle BCD + \sphericalangle DAB = \pi$ and so do the sums of the opposite (interior) angles in their semi-orthogonal paths, see Fig. 13. Hence, the semi-orthogonal path of a cyclic quadrilateral is again a cyclic quadrilateral. Numerical experiments show that each $4i$-th semi-orthogonal path of a cyclic quadrilateral is similar and perspective to the initial cyclic quadrilateral ($i \in \mathbb{N}$). The perspector is common to all pairs of quads, cf. Fig. 14.

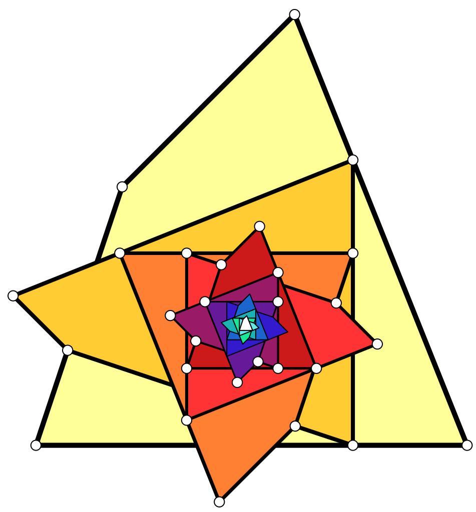

Figure 12: *A sequence of shrinking and nested closed semi-orthogonal paths in a quadrilateral. It is obvious that interior angles remain unchanged.*

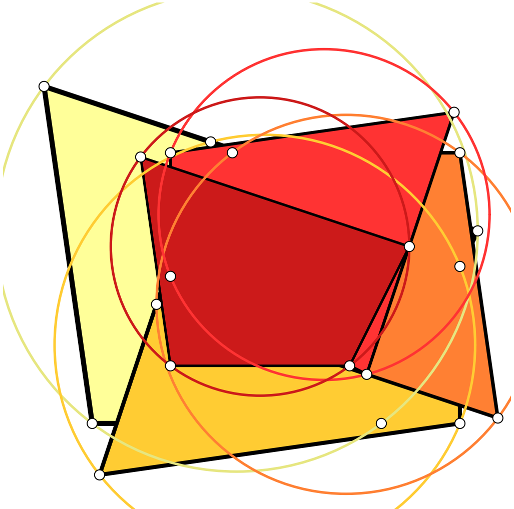

Figure 13: *The first four semi-orthogonal paths of a cyclic quadrilateral are similar and perspective to the initial quadrilateral.*

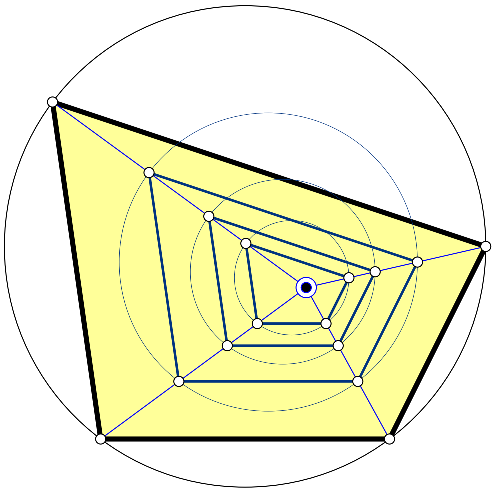

Figure 14: *The* 4th*,* 8th*, and* 12th *semi-orthogonal path of a cyclic quadrilateral are similar and perspective to the initial quadrilateral. The perspector could be see as the limit of the sequence of semi-orthogonal paths.*

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https://doi.org/10.31896/k.22.2
Original scientific paper
Accepted 16. 11. 2018.

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Jeřabek Hyperbola of a Triangle in an Isotropic Plane

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ABSTRACT

In this paper, we examine the Jeřabek hyperbola of an allowable triangle in an isotropic plane. We investigate different ways of generating this special hyperbola and derive its equation in the case of a standard triangle in an isotropic plane. We prove that some remarkable points of a triangle in an isotropic plane lie on that hyperbola whose centre is at the Feuerbach point of a triangle. We also explore other interesting properties of this hyperbola and its connection with some other significant elements of a triangle in an isotropic plane.

Key words: allowable triangle, standard triangle, Jeřabek hyperbola

MSC2010: 51N25

Jeřabekova hiperbola trokuta u izotropnoj ravnini

SAŽETAK

U radu proučavamo Jeřabekovu hiperbolu dopustivog trokuta u izotropnoj ravnini. Istražujemo različite načine generiranja ove specijalne hiperbole i izvodimo njenu jednadžbu u slučaju standardnog trokuta. Dokazujemo da neke značajne točke trokuta u izotropnoj ravnini leže na toj hiperboli čiji je centar u Feuerbachovoj točki trokuta. Proučavamo i neka druga zanimljiva svojstva ove hiperbole i njezinu vezu s nekim značajnim elementima trokuta u izotropnoj ravnini.

Ključne riječi: dopustivi trokut, standardni trokut, Jeřabekova hiperbola

In Euclidean geometry, the Jeřabek hyperbola of a triangle ABC is its circumscribed rectangular hyperbola, which is the isogonal image of the Euler line of this triangle. This hyperbola is generated by the centre of homology of the triangle ABC and the homothetic triangle to its tangential triangle $A_tB_tC_t$ with respect to the circumcentre of the triangle ABC. In Euclidean geometry, the Feuerbach hyperbola of a triangle ABC is its circumscribed rectangular hyperbola with the centre at the Feuerbach point Φ of this triangle. We will show that an analogous hyperbola exists in an isotropic plane, which unites the aforementioned properties and some other properties of these two hyperbolas. It is a special hyperbola obtained in [10], i.e., the Jeřabek hyperbola of an allowable triangle in an isotropic plane.

In an isotropic plane, a triangle is allowable if none of its sides is an isotropic line. Each allowable triangle in an isotropic plane can be set by a suitable choice of coordinates in the so–called *standard position*, where its circumscribed circle has the equation $y = x^2$, and its vertices are of the form $A = (a,a^2)$, $B = (b,b^2)$, $C = (c,c^2)$, while $a+b+c = 0$ (see [11]). By using the abbreviations

$$p = abc, \qquad q = bc+ca+ab,$$

we can get some useful expressions, e.g., $a^2+b^2+c^2 = -2q$, as well as, $q = bc-a^2$, $(c-a)(a-b) = 2q-3bc$, and identities derived therefrom by a cyclic permutation of a, b and c.

In [6], it is proved that the Brocard angle of a standard triangle ABC is given by the formula

$$\omega = -\frac{1}{3q}(b-c)(c-a)(a-b).$$

We will start with the following theorem:

Theorem 1 *Let $A_tB_tC_t$ be a tangential triangle of an allowable triangle and $A'B'C'$ the triangle obtained from the triangle $A_tB_tC_t$ by any translation into an isotropic direction. The triangles ABC and $A'B'C'$ are homologic, i.e., the lines AA', BB' and CC' pass through a point T, and the points $BC\cap B'C'$, $CA\cap C'A'$, $AB\cap A'B'$ lie on one line $\mathcal{T}$ (Figure 1).*

Proof. By [1], e.g. $A_t = \left(-\frac{a}{2}, bc\right)$, and then

$$A' = \left(-\frac{a}{2}, t+bc\right), \quad B' = \left(-\frac{b}{2}, t+ca\right),$$
$$C' = \left(-\frac{c}{2}, t+ab\right), \tag{1}$$

where t is a perimeter. The line with the equation

$$y = -\frac{2(t+q)}{3a}x + \frac{2t}{3} - \frac{q}{3} + bc$$

passes through the point $A = (a, a^2)$ and the point A' from (1) because we get

$$-\frac{2(t+q)}{3a} \cdot a + \frac{2t}{3} - \frac{q}{3} + bc = bc - q = a^2,$$
$$-\frac{2(t+q)}{3a} \cdot \left(-\frac{a}{2}\right) + \frac{2t}{3} - \frac{q}{3} + bc = t + bc,$$

and it is the line AA'. It passes through the point

$$T = \left(\frac{3p}{2(t+q)}, \frac{1}{3}(2t-q)\right) \tag{2}$$

because we get

$$-\frac{2(t+q)}{3a} \cdot \frac{3p}{2(t+q)} + \frac{2t}{3} - \frac{q}{3} + bc$$
$$= -\frac{p}{a} + bc + \frac{1}{3}(2t-q) = \frac{1}{3}(2t-q),$$

and analogously, the lines BB' and CC' pass through the point T. The line with the equation

$$y = 2ax + t - a^2 \tag{3}$$

passes through the points B' and C' from (1) because e.g. for the point B' we get

$$2a\left(-\frac{b}{2}\right) + t - a^2 = t + ca.$$

Therefore this is the line $B'C'$. From its equation (3) and the equation $y = -ax - bc$ of the line BC for the abscissa x of the point $BC \cap B'C'$ we get the equation $3ax = a^2 - t - bc$ with the solution $x = -\frac{1}{3a}(t+q)$, and then the equation of the line BC implies $y = \frac{1}{3}(t+q) - bc$. So we get

$$BC \cap B'C' = \left(-\frac{1}{3a}(t+q), \frac{1}{3}(t+q) - bc\right).$$

This point lies on the line $\mathcal{T}$ with the equation

$$\mathcal{T} \quad \dots \quad y = \frac{3p}{t+q}x + \frac{t+q}{3} \tag{4}$$

because of

$$\frac{3p}{t+q} \cdot \left(-\frac{1}{3a}\right)(t+q) + \frac{t+q}{3} = \frac{1}{3}(t+q) - bc,$$

and the line $\mathcal{T}$ also passes through the analogous points $CA \cap C'A'$ and $AB \cap A'B'$. □

Corollary 1 *In the case of a standard triangle ABC, the point T and the line $\mathcal{T}$ are given by formulas (2) and (4).*

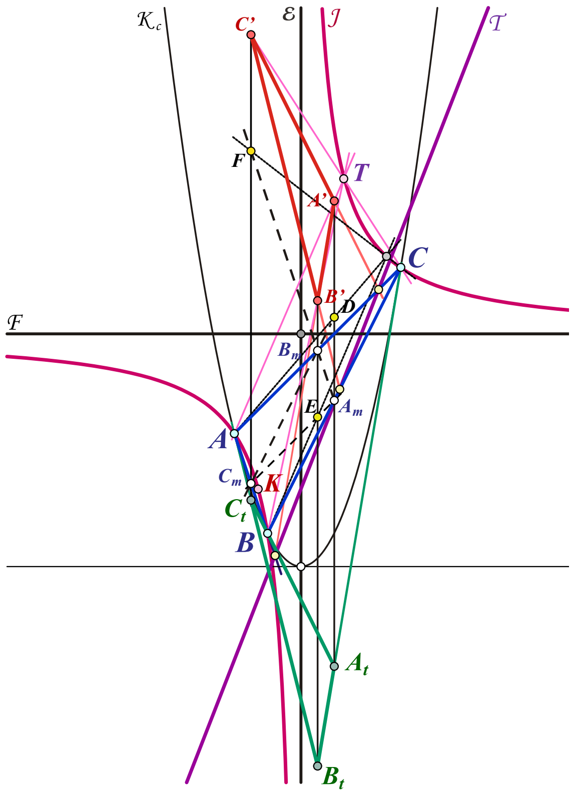

Figure 1.

Theorem 2 *The point T from Theorem 1 describes one special hyperbola $\mathcal{J}$ (Figure 1), which in the case of a standard triangle ABC has the equation*

$$xy + qx - p = 0. \tag{5}$$

Proof. The point T from (2) describes the curve $\mathcal{J}$ with the parametric equation

$$x = \frac{3p}{2(t+q)}, \quad y = \frac{1}{3}(2t-q). \tag{6}$$

This implies firstly $y + q = \frac{2}{3}(t+q)$, and then $x(y+q) = p$, i.e. the equation (5) is written in the form $y = \frac{p}{x} - q$, it implies that the curve $\mathcal{J}$ has an isotropic asymptote with the equation $x=0$, and by [11], it is the Euler line of a triangle ABC. Its nonisotropic asymptote is given by the equation $y=-q$, and by [1], it is the Feuerbach line of that triangle. The curve $\mathcal{J}$ is a special hyperbola with the centre $(0,-q)$, and by [1], it is the Feurbach point Φ of a triangle ABC. $\square$

It is shown in [10] that a hyperbola with the equation (5) is the isogonal image of the Euler line of a triangle ABC and it is called the **Jeřabek hyperbola** of that triangle.

Corollary 2 *The Jeřabek hyperbola of an allowable triangle ABC is the isogonal image of its Euler line and this line is its isotropic asymptote, while its nonisotropic asymptote is the Feuerbach line of a triangle ABC, and its centre is the Feuerbach point of that triangle (Figure 1). The Jeřabek hyperbola of a standard triangle ABC has the equation (5).*

Inserting $y=x^2$ in (5), we get the equation $x^3+qx-p=0$ for the abscissa of the intersection of the hyperbola $\mathcal{J}$ with the circumscribed circle of the triangle ABC, as the abscissas a,b,c of the points A,B,C satisfy this equation, we obtain:

Corollary 3 *The Jeřabek hyperbola of a triangle is circumscribed to this triangle, i.e., it passes through its vertices (Figure 1).*

Inserting $t=0$ in (6) we get $x=\dfrac{3p}{2q}$, $y=-\dfrac{q}{3}$ while with $t=-\dfrac{3}{2}q$, $x=-\dfrac{3p}{q}$, $y=-\dfrac{4}{3}q$ is obtained. It follows that the hyperbola $\mathcal{J}$ passes through the points

$$K=\left(\frac{3p}{2q},-\frac{q}{3}\right),\quad \Gamma=\left(-\frac{3p}{q},-\frac{4}{3}q\right),$$

being, by [8] and [2], the symmedian centre and the Gergonne point of a triangle ABC. Further, with $t=-\dfrac{3}{2}q$ in (1) we get the point $A'=\left(-\dfrac{a}{2},-\dfrac{3}{2}q+bc\right)$. This point lies on the midline B_mC_m of a triangle ABC, which, by [11], has the equation $y=-ax+\dfrac{bc}{2}-q$ and

$$-a\left(-\frac{a}{2}\right)+\frac{bc}{2}-q=\frac{1}{2}(bc-q)+\frac{bc}{2}-q=bc-\frac{3}{2}q.$$

We have obtained:

Theorem 3 *The Jeřabek hyperbola of a triangle passes through its symmedian centre and its Gergonne point. If the points D,E,F are the intersections of the perpendicular bisectors of the sides BC,CA and AB of the triangle ABC with its corresponding midlines, then the lines AD,BE and CF pass through its Gergonne point Γ.*

The geometrical results of Theorems 1, 2 and 3 and Corollaries 1, 2 and 3 are analogous to the results in the Euclidean case (see [4] and [5]).

The following two statements are analogous to Theorems 1 and 2.

Theorem 4 *Let $\mathcal{A}'$, $\mathcal{B}'$ and $\mathcal{C}'$ be the lines parallel to the bisectors of the angles A,B and C and let the distances between them be proportional to the measure of these angles of the triangle ABC. These lines determine the triangle $A''B''C''$, which is homologic to the triangle ABC. In the case of a standard triangle ABC, the centre T'' of homology is the point*

$$T''=\left(-\frac{6pt}{q(2t-3\omega)},-q-\frac{q}{6t}(2t-3\omega)\right), \tag{7}$$

and the axis of homology $\mathcal{T}''$ is given by the equation

$$\mathcal{T}'' \;\ldots\; 3q(3\omega-2t)y=9ptx+3q\omega t^2+4q^2t-3q^2\omega. \tag{8}$$

Proof. By [7], the bisector of the angle A has the equation $y=\frac{a}{2}x+\frac{a^2}{2}$, and the line $\mathcal{A}'$ has the first of the three analogous equations

$$\begin{aligned}
\mathcal{A}' \quad \ldots \quad y&=\frac{a}{2}x+\frac{a^2}{2}+\frac{1}{2}(b-c)t,\\
\mathcal{B}' \quad \ldots \quad y&=\frac{b}{2}x+\frac{b^2}{2}+\frac{1}{2}(c-a)t,\\
\mathcal{C}' \quad \ldots \quad y&=\frac{c}{2}x+\frac{c^2}{2}+\frac{1}{2}(a-b)t,
\end{aligned} \tag{9}$$

where t is the perimeter. The point

$$A''=\left(a+\frac{3at}{b-c},\frac{qt}{b-c}-\frac{bc}{2}\right) \tag{10}$$

lies on the line $\mathcal{B}'$ and $\mathcal{C}'$ from (9) because e.g. for the line $\mathcal{B}'$ we get

$$\begin{aligned}
&\frac{b}{2}\left(a+\frac{3at}{b-c}\right)+\frac{b^2}{2}+\frac{1}{2}(c-a)t\\
&=\frac{t}{2(b-c)}[3ab+(b-c)(c-a)]+\frac{b}{2}(a+b)\\
&=\frac{t}{2(b-c)}\cdot 2q-\frac{bc}{2}=\frac{qt}{b-c}-\frac{bc}{2}.
\end{aligned}$$

Hence, $A''=\mathcal{B}'\cap\mathcal{C}'$. The line with the equation

$$y=\frac{q}{6at}(2t-3\omega)x+a^2-\frac{q}{6t}(2t-3\omega)$$

obviously passes through the point$A=(a,a^2)$, as well as through the points A'' and T'' from (10) and (7) because

$$\begin{aligned}&\frac{q}{6at}(2t-3\omega)\left(a+\frac{3at}{b-c}\right)+a^2-\frac{q}{6t}(2t-3\omega)\\&=\frac{q}{2(b-c)}(2t-3\omega)+a^2\\&=\frac{qt}{b-c}+\frac{1}{2(b-c)}\cdot(b-c)(c-a)(a-b)+a^2\\&=\frac{qt}{b-c}+\frac{1}{2}(2q-3bc)+bc-q=\frac{qt}{b-c}-\frac{bc}{2},\end{aligned}$$

$$\begin{aligned}&-\frac{q}{6at}(2t-3\omega)\cdot\frac{6pt}{q(2t-3\omega)}+a^2-\frac{q}{6t}(2t-3\omega)\\&=-bc+a^2-\frac{q}{6t}(2t-3\omega)=-q-\frac{q}{6t}(2t-3\omega).\end{aligned}$$

So, the point T'' lies on the line AA'', and analogously on the lines BB'' and CC''. The point

$$L=\left(\frac{q-3bc}{3a}-\frac{b-c}{3a}t,\frac{b-c}{3}t-\frac{q}{3}\right)$$

lies on the line BC with the equation $y=-ax-bc$ and on the line $\mathcal{A}'$ from (9) because

$$\begin{aligned}&-a\left(\frac{q-3bc}{3a}-\frac{b-c}{3a}t\right)-bc=bc-\frac{q}{3}+\frac{b-c}{3}t-bc\\&=\frac{b-c}{3}t-\frac{q}{3},\\&\frac{a}{2}\left(\frac{q-3bc}{3a}-\frac{b-c}{3a}t\right)+\frac{a^2}{2}+\frac{1}{2}(b-c)t\\&=\frac{1}{6}(q-3bc)-\frac{1}{6}(b-c)t+\frac{1}{2}(b-c)t+\frac{1}{2}(bc-q)\\&=\frac{1}{3}(b-c)t-\frac{q}{3},\end{aligned}$$

and then $L=BC\cap\mathcal{A}'$. Accordingly, the point L lies on the line $\mathcal{T}''$ with the equation (8)

$$\begin{aligned}&3q(3\omega-2t)\left(\frac{b-c}{3}t-\frac{q}{3}\right)-9pt\left(\frac{q-3bc}{3a}-\frac{b-c}{3a}t\right)\\&\quad-3q\omega t^2-4q^2t+3q^2\omega\\&=-2(b-c)qt^2+3(b-c)q\omega t+2q^2t-3q^2\omega-3bc(q-3bc)t\\&\quad+3bc(b-c)t^2+(b-c)(c-a)(a-b)t^2-4q^2t+3q^2\omega\\&=[(b-c)(3bc-2q)+(b-c)(c-a)(a-b)]t^2\\&\quad+(9b^2c^2-3bcq-2q^2)t-(b-c)^2(c-a)(a-b)t\\&=[9b^2c^2-3bcq-2q^2+(q+3bc)(2q-3bc)]t=0,\end{aligned}$$

and the analogous points $CA\cap\mathcal{B}''$ and $AB\cap\mathcal{C}''$ lie on the same line. □

Theorem 5 *The point T'' from Theorem 4 describes the Jeřabek hyperbola of a triangle ABC.*

Proof. The point T'' from (7) describes the curve with the parametric equation

$$x=-\frac{6pt}{q(2t-3\omega)},\quad y=-q-\frac{q}{6t}(2t-3\omega),$$

which immediately implies $x(y+q)=p$. □

The geometrical results of Theorems 4 and 5 are analogous to the results in the Euclidean case (see [3]).

Theorem 6 *Let $A_iB_iC_i$ be a contact triangle of a triangle ABC and $A'''B'''C'''$ the triangle obtained from the triangle $A_iB_iC_i$ by any translation into an isotropic direction. The triangles ABC and $A'''B'''C'''$ are homologic, and the centre T''' of this homology describes the Jeřabek hyperbola of a triangle ABC.*

Proof. By [1], let e.g. $A_i=(-2a,bc-2q)$; so then $A'''=(-2a,t+bc-2q)$. The line with the equation

$$y=-\frac{t-q}{3a}x+\frac{t}{3}-\frac{4}{3}q+bc$$

passes through the points $A=(a,a^2)$ and A''' because

$$-\frac{t-q}{3a}\cdot a+\frac{t}{3}-\frac{4}{3}q+bc=bc-q=a^2,$$

$$-\frac{t-q}{3a}(-2a)+\frac{t}{3}-\frac{4}{3}q+bc=t+bc-2q,$$

and also through the point

$$T'''=\left(\frac{3p}{t-q},\frac{1}{3}(t-4q)\right)$$

since

$$-\frac{t-q}{3a}\cdot\frac{3p}{t-q}+\frac{t}{3}-\frac{4}{3}q+bc=-bc+\frac{t}{3}-\frac{4}{3}q+bc=\frac{1}{3}(t-4q).$$

The point T''' describes the curve with the parametric equation

$$x=\frac{3p}{t-q},\quad y=\frac{1}{3}(t-q)-q,$$

which implies $x(y+q)=p$. □

The Jeřabek hyperbola is the isogonal image of the Euler line ([10]). The following property also holds.

Theorem 7 *The Jeřabek hyperbola of an allowable triangle is the reciprocal image of the line which is anticomplementary to its Brocard diameter, i.e., the isotropic line, which passes through the Gergonne point of that triangle.*

Proof. By [13], the reciprocity with respect to the standard triangle ABC is the mapping $(x,y) \longmapsto (x',y')$, where

$$x' = -\frac{3pqx^2+4q^2xy-9py^2+9p^2x+4q^3x-12pqy-4pq^2}{q^2x^2-9pxy-3qy^2-6pqx-4q^2y+9p^2}.$$

It therefore maps the line with the equation $x' = -\frac{3p}{q}$ to the curve with the equation

$$q(3pqx^2+4q^2xy-9py^2+9p^2x+4q^3x-12pqy-4pq^2) \\ -3p(q^2x^2-9pxy-3qy^2-6pqx-4q^2y+9p^2)=0,$$

and it is the equation

$$(4q^3+27p^2)(xy+qx-p)=0,$$

i.e. equation (5) since $4q^3+27p^2 \neq 0$. □

Theorem 8 *If A_0, B_0, C_0 are the intersections of the corresponding sides of the orthic triangle $A_hB_hC_h$ and the complementary triangle $A_mB_mC_m$ of an allowable triangle ABC, then the lines A_hA_0, B_hB_0 and C_hC_0 pass through the centre Φ of the Jeřabek hyperbola of a triangle ABC.*

Proof. According to [11], the lines B_hC_h and B_mC_m have the equations

$$y = 2ax+2bc-q, \quad y=-ax+\frac{bc}{2}-q$$

and because of $bc-q=a^2$, they pass through the point

$$A_0 = \left(-\frac{bc}{2a}, a^2\right).$$

By [11], the point A_h is of the form $A_h=(a,q-2bc)$. The line with the equation

$$y=-2ax-q$$

passes through the points A_h and A_0 because of

$$-2a^2-q=-2(bc-q)-q=q-2bc,$$
$$-2a\left(-\frac{bc}{2a}\right)-q=bc-q=a^2,$$

which is the line A_hA_0. However, this line obviously passes through the point $\Phi=(0,-q)$. □

An analogous statement of Theorem 8 in the Euclidean case is given in [12].

Now we will use the parametric equation (6) of the hyperbola $\mathcal{J}$.

Theorem 9 *The points T_1 and T_2 of the Jeřabek hyperbola with the parametric equation (6), which correspond to the values t_1 and t_2 of the parameter t, have the joint line with the equation*

$$y=-\frac{4}{9p}(t_1+q)(t_2+q)x+\frac{1}{3}(2t_1+2t_2+q). \tag{11}$$

Proof. The point T from (2) with $t=t_1$ lies on the line (11) since we get

$$-\frac{4}{9p}(t_1+q)(t_2+q)\cdot\frac{3p}{2(t_1+q)}+\frac{1}{3}(2t_1+2t_2+q)$$
$$=-\frac{2}{3}(t_2+q)+\frac{1}{3}(2t_1+2t_2+q)=\frac{1}{3}(2t_1-q). \quad □$$

Corollary 4 *At the point T given by equality (2), the Jeřabek hyperbola with the equations (6) has the tangent with the equation*

$$y=-\frac{4}{9p}(t+q)^2x+\frac{1}{3}(4t+q). \tag{12}$$

Theorem 10 *The point T given by equality (2) is isogonal, with respect to a triangle ABC, to the point*

$$T'=\left(0,-\frac{2}{3}(t+q)\right). \tag{13}$$

Proof. With $x'=0$, $y'=-\frac{2}{3}(t+q)$ we get

$$y'-x'^2=-\frac{2}{3}(t+q), \qquad x'y'+qx'-p=-p,$$
$$px'-qy'-y'^2=\frac{2}{3}(t+q)\cdot\frac{1}{3}(q-2t),$$

and by [10], the point isogonal to the point T' has the coordinates

$$x=\frac{x'y'+qy'-p}{y'-x'^2}=\frac{3p}{2(t+q)},$$
$$y=\frac{px'-qy'-y'^2}{y'-x'^2}=\frac{1}{3}(2t-q),$$

i.e., that is the point T from (2). □

Theorem 11 *Let T_1, T_2 and T_3 be the points on the Jeřabek hyperbola with the equations (6), which correspond to the values t_1, t_2 and t_3 of the perimeter t, and let T_3' be the point isogonal to the point T_3 with respect to the triangle ABC. The points T_1, T_2 and T_3' are collinear if and only if*

$$t_1+t_2+t_3=-\frac{3}{2}q. \tag{14}$$

Proof. The point T_3' given by the equality (13) with $t=t_3$ lies on the line T_1T_2 with equation (11) supposing that

$$-\frac{2}{3}(t_3+q)=\frac{1}{3}(2t_1+2t_2+q),$$

that is the condition (14). □

Symmetry by t_1, t_2 and t_3 of the condition (14) gives the following statement.

Corollary 5 *Let T_1, T_2 and T_3 be the points on the Jeřabek hyperbola with the equations (6), which correspond to the values t_1, t_2 and t_3 of the perimeter t so that the equality (14) holds, and let T_1', T_2' and T_3' be the points on the Euler line of the triangle ABC isogonal, with respect to that triangle, to the points T_1, T_2 and T_3. Then the points T_1', T_2, T_3; T_1, T_2', T_3; T_1, T_2, T_3' are the triples of the collinear points, i.e., T_1, T_1'; T_2, T_2'; T_3, T_3' are the pairs of the opposite vertices of one complete quadrilateral.*

The points $\Phi = (0, -q)$ and $G = (0, -\frac{2}{3}q)$ lie on the line (11) under the condition

$$\frac{1}{3}(2t_1 + 2t_2 + q) = -q, \qquad \frac{1}{3}(2t_1 + 2t_2 + q) = -\frac{2}{3}q$$

respectively.
Hence, we get:

Corollary 6 *Points T_1 and T_2 on the Jeřabek hyperbola with the equations (6), which correspond to the values t_1 and t_2 of the perimeter t, are diametrically opposite on this hyperbola supposing that $t_1 + t_2 = -2q$, and they are collinear with the centroid G of the triangle ABC under the condition $t_1 + t_2 = -\frac{3}{2}q$.*

The value of the perimeter t, which is associated with the point A on the hyperbola (6) follows from the equality

$$\frac{3p}{2(t+q)} = a,$$

i.e., the equality $2(t+q) = 3bc$, and then $t = \frac{3}{2}bc - q$. With this value of t the right-hand side of the equation (12) gets the form

$$-\frac{4}{9p}\left(\frac{3}{2}bc\right)^2 x + \frac{1}{3}(6bc - 3q) = -\frac{bc}{a}x + 2bc - q,$$

and the following statement follows.

Theorem 12 *Tangents $\mathcal{A}, \mathcal{B}$ and $\mathcal{C}$ of the Jeřabek hyperbola of a standard triangle ABC at its vertices A, B and C have the equations*

$$y = -\frac{bc}{a}x + 2bc - q, \qquad y = -\frac{ca}{b}x + 2ca - q,$$
$$y = -\frac{ab}{c}x + 2ab - q. \tag{15}$$

Theorem 13 *If $A_iB_iC_i$ and $A_hB_hC_h$ are the contact triangle and the orthic triangle of an allowable triangle ABC, respectively, then the points $D = B_iC_i \cap B_hC_h$, $E = C_iA_i \cap C_hA_h$ and $F = A_iB_i \cap A_hB_h$ are the poles of the lines BC, CA, and AB with respect to the Jeřabek hyperbola of a triangle ABC.*

Proof. The point

$$D = \left(-\frac{2bc}{a}, -q - 2bc\right)$$

lies on the lines $\mathcal{B}$ and $\mathcal{C}$ because e.g. for the line $\mathcal{B}$ with the second equation (15) we get

$$-\frac{ca}{b}\left(-\frac{2bc}{a}\right) + 2ca - q = 2c^2 + 2ca - q = -2bc - q.$$

Therefore the point D is the pole of the line BC. According to [11] and [1], the lines B_hC_h and B_iC_i have the equations

$$y = 2ax + 2bc - q, \quad y = \frac{a}{2}x - q - bc.$$

The point D lies on these lines because of the following:

$$2a\left(-\frac{2bc}{a}\right) + 2bc - q = -q - 2bc,$$
$$\frac{a}{2}\left(-\frac{2bc}{a}\right) - bc - q = -q - 2bc. \qquad \square$$

With $t = 0$ from (12), we get the equation $y = -\frac{4q^2}{9p}x + \frac{q}{3}$ of the tangent of the Jeřabek hyperbola $\mathcal{J}$ at the symmedian centre K of a triangle ABC. This tangent obviously passes through the point $G_t = \left(0, \frac{q}{3}\right)$, which is, by [1], the centroid of a tangential triangle $A_tB_tC_t$ of a triangle ABC.
With $t = -\frac{3}{2}q$ from (12), we get the equation $y = -\frac{q^2}{9p}x - \frac{5}{3}q$ of the tangent of a hyperbola $\mathcal{J}$ at the Gergonne point Γ of a triangle ABC. This tangent obviously passes through the point $G_t = \left(0, -\frac{5}{3}q\right)$, which is, by [2], the centroid of the contact triangle $A_iB_iC_i$ of a triangle ABC, i.e., the Cevian triangle of a point Γ for a triangle ABC.

Theorem 14 *Lines parallel with the lines AP, BP and CP through the vertices A_t, B_t and C_t of the tangential triangle $A_tB_tC_t$ of a triangle ABC pass through one point P' if and only if the point P lies on the Jeřabek hyperbola $\mathcal{J}$ of a triangle ABC.*

Proof. According to [1], we get e.g. $A_t = \left(-\frac{a}{2}, bc\right)$. Let $P = (u, v)$. The line AP has the slope $(v - a^2) : (u - a)$. Its parallel line given by the equation

$$2(u-a)y = 2(v - a^2)x + 2bcu + av + aq - 3p. \tag{16}$$

goes through the point A_t because of

$$\begin{aligned}&(v - a^2)(-a) + 2bcu + av + aq - 3p\\ &\quad = a(a^2 + q) + 2bcu - 3p = abc + 2bcu - 3p\\ &\quad = 2(u-a)bc.\end{aligned}$$

The line from the equation (16) and two more analogous lines pass through one point under the condition

$$\begin{vmatrix} u-a & v-a^2 & 2bcu + av + aq - 3p \\ u-b & v-b^2 & 2cau + bv + bq - 3p \\ u-c & v-c^2 & 2abu + cv + cq - 3p \end{vmatrix} = 0.$$

As e.g.

$$\begin{aligned}&(u-b)(v-c^2)-(u-c)(v-b^2)\\&\quad=(b^2-c^2)u-(b-c)v-bc(b-c)\\&\quad=-(au+v+bc)(b-c),\end{aligned}$$

this condition can be written in the form

$$\Sigma(au+v+bc)(2bcu+av+aq-3p)(b-c)=0,$$

where Σ represents the sum of three addends, where one is always written, and the other two are obtained therefrom by a cyclic permutation of the letters a,b and c. The same condition can also be written as follows:

$$\begin{aligned}&2pu^2\Sigma(b-c)+2uv\Sigma bc(b-c)+2u\Sigma b^2c^2(b-c)\\&\quad+uv\Sigma a^2(b-c)+v^2\Sigma a(b-c)+pv\Sigma(b-c)\\&\quad+qu\Sigma a^2(b-c)+qv\Sigma a(b-c)+pq\Sigma(b-c)\\&\quad-3pu\Sigma a(b-c)-3pv\Sigma(b-c)-3p\Sigma bc(b-c)=0.\end{aligned}$$

As we have $\Sigma(b-c)=0$, $\Sigma a(b-c)=0$, $\Sigma a^2(b-c)=-(b-c)(c-a)(a-b)$, $\Sigma bc(b-c)=-(b-c)(c-a)(a-b)$, we obtain

$$\begin{aligned}&\Sigma b^2c^2(b-c)=\Sigma bc(a^2+q)(b-c)\\&\quad=p\Sigma a(b-c)+q\Sigma bc(b-c)=-q(b-c)(c-a)(a-b).\end{aligned}$$

Then the last condition, without the factor $(b-c)(c-a)(a-b)$, has the form $-2uv-2qu-uv-qu+3p=0$, i.e., in the end it has the form $uv+qu-p=0$. It means that the point P lies on the hyperbola $\mathcal{J}$. □

Theorem 15 *With the labels from Theorem 14, the point P' describes the Jeřabek hyperbola $\mathcal{J}_t$ of the tangential triangle $A_tB_tC_t$ of a triangle ABC, which in the case of the standard triangle ABC has the equation*

$$2xy+p=0. \tag{17}$$

Proof. Let $P'=(u,v)$. The line A_tP' has the slope

$$\frac{v-bc}{u+\frac{a}{2}}=\frac{2(v-bc)}{2u+a}.$$

Its parallel line given by the equation

$$(2u+a)y=2(v-bc)x+2a^2u-2av-aq+3p \tag{18}$$

passes through the point $A=(a,a^2)$ because of

$$\begin{aligned}&2(v-bc)a+2a^2u-2av-aq+3p=2a^2u-aq+abc\\&\quad=2a^2u+a\cdot a^2=(2u+a)a^2.\end{aligned}$$

The line with the equation (18) and two more analogous lines pass through one point supposing that

$$\begin{vmatrix}2u+a & v-bc & 2a^2u-2av-aq+3p\\2u+b & v-ca & 2b^2u-2bv-bq+3p\\2u+c & v-ab & 2c^2u-2cv-cq+3p\end{vmatrix}=0.$$

As e.g.

$$\begin{aligned}&(2u+b)(v-ab)-(2u+c)(v-ca)\\&\quad=-2au(b-c)+v(b-c)-a(b^2-c^2)\\&\quad=-(2au-v-a^2)(b-c),\end{aligned}$$

this condition can also be written in the form

$$\begin{aligned}&\Sigma(2au-v-a^2)(2a^2u-2av-aq+3p)(b-c)=0,\\&4u^2\Sigma a^3(b-c)-2uv\Sigma a^2(b-c)-2u\Sigma a^4(b-c)\\&\quad-4uv\Sigma a^2(b-c)+2v^2\Sigma a(b-c)+2v\Sigma a^3(b-c)\\&\quad-2qu\Sigma a^2(b-c)+qv\Sigma a(b-c)+q\Sigma a^3(b-c)\\&\quad+6pu\Sigma a(b-c)-3pv\Sigma(b-c)-3p\Sigma a^2(b-c)=0.\end{aligned}$$

In addition to the aforementioned equations from the proof of Theorem 14, the following equations also hold:

$$\begin{aligned}&\Sigma a^3(b-c)=\Sigma a(bc-q)(b-c)=p\Sigma(b-c)-q\Sigma a(b-c)=0,\\&\Sigma a^4(b-c)=\Sigma a^2(bc-q)(b-c)\\&\quad=p\Sigma a(b-c)-q\Sigma a^2(b-c)=q(b-c)(c-a)(a-b),\end{aligned}$$

where, without the factor $(b-c)(c-a)(a-b)$, the last condition gets the form $2uv-2qu+4uv+2qu+3p=0$, i.e., we have $2uv+p=0$, meaning that the point P' lies on the curve $\mathcal{J}_t$ with the equation (17). This curve is a special hyperbola with the asymptotes $x=0$ and $y=0$ and its centre is at $(0,0)$, which is, by [1], the Feuerbach point Φ_t of a triangle $A_tB_tC_t$. By [1], the circumscribed circle of that triangle has the equation $y=4x^2+q$. From this equation and the equation (17) we get the equation $8x^3+2qx+p=0$ for the abscissa of the intersection of these two curves. The solutions of this equation are the abscissas $-\frac{a}{2}$, $-\frac{b}{2}$, $-\frac{c}{2}$ of the points A_t,B_t,C_t because of

$$-\frac{a}{2}-\frac{b}{2}-\frac{c}{2}=0,\quad \frac{1}{4}(bc+ca+ab)=\frac{1}{4}q,\quad -\frac{1}{8}abc=-\frac{1}{8}p.$$

So the hyperbola $\mathcal{J}_t$ is circumscribed to a triangle $A_tB_tC_t$, and since it has a centre Φ_t, it is the Jeřabek hyperbola of that triangle. □

The symmedian centre $K=\left(\frac{3p}{2q},-\frac{q}{3}\right)$ of a triangle ABC lies on the hyperbola (17), which is in line with the fact that K is the Gergonne point of a triangle $A_tB_tC_t$.

At its point (x_0,y_0), hyperbola (17) has the tangent with the equation $x_0y+y_0x+p=0$, which in the case of the point

$K = \left(\frac{3p}{2q}, -\frac{q}{3}\right)$ gets the form

$$\frac{3p}{2q}y - \frac{q}{3}x + p = 0, \qquad \text{i.e.,} \qquad y = \frac{2q^2}{9p}x - \frac{2}{3}q,$$

and it obviously passes through the point $G = \left(0, -\frac{2}{3}q\right)$ of a triangle ABC. So we get:

Theorem 16 *The Jeřabek hyperbola of a tangential triangle of an allowable triangle passes through its symmedian centre and at this point it touches its joint line with the centroid of the given triangle.*

On the basis of Corollary 2, hyperbola $\mathcal{J}_t$ has the Euler line and the Feuerbach line of a triangle $A_tB_tC_t$ as asymptotes, i.e., the lines with the equations $x = 0$ and $y = 0$, by [9], the Euler and the dual Feuerbach line of a triangle ABC. Therefore

Corollary 7 *The Jeřabek hyperbola of a tangential triangle of an allowable triangle has the Euler and the dual Feuerbach line of that triangle as asymptotes.*

Theorem 17 *The Jeřabek hyperbola of a tangential triangle of an allowable triangle is circumscribed to its symmetrical triangle.*

Proof. According to [7], the symmetrical triangle $A_sB_sC_s$ of a triangle ABC has e.g. the vertex $A_s = \left(a, -\frac{bc}{2}\right)$, which obviously satisfies equation (17). □

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https://doi.org/10.31896/k.22.3
Original scientific paper
Accepted 30. 11. 2018.

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Curve of Brocard Points in Triangle Pencils in Isotropic Plane

Curve of Brocard Points in Triangle Pencils in Isotropic Plane

ABSTRACT

In this paper we consider a triangle pencil in an isotropic plane consisting of the triangles that have the same circumscribed circle. We study the locus of their Brocard points, two curves of order 4.

Key words: isotropic plane, triangle pencil, Brocard points

MSC2010: 51N25

Krivulje Brocardovih točaka u pramenovima trokuta u izotropnoj ravnini

SAŽETAK

U radu se promatra pramen trokuta sa zajedničkom opisanom kružnicom. Pokazuje se da Brocardove točke tih trokuta leže na dvije krivulje 4. reda.

Ključne riječi: izotropna ravnina, pramen trokuta, Brocardove točke

1 Introduction

In [1] the author gave a historical overview and presented many results regarding the Brocard points of polygons in the Euclidean plane. The Brocard points of the triangles in the isotropic plane were introduced and studied in [3], [7] and [8], while such points for harmonic quadrangles were observed in [5] and [6].

In this paper we study the pencil of triangles having the same circumscribed circle and determine the locus of their Brocard points. In a way this paper is a sequel of [2] where a similar study for curves of centroids, Gergonne points and symmedian centers in triangle pencils in the isotropic plane was given.

Let us start by recalling some basic facts about the isotropic plane. It is a real projective plane where the metric is induced by a real line f and a real point F incident with it. All straight lines through the absolute point F are called isotropic lines, and all points incident with the absolute line f are called isotropic points. Two lines are parallel if they are incident with the same isotropic point, and two points are parallel if they lie on the same isotropic line. In the affine model of the isotropic plane where the coordinates of the points are defined by $x = \frac{x_1}{x_0}$, $y = \frac{x_2}{x_0}$, the absolute line has the equation $x_0 = 0$ and the absolute point has the coordinates $(0,0,1)$. For two non-parallel points $A = (x_A, y_A)$ and $B = (x_B, y_B)$ a distance is defined by $d(A,B) = x_B - x_A$, and for two non-parallel lines p and q, given by the equations $y = k_p x + l_p$ and $y = k_q x + l_q$, an angle is defined by $\angle(p,q) = k_q - k_p$, [7]. As a circle is defined as a conic touching the absolute line at the absolute point, it has an equation of a form $y = ax^2 + bx + c$, $a,b,c \in \mathbb{R}$.

2 Brocard points

Let a triangle ABC having the circumscribed circle k with equation $y = x^2$ be given (see [4]). The triangle vertices are of the form:

$$A(a,a^2), \quad B(b,b^2), \quad C(c,c^2) \tag{1}$$

and its sides have the equations

$$\begin{aligned} AB \quad &\dots \quad y = (a+b)x - ab \\ BC \quad &\dots \quad y = (b+c)x - bc \\ CA \quad &\dots \quad y = (a+c)x - ac. \end{aligned} \tag{2}$$

The tangent lines to k at the points A, B and C are given by equations:

$$\begin{array}{lll} t_A & \dots & y=2ax-a^2 \\ t_B & \dots & y=2bx-b^2 \\ t_C & \dots & y=2cx-c^2. \end{array} \tag{3}$$

Theorem 1 *Let ABC be a triangle and let the lines a',b',c' be incident with the vertices A,B,C and form equal angles with the sides AB,BC,CA, respectively. For the triangle $A'B'C'$, where $A'=c'\cap a'$, $B'=a'\cap b'$ and $C'=b'\cap c'$, the following equalities hold:*

$$\begin{aligned} \angle(CA,AB) &= \angle(C'A',A'B') \\ \angle(AB,BC) &= \angle(A'B',B'C') \\ \angle(BC,CA) &= \angle(B'C',C'A') \end{aligned} \tag{4}$$

and

$$\frac{d(A',B')}{d(A,B)}=\frac{d(B',C')}{d(B,C)}=\frac{d(C',A')}{d(C,A)}. \tag{5}$$

Proof. Let the angle from the theorem be denoted by h, i.e.

$\angle(a',AB)=\angle(b',BC)=\angle(c',CA)=h.$

Then the lines a', b', c' are given by:

$$\begin{array}{lll} a' & \dots & y=(a+b-h)x+a(h-b) \\ b' & \dots & y=(b+c-h)x+b(h-c) \\ c' & \dots & y=(c+a-h)x+c(h-a) \end{array} \tag{6}$$

and their intersections are

$$\begin{array}{l} A'\left(a-\dfrac{a-c}{b-c}h,\quad a^2-\dfrac{(a+b)(a-c)}{b-c}h+\dfrac{a-c}{b-c}h^2\right) \\ B'\left(b-\dfrac{b-a}{c-a}h,\quad b^2-\dfrac{(b+c)(b-a)}{c-a}h+\dfrac{b-a}{c-a}h^2\right) \\ C'\left(c-\dfrac{c-b}{a-b}h,\quad c^2-\dfrac{(c+a)(c-b)}{a-b}h+\dfrac{c-b}{a-b}h^2\right). \end{array} \tag{7}$$

It follows from (2) and (6) that:

$\angle(CA,AB)=(a+b)-(c+a)=b-c$

and

$\angle(C'A',A'B')=\angle(c',a')=(a+b-h)-(c+a-h)=b-c.$

Therefore, $\angle(CA,AB)=\angle(C'A',A'B')$. The other two equalities of (4) can be proved analogously. From (7) we get:

$$\begin{aligned} d(A',B') &= b-\frac{b-a}{c-a}h-a+\frac{a-c}{b-c}h \\ &= (b-a)+h\cdot\frac{ab+bc+ac-a^2-b^2-c^2}{(b-c)(c-a)} \\ &= (b-a)\left[1-h\cdot\frac{ab+bc+ac-a^2-b^2-c^2}{(a-b)(b-c)(c-a)}\right]. \end{aligned}$$

Thus,

$$\frac{d(A',B')}{d(A,B)}=1-h\cdot\frac{ab+bc+ca-a^2-b^2-c^2}{(a-b)(b-c)(c-a)}. \tag{8}$$

Similarly we get that the ratios $\dfrac{d(B',C')}{d(B,C)}$ and $\dfrac{d(C',A')}{d(C,A)}$ take the same value. □

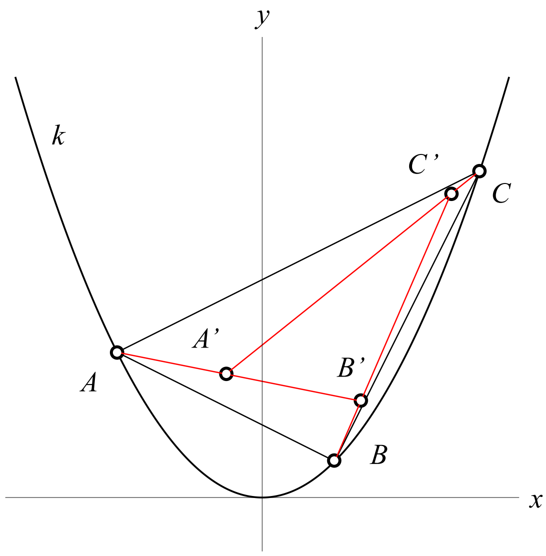

Figure 1: *Visualization of Theorem 1*

It follows immediately from (8):

Corollary 1 *Let ABC be a triangle and let the lines a',b',c' be incident with the vertices A, B, C and form equal angles h with the sides AB, BC, CA, respectively. The lines lines a', b', c' are concorrent if and only iff*

$$h=\frac{(a-b)(b-c)(c-a)}{ab+bc+ca-a^2-b^2-c^2}. \tag{9}$$

The angle h from Corollary 1 is called the Brocard angle, and the point P_1 incident with the lines a', b', c', is called the *first Brocard point* of the triangle ABC.
From (7) we get the coordinates of P_1:

$$\begin{aligned} x &= \frac{a+b+c}{3}-\frac{h}{3}\left(\frac{a-c}{b-c}+\frac{b-a}{c-a}+\frac{c-b}{a-b}\right) \\ y &= \frac{a^2+b^2+c^2}{3}+\frac{h^2}{3}\left(\frac{a-c}{b-c}+\frac{b-a}{c-a}+\frac{c-b}{a-b}\right) \\ &\quad -\frac{h}{3}\left(\frac{(a+b)(a-c)}{b-c}+\frac{(b+c)(b-a)}{c-a}+\frac{(c+a)(c-b)}{a-b}\right), \end{aligned} \tag{10}$$

where h is given by (9).

Similarly, the *second Brocard point* P_2 is defined as the point such that its connection lines with the vertices A,B,C form the equal angles with the sides AC, CB, and BA, respectively. These angles equal $-h$.

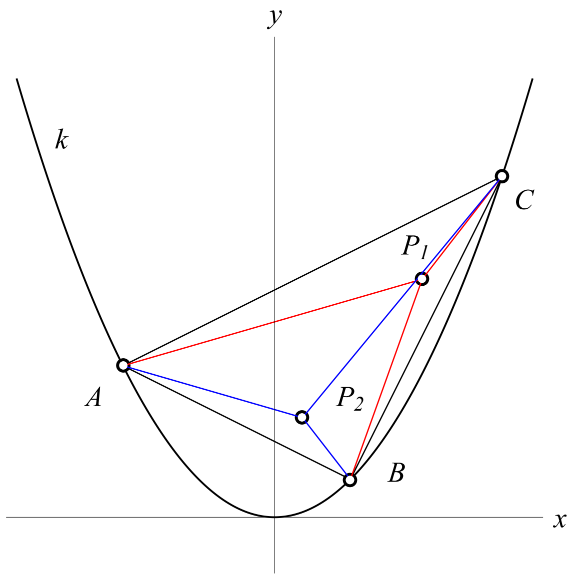

Figure 2: *The Brocard points P_1 and P_2 of the triangle ABC*

3 Triangle pencils

In order to get the pencil of the triangles, we will keep the vertices A and B fixed and move vertex C along the circumscribed circle k. Now the expressions (10) present the parametric equation of the locus of the first Brocard points, a curve k_1. By eliminating c we get an implicit equation of k_1:

$$x^4 - 2ax^3 - x^2y + (2a^2 + 2ab + b^2)x^2 - 2bxy + y^2 - 2ab(a+b)x + a(2b-a)y + a^2b^2 = 0. \tag{11}$$

It is of degree 4, so k_1 is a curve of order 4. Its only intersection point with the absolute line is the absolute point. It intersects the circle k in two basic points A and B, both with intersection multiplicity 2. It can be easily checked that A is a cusp of k_1 with tangent line AB, while B is a regular point at which k_1 touches k. The points A and B are Brocard points of two degenerated triangles of the pencil obtained when the third vertex C coincide with A and B, respectively. This observation can be summerized in:

Theorem 2 *Let the points A and B on the circle k be given. The curve of the first Brocard points of all triangles ABC having the same circumscribed circle k is a curve of order 4. It has a cusp in the point A and touches k at the point B.*

Analogously, it can be shown that the equation of that locus of the second Brocard points is

$$x^4 - 2bx^3 - x^2y + (a^2 + 2ab + 2b^2)x^2 - 2axy + y^2 - 2ab(a+b)x + 2aby - b^2y + a^2b^2 = 0 \tag{12}$$

and that the following theorem holds:

Theorem 3 *Let the points A and B on the circle k be given. The curve of the second Brocard points of all triangles ABC having the same circumscribed circle k is a curve of order 4. It has a cusp in the point B and touches k at the point A.*

Two Brocard curves k_1 and k_2 given by (11) and (12) intersect in points $\left(\frac{a+b}{2}, \frac{5a^2+5b^2-2ab\pm\sqrt{5}(a-b)^2}{8}\right)$ parallel to the midpoint $M_{AB}\left(\frac{a+b}{2}, \frac{a^2+b^2}{2}\right)$ of the points A and B.

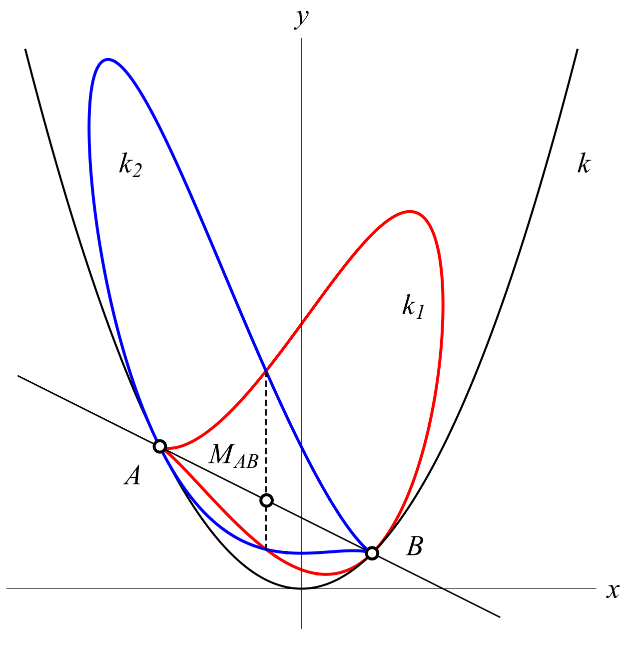

Figure 3: *The curves k_1 and k_2 of the first and second Brocard points for a pencil of triangles with the same circumscribed circle k*

Now, we will study the Brocard curves of the pencil of tangential triangles. The tangential triangle $A_tB_tC_t$ of a given triangle ABC is a triangle formed by the tangent lines to the circumscribed circle k of the triangle ABC at its vertices. The equations of the tangent lines are given by (3) and they intersect in the points $A_t = \left(\frac{b+c}{2}, bc\right)$, $B_t = \left(\frac{a+c}{2}, ac\right)$ and $C_t = \left(\frac{a+b}{2}, ab\right)$, parallel to the midpoints of the sides BC, CA and AB, respectively.
Keeping the points A and B fixed and moving C on the circle k, we obtain the pencil of tangential triangles $A_tB_tC_t$. The triangles of this pencil have the same inscribed circle, the circle k. They have one fixed vertex C_t, and two fixed sides t_A and t_B. Repeating the procedure from above we

calculate the locus of the first Brocard points of the pencil of tangential triangles:

$$
\begin{aligned}
&32ax^5-16x^4y-16ab(5a+3b)x^4+24(a+b)x^3y+4x^2y^2\\
&+8(10a^3+13a^2b+3ab^2+b^3)x^3-4(3a^2+8ab+6b^2)x^2y\\
&-2(2a-b)xy^2-y^3-4(10a^4+21a^3b+11a^2b^2+ab^3+3b^4)x^2\\
&+2(a^3+7a^2b+7ab^2+7b^3)xy+(a^2+2ab-2b^2)y^2\\
&+2a^3(5a^2+15ab+13b^2)x-b(2a^3+4a^2b+2ab^2+3b^3)y\\
&-a^6-4a^5b-5a^4b^2-2a^3b^3-b^6=0. \qquad (13)
\end{aligned}
$$

Thus, the curve of the first Brocard points k_{t1} is a curve of order 5. It intersects the absolute line at the absolute point with the intersection multiplicity 4, and at the isotropic point of the line t_A. It has a singular point at C_t since every line through C_t intersects t_A in the point C_t counted three times and two further points. This fact can be proved by putting $y=m(x-\frac{a+b}{2})+ab$, $m\in\mathbb{R}$, into (13), which than becomes an equation in x with a triple root $x=\frac{a+b}{2}$. Only in the special case when $y=2bx-b^2$, the equation (13) takes the form $(a-b)(a+b-2x)^5=0$ and $x=\frac{a+b}{2}$ is its fivefold root. Thus, all tangents to k_{t1} at C_t coincide with t_B, Figure 4.

The similar study can be done for the curve of the second Brocard points k_{t2} and the analogous results would be obtained. Therefore, we can conclude our observation with the following:

Theorem 4 *The curves of Brocard points of all the tangential triangles in the pencil of triangles having the same circumscribed circle are the curves of order 5.*

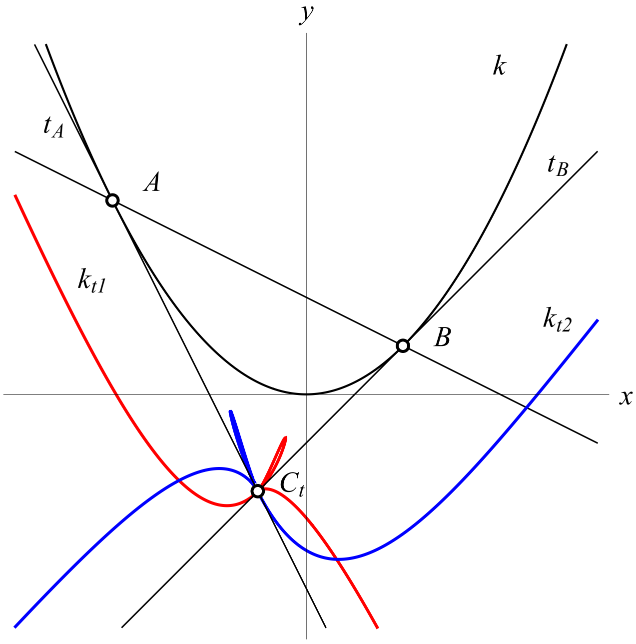

Figure 4: *The Brocard curves k_{t1} and k_{t2} in the pencil of tangential triangles of the triangles with the same circumscribed circle k*

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https://doi.org/10.31896/k.22.4
Original scientific paper
Accepted 19. 10. 2018.

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The Universal Parabola

The Universal Parabola

ABSTRACT

We develop classical properties, as well as some novel facts, for the parabola using the more general framework of rational trigonometry. This extends the study of this conic to general fields.

Key words: parabola, rational trigonometry, conic

MSC2010: 51N20, 14H50

Univerzalna parabola

SAŽETAK

Dokazujemo neka klasična svojstva kao i neke nove činjenice o paraboli koristeći okvir racionalne trigonometrije. Proširujemo proučavanje konika na opća polja.

Ključne riječi: parabola, racionalna trigonometrija, konika

1 Introduction

Next to the circle, the parabola is perhaps the most accessible conic section. It was studied by Menaechmus, who used it to duplicate the volume of a cube. Apollonius gave it its name, and deduced many important properties. Archimedes studied areas of parabolic arcs, Euclid mentions the parabola, and Pappus investigated the focus and directrix. Galileo showed that projectiles follow parabolic arcs. The reflective property was studied by Gregory and Newton. The parabola appears in car headlights, solar ovens, telescopes, astronomical radio dishes, the orbits of comets, architecture and whenever one variable is proportional to the square of another.

Classical geometry considers the parabola to be an element of Euclidean geometry over the field of decimal or 'real' numbers. Treatises which establish some of these properties include [3], [4], [6], [7], [8], [9], [10], [11], [12], [13] and [14]. From our point of view, this traditional aspect is but a shadow of the *true parabola*, which is an object that properly lives in *universal geometry*, a form of Euclidean metrical geometry that is valid over a *general field*. There are parabolas defined over finite fields, over the complex numbers, and over the p-adic numbers. With universal geometry we may investigate properties of parabolas that are shared in these different contexts, in other words that hold in complete generality.

This may well strike the reader as curious. One of the most familiar properties of a parabola is the reflective property—any light beam coming in parallel to the axis and reflected off the parabola so as to make equal angles with the tangent line passes through the focus. How is one to even state such a fact over say the finite field $\mathbb{F}_{11}$ where angles make no sense?

The answer is to free oneself from the straightjacket of traditional geometric thought. Distance and angle are not really the mathematically fundamental concepts that we like to believe. Euclid carefully avoided mentioning these metrical notions because of their attendant irrationalities. Make the shift to *quadrance* and *spread*, and you have an entirely new and simplified way of thinking about metrical geometry, which allows you to study parabolas and other conic sections in the universal setting, as well as much else besides, as shown in the recent book [16]. This can then be extended also to hyperbolic geometry, as in [1] and [2].

This paper derives numerous properties of the universal parabola. Some of the theorems are extensions of familiar and classical results, suitably restated in the new language to hold in an arbitrary field. Others are new even in the familiar setting. The diagrams mostly illustrate the situation in the familiar domain of the rational numbers, or numerically the decimal number field.

2 Universal geometry

Universal geometry, introduced in [16], is a form of Euclidean geometry that holds over any field, *characteristic two (and sometimes three) excluded.* Distance and angle are replaced by algebraic analogs; the separation of points is measured using *quadrance*, and the separation of lines is measured using *spread*. The following definitions and results are taken from [16].

Fix a field $\mathbb{F}$ whose elements are called **numbers**. We will throughout assume that the characteristic of this field *is not two*. That means that the number 2 is always invertible in $\mathbb{F}$. A **point** is an ordered pair of numbers, denoted $A \equiv [x,y]$. A **side** $\overline{A_1A_2}$ is a set consisting of two points A_1 and A_2. The **midpoint** of the side $\overline{A_1A_2}$, where $A_1 = [x_1,y_1]$ and $A_2 = [x_2,y_2]$, is the point

$$M = \left[\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right].$$

A **line** is a proportion $(a:b:c)$ of numbers with at least one of a and b non-zero. The point $A \equiv [x,y]$ **lies on** the line $l \equiv (a:b:c)$ precisely when $ax+by+c=0$, which is called the **equation** of the line. Equivalently we say l **passes through** A.

A line $(a:b:c)$ is a **null line** precisely when $a^2+b^2=0$. Null lines do not occur over the rational or decimal number fields, but they occur whenever -1 is a square. In fact if $i^2=-1$ then any null line has the form $(1:\pm i:c)$ for some c.

Theorem 1 (Collinear points) *The points $[x_1,y_1]$, $[x_2,y_2]$ and $[x_3,y_3]$ are collinear (meaning they lie on the same line) precisely when*

$$x_1y_2 - x_1y_3 + x_2y_3 - x_3y_2 + x_3y_1 - x_2y_1 = 0.$$

Theorem 2 (Concurrent lines) *If the lines $(a_1:b_1:c_1)$, $(a_2:b_2:c_2)$ and $(a_3:b_3:c_3)$ are concurrent (meaning they pass through the same point) then*

$$a_1b_2c_3 - a_1b_3c_2 + a_2b_3c_1 - a_3b_2c_1 + a_3b_1c_2 - a_2b_1c_3 = 0.$$

Definition 1 *The lines $l_1 = (a_1:b_1:c_1)$ and $l_2 \equiv (a_2:b_2:c_2)$ are* ***parallel*** *precisely when*

$$a_1b_2 - a_2b_1 = 0.$$

Definition 2 *The lines $l_1 = (a_1:b_1:c_1)$ and $l_2 \equiv (a_2:b_2:c_2)$ are* ***perpendicular*** *precisely when*

$$a_1a_2 + b_1b_2 = 0.$$

Definition 3 *The* ***quadrance*** *$Q(A_1,A_2)$ between the points $A_1 \equiv [x_1,y_1]$ and $A_2 \equiv [x_2,y_2]$ is the number*

$$Q(A_1,A_2) \equiv (x_2-x_1)^2 + (y_2-y_1)^2.$$

Definition 4 *The* ***spread*** *$s(l_1,l_2)$ between the non-null lines $l_1 = (a_1:b_1:c_1)$ and $l_2 \equiv (a_2:b_2:c_2)$ is the number*

$$s(l_1,l_2) \equiv \frac{(a_1b_2-a_2b_1)^2}{(a_1^2+b_1^2)(a_2^2+b_2^2)}.$$

Two lines l_1 and l_2 are perpendicular precisely when $s(l_1,l_2)=1$. If $\overline{A_1A_2A_3}$ is a right triangle with A_1A_3 perpendicular to A_2A_3, then the Spread ratio theorem asserts that

$$s(A_1A_3, A_1A_2) = \frac{Q_1}{Q_3}.$$

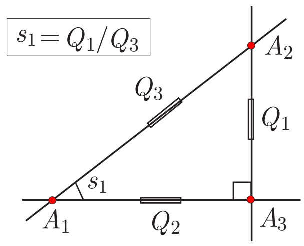

Figure 1: *Spread as ratio*

For any point $A \equiv [x,y]$ and any line $l \equiv (a:b:c)$ there is a unique line n, called the **altitude from** A **to** l, which passes through A and is perpendicular to l, namely

$$n = (-b : a : bx - ay).$$

If l is non-null, then this altitude intersects l at the point

$$N \equiv \left[\frac{b^2x - aby - ac}{a^2+b^2}, \frac{-abx + a^2y - bc}{a^2+b^2}\right]$$

called the **foot** of the altitude. The **quadrance** $Q(A,l)$ between the point A and the line l is then defined to be $Q(A,N)$. This turns out to be

$$Q(A,l) = \frac{(ax+by+c)^2}{a^2+b^2}.$$

3 Isometries and similarities

Definition 5 *An* ***isometry*** *σ is a function that inputs and outputs points, such that for any points A_1 and A_2*

$$Q(A_1,A_2) = Q(\sigma(A_1), \sigma(A_2)).$$

Definition 6 *A* ***similarity*** *τ is a function that inputs and outputs points, such that for any points A_1, A_2 and A_3,*

$$Q(A_1,A_2)\,Q(\tau(A_2),\tau(A_3)) = Q(A_2,A_3)\,Q(\tau(A_1),\tau(A_2)).$$

Every isometry is a similarity, but scalings are similarities which are not generally isometries. A similarity preserves the ratio of quadrances, so it preserves the spread between lines.

Theorem 3 *For any numbers a and b, the function*

$\tau([x,y]) = [ax+by, bx-ay]$

is a similarity.

Proof. Suppose that $A_1 = [x_1, y_1]$, $A_2 = [x_2, y_2]$. Then

$$\begin{aligned}
&Q(\tau(A_1), \tau(A_2)) \\
&= Q([ax_1+by_1, bx_1-ay_1], [ax_2+by_2, bx_2-ay_2]) \\
&= (ax_2+by_2-ax_1-by_1)^2 + (bx_2-ay_2-bx_1+ay_1)^2 \\
&= \left(a^2+b^2\right)\left(x_1^2 - 2x_1x_2 + x_2^2 + y_1^2 - 2y_1y_2 + y_2^2\right) \\
&= \left(a^2+b^2\right) Q(A_1, A_2)
\end{aligned}$$

from which the result follows. □

The transformation τ above is affine, implying that lines are sent to lines. The point $[x,y]$ lies on the line $ax+by+c=0$ precisely when $\tau([x,y])$ lies on the line $x+c=0$. The inverse transformation to τ is defined only if $a^2+b^2 \neq 0$ and is

$$\tau^{-1}([x,y]) = \left[\frac{ax+by}{a^2+b^2}, \frac{bx-ay}{a^2+b^2}\right].$$

4 Parabolas

A conic is given by a polynomial equation in x and y of degree exactly two. A **circle** is a conic whose equation in $X \equiv [x,y]$ has the form $Q(X,C) = R$ for some point C and some number R. Some basic facts about circles will be taken from [16], such as the fact that if A and B lie on the circle, then the spread $s(AX, BX)$ is constant for all X lying on the circle, and is equal to 1 precisely when AB is a diameter of the circle (i.e. passes through the center C).

Definition 7 *A* ***parabola*** *p is a conic whose equation in* $X \equiv [x,y]$ *has the form*

$Q(X,F) = Q(X,l)$

for some fixed point F and some fixed non-null line l not passing through F.

A point $A = [x,y]$ **lies on** p precisely when x and y satisfy the equation of p. It is a theorem in [16] that the point F and the line l are determined by p. The point F is the **focus** and the line l the **directrix** of p. The **axis** of the parabola p is the altitude from F to l. The **vertex** of the parabola is the midpoint of the side $\overline{FP}$ where P is the foot of the altitude from F to the directrix l. It lies on the axis and also on the parabola p.

Example 1 *If* $\lambda \neq 0$ *then the parabola p with focus* $F \equiv [\lambda, 0]$ *and directrix* $x+\lambda = 0$ *has equation*

$$(x-\lambda)^2 + y^2 = (x+\lambda)^2 \quad \text{or} \quad y^2 = 4\lambda x.$$

Every point lying on this parabola has the form $[\lambda a^2, 2\lambda a]$ *for some number a, and every such point lies on the parabola.*

To prove theorems about parabolas, it is very convenient to work with this simple equation. The following useful result shows how this can be done.

Theorem 4 *Any parabola can be transformed by a similarity to one of the form* $y^2 = 4\lambda x$ *for some number* λ.

Proof. If the directrix of the parabola p is $ax+by+c=0$ then the similarity

$$\tau([x,y]) = [ax+by, bx-ay]$$

takes this line to $x+c=0$. This transformation is invertible since the directrix of a parabola is a non-null line, so that $a^2+b^2 \neq 0$. A translation (in the y direction) now moves the focus to the form $[\beta, 0]$ and leaves the directrix unchanged. Then the translation (in the x direction)

$$[x,y] \to [x-(\beta-c)/2, y]$$

takes the focus to $[(\beta+c)/2, 0]$ and the directrix to $x+(\beta+c)/2 = 0$ which has the required form, with $\lambda = (\beta+c)/2$. □

Although the transformation used here is a similarity, not in general an isometry, it is sufficient to prove all the theorems in this paper, which are ultimately not about individual quadrances, but always only about the *proportions* between quadrances.

The parabola p with equation $y^2 = 4\lambda x$ has focus $F = [\lambda, 0]$, directrix $(1:0:\lambda)$, axis the line $(0:1:0)$ and vertex the point $[0,0]$. If A and B are two distinct points lying on p, then the line AB is a **chord** of the parabola, while the side $\overline{AB}$ is a **side** of the parabola. A chord passing through the focus F is a **focal chord**. The chord of a parabola through its focus which is perpendicular to its axis is called the **latus rectum**. A **null point of** p is a point N lying on p such that $Q(N,F) = 0$.

Tangent lines to conics are best defined algebraically, since limiting procedures are not available for general fields. Given a conic which passes through the origin with equation such as $x^2 - 3xy + 5y^2 + 3x - 2y = 0$, the **tangent line** at the origin is defined to be just the linear part of this expression, that is the line $3x - 2y = 0$. To define the tangent at a general point of a general conic, first translate so that the point is at the origin, then take the linear part, and translate back. This is explained in more detail in [16], which also contains the following result, generalizing the most well-known property of a parabola to the universal case.

Theorem 5 *Suppose k is the tangent line to a parabola p at a point A lying on it. Then $s(k,m) = s(k,n)$ where $m = AF$ with F the focus of p, and n is the altitude from A to the directrix of p.*

The general point on $y^2 = 4\lambda x$ is $[x_1, y_1] = [\lambda t^2, 2\lambda t]$ in terms of a parameter t. The **tangent** to p at this point is the line $(2\lambda : -y_1 : 2\lambda x_1) = \left(1 : -t : \lambda t^2\right)$. A **null tangent** to a parabola is a null line that is tangent to the parabola, clearly this occurs precisely when $t^2 = -1$. For two different values of t, say $t = a$ and $t = b$ we get two points $A \equiv [\lambda a^2, 2\lambda a]$ and $B \equiv [\lambda b^2, 2\lambda b]$, which determine the chord $(2 : -(a+b) : 2\lambda ab)$. The tangents to these points intersect at the point

$$X = [x_0, y_0] = [\lambda ab, \lambda(a+b)]$$

which is the **external point** of the chord AB. The side $\overline{AB}$ is the **side** determined by X, and the chord AB is the **chord of contact** of X; it has the form $(2\lambda : -y_0 : 2\lambda x_0)$.

Summary of formulae Let p be the parabola with equation $y^2 = 4\lambda x$. Let $[x_1, y_1]$ denote a general point on the parabola, and $A \equiv [\lambda a^2, 2\lambda a]$ and $B \equiv [\lambda b^2, 2\lambda b]$ two specific points on the parabola. Let $[x_0, y_0]$ be the external point of the chord AB. Let $[\lambda t^2, 2\lambda t]$ be the parametric form of p.

The tangent to p at $[x_1, y_1]$ in Cartesian form

$$(2\lambda : -y_1 : 2\lambda x_1) \tag{1}$$

The tangent to p at $A = [\lambda a^2, 2\lambda a]$ in parametric form

$$\left(1 : -t : \lambda t^2\right) \tag{2}$$

The chord AB in Cartesian form

$$(2\lambda : -y_0 : 2\lambda x_0) \tag{3}$$

The chord AB in parametric form

$$(2 : -(a+b) : 2\lambda ab) \tag{4}$$

The external point of the chord AB in parametric form

$$[\lambda ab, \lambda(a+b)]. \tag{5}$$

5 Tangents and external points

Theorem 6 *No two tangents to a parabola are parallel.*

Proof. Let the equation of the parabola be

$$y^2 = 4\lambda x$$

for some $\lambda \neq 0$. Then the tangents to the parabola at two distinct points $[x_1, y_1]$, $[x_2, y_2]$ are

$$(2\lambda : -y_1 : 2\lambda x_1) \quad \text{and} \quad (2\lambda : -y_2 : 2\lambda x_2)$$

respectively. But since $y_1 \neq y_2$, these lines are not parallel. □

Theorem 7 *No three tangents to a parabola are concurrent.*

Proof. Since affine transformations preserve concurrence of lines, it suffices to prove this for the parabola

$$y^2 = 4\lambda x.$$

Let

$$A \equiv [\lambda a^2, 2\lambda a] \quad B \equiv [\lambda b^2, 2\lambda b] \quad \text{and} \quad C \equiv [\lambda c^2, 2\lambda c].$$

The tangents to the parabola at these points are

$$\left(1 : -a : \lambda a^2\right) \quad \left(1 : -b : \lambda b^2\right) \quad \text{and} \quad \left(1 : -c : \lambda c^2\right)$$

respectively. Suppose the three tangents are concurrent, then by the Concurrent lines theorem

$$\begin{vmatrix} 1 & -a & \lambda a^2 \\ 1 & -b & \lambda b^2 \\ 1 & -c & \lambda c^2 \end{vmatrix} = 0.$$

Expand the determinant to get

$$\lambda(a-b)(c-a)(c-b) = 0 \tag{6}$$

Since A, B and C are all distinct points, a, b and c are also distinct. This implies $\lambda = 0$, which is impossible. □

Theorem 8 *There is a unique pair of tangents to a parabola through an external point.*

Proof. By Theorem 7 no three tangents are concurrent, hence there is no external point with two distinct pairs of tangents to the parabola. □

6 Chords of a parabola

Theorem 9 *Suppose A and B are two points on a parabola. Then the following are equivalent.*

1. *AB is a focal chord*
2. *The tangents at A and B are perpendicular*
3. *The tangents at A and B intersect on the directrix.*

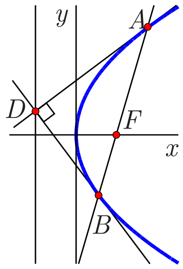

Figure 2.

Proof. $1 \Rightarrow 2$) Let $A \equiv [\lambda a^2, 2\lambda a]$ and $B \equiv [\lambda b^2, 2\lambda b]$ and $[x_0, y_0]$ be the external point of the chord AB. By (2) the tangents to p at A and B are

$$\left(1 : -a : \lambda a^2\right) \quad \text{and} \quad \left(1 : -b : \lambda b^2\right)$$

respectively. Since the line AB passes through the focus $[\lambda, 0]$, from (4) we deduce that

$$ab = -1.$$

Hence the two tangents are perpendicular.
$2 \Rightarrow 3$) If the tangents are perpendicular, then from (2) $ab = -1$. By comparing (4) with (3), $x_0 = -\lambda$. Therefore the external point of the chord AB lies on the directrix. Hence the two tangents intersect on the directrix.
$3 \Rightarrow 1$) Suppose two tangents to the parabola intersect on the directrix i.e. $x_0 = -\lambda$. Then from (3) the chord AB is

$$\left(2\lambda : -y_0 : -2\lambda^2\right)$$

which passes through the point $[\lambda, 0]$. Hence AB is a focal chord. □

Theorem 10 *Any circle whose diameter is a focal chord of a parabola touches the directrix.*

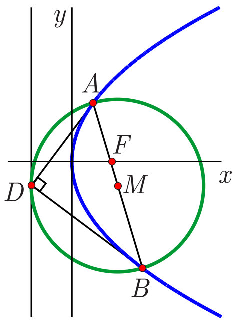

Figure 3.

Proof. Since the tangents to p at $A = [\lambda a^2, 2\lambda a]$ and $B = [\lambda b^2, 2\lambda b]$ intersect at

$$D \equiv [\lambda ab, \lambda(a+b)]$$

we may compute that

$$AD = \left(1 : -a : a^2\lambda\right) \quad \text{and} \quad BD = \left(1 : -b : b^2\lambda\right).$$

If AB is a focal chord then from the proof of the previous proposition, $ab = -1$,so that AD and BD are perpendicular. Then D lies on the circle with AB as its diameter. In particular, the midpoint

$$M \equiv \left[\frac{\lambda(a^2+b^2)}{2}, \lambda(a+b)\right]$$

of the side $\overline{AB}$ is the center of the circle. The equation of the line MD is

$$(0 : 1 : -\lambda(a+b))$$

which is perpendicular to the directrix. Therefore the directrix is tangent to the circle. □

Theorem 11 *Let A be a point on a parabola with focus F. Let D be the foot of the altitude from A to the directrix, and T be the point where the tangent at A meets the axis of the parabola. Then the side $\overline{FD}$ is perpendicular to the side $\overline{AT}$ and they share a common midpoint.*

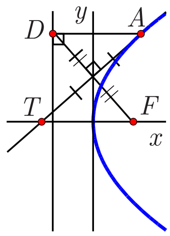

Figure 4.

Proof. Let $A \equiv [\lambda a^2, 2\lambda a]$. By (2) the tangent at A is

$$\left(1 : -a : \lambda a^2\right)$$

which meets the axis at

$$T = [-\lambda a^2, 0].$$

By the foot of an altitude formula

$$D = [-\lambda, 2\lambda a].$$

So the line AT and the line

$$DF = (a : 1 : -\lambda a)$$

are perpendicular. Moreover the midpoints of the side $\overline{AT}$ and $\overline{DF}$ are both

$$[0, \lambda a].$$ □

Theorem 12 *Let AB be a chord of a parabola which is parallel to the directrix. Let E be a third point on the parabola. Suppose the lines EA and EB intersect the axis of the parabola at C and D respectively. Then the midpoint of the side $\overline{CD}$ is the vertex of the parabola.*

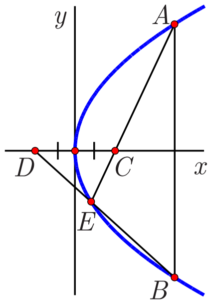

Figure 5.

Proof. Let

$$A \equiv [\lambda a^2, 2\lambda a] \quad B \equiv [\lambda b^2, 2\lambda b] \quad \text{and} \quad E \equiv [\lambda e^2, 2\lambda e].$$

Since the line AB is parallel to the directrix by (4)

$$a+b=0.$$

From (4) the line EA intersects the axis of parabola at

$$C \equiv [-\lambda ea, 0].$$

Similarly the line EB intersects the axis of the parabola at

$$D \equiv [-\lambda eb, 0].$$

Therefore the midpoint of $\overline{CD}$ is

$$\left[\frac{-\lambda e(a+b)}{2}, 0\right] = [0,0]$$

which is the vertex of the parabola. □

Theorem 13 *Let $\overline{AB}$ be the side of an external point X to a parabola. Let M be the midpoint of the side $\overline{AB}$. Then*

1. *MX is parallel to the axis of the parabola.*
2. *The midpoint T of the side $\overline{MX}$ lies on the parabola.*
3. *The tangent at T is parallel to the chord AB.*

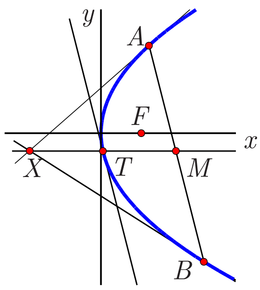

Figure 6.

Proof. 1.) Let $A \equiv [\lambda a^2, 2\lambda a]$ and $B \equiv [\lambda b^2, 2\lambda b]$. Then the midpoint of the side $\overline{AB}$ is

$$M = \left[\frac{\lambda(a^2+b^2)}{2}, \lambda(a+b)\right].$$

By (5) the external point of $\overline{AB}$ is

$$X = [\lambda ab, \lambda(a+b)]$$

which has the same y component as M. Hence MX is parallel to the axis $(0:1:0)$ of the parabola.
2.) The midpoint T of the side $\overline{MX}$ is

$$\left[\frac{\lambda(a+b)^2}{4}, \lambda(a+b)\right]$$

which lies on the parabola, with parameter $t=(a+b)/2$.
3.) By (2) the tangent at T is

$$\left(4 : -2(a+b) : \lambda(a+b)^2\right).$$

By (4) the line containing the chord is

$$(2 : -(a+b) : 2\lambda ab)$$

which is parallel to the tangent at T. □

Theorem 14 *Let AB be the chord from an external point X to a parabola with focus F. Suppose the line AB meets the axis at I, and that the line through X parallel to the axis meets the parabola at N. Then*

1. *The midpoint of the side $\overline{XI}$ lies on the tangent at the vertex of parabola.*
2. *The tangent at N is parallel to the chord AB.*

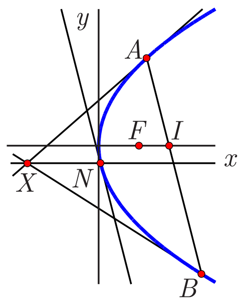

Figure 7.

Proof. Let $A \equiv [\lambda a^2, 2\lambda a]$ and $B \equiv [\lambda b^2, 2\lambda b]$ as usual.
1.) By (4)

$$I = [-\lambda ab, 0]$$

and by (5)

$$X = [\lambda ab, \lambda(a+b)].$$

Therefore the midpoint of the side $\overline{XI}$ is

$$\left[0, \frac{\lambda(a+b)}{2}\right]$$

which lies on $(1:0:0)$, the tangent at the vertex to the parabola.
2.) The line through X parallel to the axis of the parabola intersects the parabola at

$$N \equiv \left[\frac{\lambda(a+b)^2}{4}, \lambda(a+b)\right].$$

By (1) the tangent at N is

$$\left(4 : -2(a+b) : \lambda(a+b)^2\right)$$

which is parallel to line AB from (4). □

Theorem 15 *For any point C there is a line $l = l(C)$ with the property that if a chord AB of the parabola p passes through C, then the external point of the chord lies on l.*

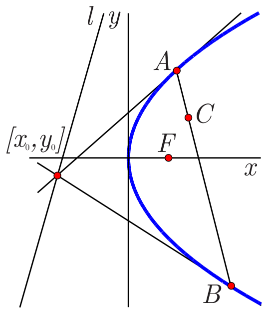

Figure 8.

Proof. Let the external point of the chord be $[x_0, y_0]$ and $C \equiv [h,k]$. By assumption the chord passes through $[h,k]$, so by (3)

$$2\lambda h - y_0 k + 2\lambda x_0 = 0.$$

Hence $[x_0, y_0]$ lies on the line

$$l \equiv (2\lambda : -k : 2\lambda h).$$ □

C is called the **pole** of l, while l is called the **polar** of C.

Theorem 16 *Let $A_1 \equiv [x_1, y_1]$ and $A_2 \equiv [x_2, y_2]$ be two external points to a parabola. If the line containing the chord of contact from A_1 passes through A_2, then the line containing the chord of contact from A_2 passes through A_1.*

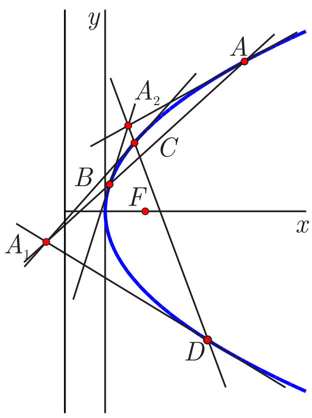

Figure 9.

Proof. Since the chord of contact from the point A_1, shown as CD in the Figure, passes through A_2 by (3), we have

$$2\lambda x_2 - y_1 y_2 + 2\lambda x_1 = 0.$$

Rearranging the equation shows that $A_1 \equiv [x_1, y_1]$ lies on

$$(2\lambda : -y_2 : 2\lambda x_2)$$

which is the chord of contact from the point A_2, shown as AB in the Figure. □

7 Quadrance properties of a parabola

Theorem 17 *Let A lie on a parabola with focus F. Let T be the point where the tangent at A meets the axis of the parabola. Then $Q(A,F) = Q(F,T)$.*

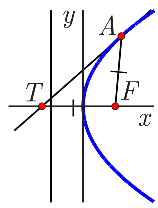

Figure 10.

Proof. Let $A \equiv [\lambda a^2, 2\lambda a]$. By (2) the tangent at A meets the axis at the point

$$T = [-\lambda a^2, 0].$$

Now

$$Q(A,F) = (\lambda a^2 - \lambda)^2 + 4\lambda^2 a^2 = \lambda^2(a^2+1)^2$$

and

$$Q(F,T) = (\lambda + \lambda a^2)^2 = \lambda^2(a^2+1)^2.$$

Hence $Q(A,F) = Q(F,T)$. □

Theorem 18 *Let AB be a chord of the parabola p from the external point X. Let F be the focus of p. Then*

$$Q(X,F)^2 = Q(A,F)\,Q(F,B).$$

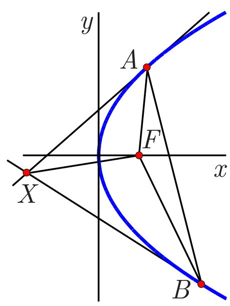

Figure 11.

Proof. Let $A \equiv [\lambda a^2, 2\lambda a]$ and $B \equiv [\lambda b^2, 2\lambda b]$. By (5) the external point of the chord AB is $X = [\lambda ab, \lambda(a+b)]$. Now

$$\begin{aligned} Q(X,F)^2 &= (\lambda^2(ab-1)^2 + \lambda^2(a+b)^2)^2 \\ &= \lambda^4(a^2+b^2+a^2b^2+1)^2 \\ &= \lambda^4((a^2+1)(b^2+1))^2. \end{aligned}$$

Moreover

$$Q(A,F) = (\lambda a^2 - \lambda)^2 + 4\lambda^2 a^2 = \lambda^2(a^2+1)^2.$$

Similarly

$$Q(F,B) = \lambda^2(b^2+1)^2.$$

Hence $Q(X,F)^2 = Q(A,F)\,Q(F,B)$. □

Theorem 19 *If a focal chord AB of a parabola with focus F meets the directrix at D, then*

$$Q(A,F) : Q(F,B) = Q(A,D) : Q(D,B).$$

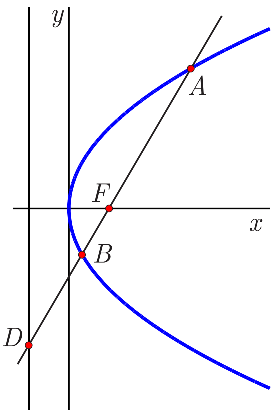

Figure 12.

Proof. Let $A \equiv [\lambda a^2, 2\lambda a]$ and $B \equiv [\lambda b^2, 2\lambda b]$. Then

$$\begin{aligned} Q(A,F) &= (\lambda a^2 - \lambda)^2 + 4\lambda^2 a^2 = \lambda^2(a^2+1)^2 \\ Q(F,B) &= (\lambda b^2 - \lambda)^2 + 4\lambda^2 b^2 = \lambda^2(b^2+1)^2. \end{aligned}$$

Therefore

$$Q(A,F) : Q(F,B) = (1+a^2)^2 : (1+b^2)^2$$

By (4) the chord AB intersects the directrix at

$$D = \left[-\lambda, \frac{2\lambda(ab-1)}{a+b}\right]$$

where $a \neq -b$. Therefore

$$\begin{aligned} Q(A,D) &= (\lambda a^2 + \lambda)^2 + \left(2\lambda a - \frac{2\lambda(ab-1)}{a+b}\right)^2 \\ &= \lambda^2(a^2+1)^2\left(1 + \frac{4}{(a+b)^2}\right). \end{aligned}$$

Similarly

$$Q(D,B) = \lambda^2(b^2+1)^2\left(1 + \frac{4}{(a+b)^2}\right).$$

Hence

$$Q(A,F) : Q(F,B) = Q(A,D) : Q(D,B).$$ □

Theorem 20 *Let A and B be two non-null points on a parabola with focus F. If AB is a focal chord and the tangent at A meets the latus rectum at C. Then*

$$Q(C,F)^2 = Q(A,F)\,Q(F,B).$$

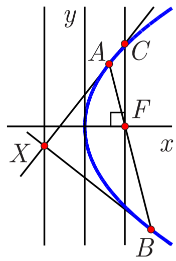

Figure 13.

Proof. Let $A \equiv [\lambda a^2, 2\lambda a]$. The latus rectum is

$$(1 : 0 : -\lambda)$$

which intersects the tangent at A

$$(1 : -a : \lambda a^2)$$

at

$$C \equiv \left[\lambda, \frac{\lambda(1+a^2)}{a}\right].$$

Now

$$Q(C,F)^2 = \left(\left(\frac{\lambda(1+a^2)}{a}\right)^2\right)^2 = \frac{\lambda^4(1+a^2)^4}{a^4}$$

Moreover

$$Q(A,F) = (\lambda a^2 - \lambda)^2 + 4\lambda^2 a^2 = \lambda^2(a^2+1)^2.$$

Similarly with $B \equiv [\lambda b^2, 2\lambda b]$,

$$Q(F,B) = \lambda^2(b^2+1)^2.$$

Since AB is a focal chord, it passes through F. By (4) $ab = -1$. Therefore

$$\begin{aligned} Q(A,F)\,Q(F,B) &= \lambda^2(a^2+1)^2\lambda^2(b^2+1)^2 \\ &= \lambda^4(a^2+1)^2\left(\frac{1}{a^2}+1\right)^2 \\ &= \lambda^4(a^2+1)^2\left(\frac{1+a^2}{a^2}\right)^2 \\ &= \frac{\lambda^4(a^2+1)^4}{a^4}. \end{aligned}$$

Hence

$$Q(C,F)^2 = Q(A,F)\,Q(F,B).$$ □

8 Spread properties of a parabola

Theorem 21 *Let A lie on a parabola with focus F. Suppose the tangent at A meets the directrix at D. Then AF is perpendicular to DF.*

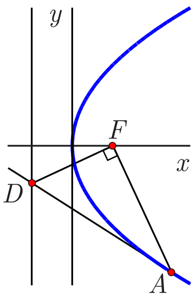

Figure 14.

Proof. Let $A \equiv [\lambda a^2, 2\lambda a]$. From (2) the tangent at A meets the directrix at

$$D = \left[-\lambda, \frac{\lambda(a^2-1)}{a}\right]$$

where $a \neq 0$. Now the line AF is

$$\left(2a : 1-a^2 : -2\lambda a\right)$$

and the line DF

$$\left(a^2-1 : 2a : -\lambda(a^2-1)\right).$$

So AF is perpendicular to DF. □

Theorem 22 *Let A be a non-null point on a parabola with focus F. Then the spread between the line AF and the axis of the parabola is a square number.*

Proof. Let $A \equiv [\lambda a^2, 2\lambda a]$. Then the line AF has equation

$$\left(2a : 1-a^2 : -2\lambda a\right).$$

The axis of the parabola is $(0:1:0)$. The spread between these two lines is

$$\frac{(2a)^2}{(2a)^2+(1-a^2)^2} = \left(\frac{2a}{1+a^2}\right)^2$$

which is a square. □

Theorem 23 *Let A and B be two non-null points on a parabola with focus F. Then $\overline{AB}$ subtends a square spread at the focus.*

Proof. Let $A \equiv [\lambda a^2, 2\lambda a], B \equiv [\lambda b^2, 2\lambda b]$ Then the lines AF and BF are

$$\left(2a : 1-a^2 : -2\lambda a\right) \quad \text{and} \quad \left(2b : 1-b^2 : -2\lambda b\right)$$

respectively. Now the spread between AF and BF is

$$s(AF,BF) = \frac{(2a(1-b^2)-2b(1-a^2))^2}{((2a)^2+(1-a^2)^2)((2b)^2+(1-b^2)^2)} = \frac{(2(a-b)(1+ab))^2}{(1+a^2)^2(1+b^2)^2}$$

which is a square. □

Theorem 24 *Let A and B be two points on a parabola with focus F with at least one of the points non-null. Let X be the external point of the chord AB, and suppose the line AB meets the directrix of the parabola at D. Then DF is perpendicular to FX.*

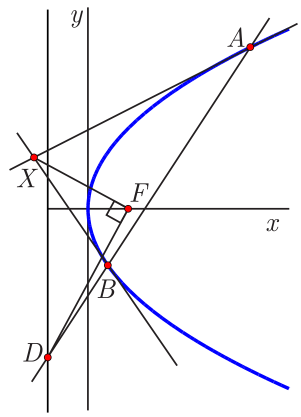

Figure 15.

Proof. With A and B the usual points, by (4) the line AB intersects the directrix at

$$D = \left[-\lambda, \frac{2\lambda(ab-1)}{a+b}\right]$$

where $a \neq -b$. By (5)

$$X = [\lambda ab, \lambda(a+b)].$$

Now the line DF is

$$(ab-1 : a+b : -\lambda(ab-1))$$

and the line FX is

$$(a+b : 1-ab : -\lambda(a+b)).$$

Thus DF is perpendicular to FX. □

Theorem 25 *Let A and B be two non-null points lying on a parabola with focus F. Let the tangents at A and B intersect the directrix at C and D respectively. Then $\overline{AB}$ and $\overline{CD}$ subtend equal spreads at the focus.*

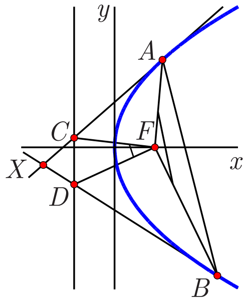

Figure 16.

Proof. By (2) the tangents at the usual points A and B are

$$\left(1 : -a : \lambda a^2\right) \qquad \text{and} \qquad \left(1 : -b : \lambda b^2\right)$$

respectively. They intersect the directrix at

$$C \equiv \left[-\lambda, \frac{\lambda(a^2-1)}{a}\right] \quad \text{and} \quad D \equiv \left[-\lambda, \frac{\lambda(b^2-1)}{b}\right]$$

respectively. The line AF is

$$\left(2a : 1-a^2 : -2\lambda a\right).$$

Similarly the line BF is $\left(2b : 1-b^2 : -2\lambda b\right)$. Thus

$$\begin{aligned} s(AF,BF) &= \frac{(2a(1-b^2)-2b(1-a^2))^2}{(4a^2+(1-a^2)^2)(4b^2+(1-b^2)^2)} \\ &= \frac{4((a-b)(ab+1))^2}{(1+a^2)^2(1+b^2)^2}. \end{aligned}$$

The line CF is $\left(a^2-1 : 2a : -\lambda(a^2-1)\right)$ and similarly the line DF is $\left(b^2-1 : 2b : -\lambda(b^2-1)\right)$. Thus

$$\begin{aligned} s(CF,DF) &= \frac{(2b(a^2-1)-2a(b^2-1))^2}{((a^2-1)^2+4a^2)((b^2-1)^2+4b^2)} \\ &= \frac{4((a-b)(1+ab))^2}{(1+a^2)^2(1+b^2)^2}. \end{aligned}$$

Hence $s(AF,BF) = s(CF,DF)$. □

Theorem 26 *Suppose A and B are two non-null points on a parabola. If $\overline{AB}$ is a side of the parabola with the external point X, then the sides $\overline{AX}$ and $\overline{BX}$ subtend equal spreads at the focus.*

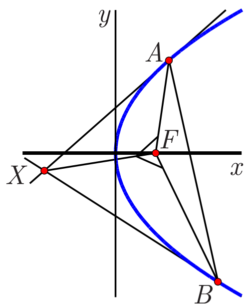

Figure 17.

Proof. Let $A \equiv [\lambda a^2, 2\lambda a]$ and $B \equiv [\lambda b^2, 2\lambda b]$. By (5) the tangents intersect at

$$X = [\lambda ab, \lambda(a+b)].$$

Then

$$\begin{aligned} AF &= \left(2a : 1-a^2 : -2\lambda a\right) \\ FB &= \left(2b : 1-b^2 : -2\lambda b\right) \\ XF &= \left(a+b : 1-ab : -\lambda(a+b)\right). \end{aligned}$$

Therefore

$$\begin{aligned} s(AF,FX) &= \frac{(2a(1-ab)-(1-a^2)(a+b))^2}{(4a^2+(1-a^2)^2)((a+b)^2+(1-ab)^2)} \\ &= \frac{(a-a^2b+a^3-b)^2}{(1+a^2)^2(a^2+b^2+a^2b^2+1)} \\ &= \frac{((a-b)(1+a^2))^2}{(1+a^2)^2(1+a^2)(1+b^2)} \\ &= \frac{(a-b)^2}{(1+a^2)(1+b^2)}. \end{aligned}$$

Since this expression is symmetrical in a and b,

$$s(AF,FX) = s(XF,FB).$$ □

Theorem 27 *Suppose the non-null tangent at A to a parabola with focus F passes through an external point X to a parabola. Then the spread between the tangent to the parabola at A and its axis is equal to the spread between the other non-null tangent and line XF.*

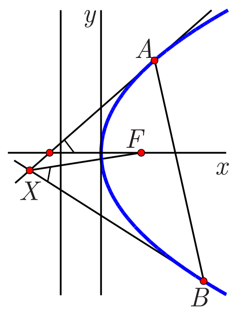

Figure 18.

Proof. Let $A \equiv [\lambda a^2, 2\lambda a]$ and $B \equiv [\lambda b^2, 2\lambda b]$ lie on the parabola. By (5) the tangents intersect at

$$X = [\lambda ab, \lambda(a+b)].$$

The axis of the parabola is

$$n \equiv (0:1:0).$$

Then by (2)

$$AX = \left(1 : -a : \lambda a^2\right) \qquad \text{and} \qquad BX = \left(1 : -b : \lambda b^2\right).$$

The equation of the line through X and the focus F is

$$XF = (a+b : 1-ab : -\lambda(a+b)).$$

Therefore

$$s(AX,n) = \frac{1}{1+a^2}.$$

Moreover

$$\begin{aligned} s(BX,XF) &= \frac{((1-ab)+b(a+b))^2}{(1+b^2)((a+b)^2+(1-ab)^2)} \\ &= \frac{(1+b^2)^2}{(1+b^2)(a^2+b^2+a^2b^2+1)} \\ &= \frac{(1+b^2)^2}{(1+b^2)(1+a^2)(1+b^2)} \\ &= \frac{1}{1+a^2}. \end{aligned}$$

Hence $s(AX,n) = s(BX,XF)$. $\square$

Theorem 28 *Let A and B be two non-null points on a parabola with focus F. Let the tangents at A and B intersect at X. Let the spread subtended by the side $\overline{AB}$ at X be s. Then the spread subtended by $\overline{AB}$ at the focus F is equal to $S_2(s) = 4s(1-s)$.*

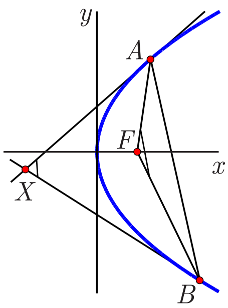

Figure 19.

Proof. Suppose $A \equiv [\lambda a^2, 2\lambda a]$ and $B \equiv [\lambda b^2, 2\lambda b]$ are points on the parabola. By (5) the tangents intersect at

$$X = [\lambda ab, \lambda(a+b)].$$

The spread between the tangents is

$$s \equiv \frac{(a-b)^2}{(1+a^2)(1+b^2)}.$$

Therefore

$$\begin{aligned} S_2(s) &= 4s(1-s) \\ &= 4\frac{(a-b)^2}{(1+a^2)(1+b^2)}\left(1-\frac{(a-b)^2}{(1+a^2)(1+b^2)}\right) \\ &= \frac{4(a-b)^2}{(1+a^2)(1+b^2)}\left(\frac{1+a^2b^2+2ab}{(1+a^2)(1+b^2)}\right) \\ &= \frac{4(a-b)^2(1+ab)^2}{(1+a^2)^2(1+b^2)^2}. \end{aligned}$$

The lines AF and FB are

$$\left(2a : 1-a^2 : -2\lambda a\right) \quad \left(2b : 1-b^2 : -2\lambda b\right)$$

respectively. Therefore

$$\begin{aligned} s(AF,FB) &= \frac{(2a(1-b^2)-2b(1-a^2))^2}{(4a^2+(1-a^2)^2)(4b^2+(1-b^2)^2)} \\ &= \frac{4((a-b)(1+ab))^2}{(1+a^2)^2(1+b^2)^2}. \end{aligned}$$

Hence $s(AF,FB) = S_2(s) = 4s(1-s)$. $\square$

Theorem 29 *Suppose that two congruent parabolas have the same vertex, and that their axes are perpendicular to each other. If the characteristic of the field is not 5, then the parabolas intersect with a spread of $9/25$. Otherwise the two parabolas intersect at one of the null points on the two parabolas.*

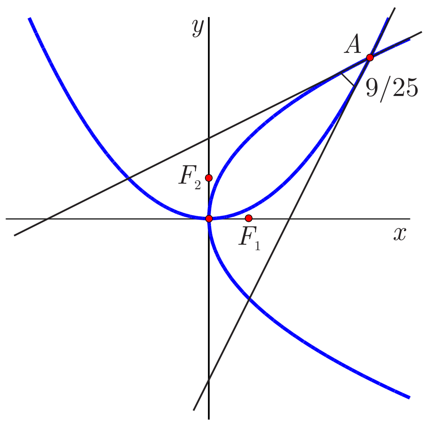

Figure 20.

Proof. Let the two congruent parabolas be

$$y^2 = 4\lambda x \qquad \text{and} \qquad x^2 = 4\lambda y \tag{7}$$

with foci F_1 and F_2 respectively.

Suppose first that the two parabolas are defined over a field of characteristic other than 5. Then they intersect at

$$A \equiv [4\lambda, 4\lambda].$$

The tangents at A in (7) are

$$(1 : -2 : 4\lambda) \qquad \text{and} \qquad (2 : -1 : -4\lambda)$$

respectively. Hence their spread is

$$\frac{((1)(-1)-(2)(-2))^2}{(1^2+(-2)^2)(2^2+(-1)^2)} = \frac{3^2}{5^2} = \frac{9}{25}.$$

On the other hand, if the characteristic of the field is 5, then

$$Q(A,F_1) = (4\lambda-\lambda)^2 + (4\lambda)^2 = 25\lambda^2 = 0$$

$$Q(A,F_2) = (4\lambda)^2 + (4\lambda-\lambda)^2 = 25\lambda^2 = 0$$

which shows that the point A is a null point on the parabolas. □

Theorem 30 *Suppose that two congruent parabolas have a common focus but different vertices. If their axes are the same, then the parabolas intersect perpendicularly.*

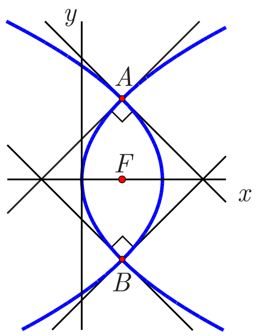

Figure 21.

Proof. Let the parabolas be

$$y^2 = 4\lambda x \qquad \text{and} \qquad y^2 = -4\lambda(x-2\lambda). \tag{8}$$

They intersect at

$$A = [\lambda, 2\lambda] \qquad \text{and} \qquad B = [\lambda, -2\lambda].$$

The tangents at A for (8) are

$$(1:-1:\lambda) \qquad \text{and} \qquad (1:1:-3\lambda)$$

respectively, which are perpendicular. Similarly the tangents at B in (8) are

$$(1:1:\lambda) \qquad \text{and} \qquad (1:-1:-3\lambda)$$

respectively, which are also perpendicular. □

9 Signed areas and anti-symmetric polynomials

In this section we review some definitions, notation and results from [16] which we quickly review. If $A_1 \equiv [x_1,y_1]$, $A_2 \equiv [x_2,y_2]$ and $A_3 \equiv [x_3,y_3]$, then the **signed area** of the oriented triangle $\overrightarrow{A_1A_2A_3}$ is the number

$$a\left(\overrightarrow{A_1A_2A_3}\right) \equiv \frac{x_1y_2 - x_2y_1 + x_2y_3 - x_3y_2 + x_3y_1 - x_1y_3}{2} = \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}.$$

This concept is extended to more general n-gons, by for example defining the signed area of an oriented quadrilateral $\overrightarrow{A_1A_2A_3A_4}$ to be

$$a\left(\overrightarrow{A_1A_2A_3A_4}\right) = a\left(\overrightarrow{A_1A_2A_3}\right) + a\left(\overrightarrow{A_1A_3A_4}\right).$$

To deal with a wide variety of anti-symmetric expressions in the variable x_i and y_j for $i,j = 1,2$ and 3, introduce the following notation. For any polynomial r in these variables, define $[r]_3^-$ to be the alternating sum of the six terms obtained from r by applying the following changes to the indices:

$$2 \longleftrightarrow 3 \quad 1 \longleftrightarrow 2 \quad 2 \longleftrightarrow 3 \quad 1 \longleftrightarrow 2 \quad 2 \longleftrightarrow 3.$$

in this order. The expression in the numerator of $a\left(\overrightarrow{A_1A_2A_3}\right)$ is $[x_1y_2]_3^-$. Here is another example:

$$[x_1x_2^3y_2]_3^- = x_1x_2^3y_2 - x_1x_3^3y_3 + x_2x_3^3y_3 - x_3x_2^3y_2 + x_3x_1^3y_1 - x_2x_1^3y_1.$$

Theorem 31 (Circumcenter formula) *If $A_1 \equiv [x_1,y_1]$, $A_2 \equiv [x_2,y_2]$ and $A_3 \equiv [x_3,y_3]$, then the circumcenter C of the triangle $\overline{A_1A_2A_3}$ is*

$$C = \left[\frac{[x_1^2y_2]_3^- + [y_1^2y_2]_3^-}{2\,[x_1y_2]_3^-}, \frac{[x_1x_2^2]_3^- + [x_1y_2^2]_3^-}{2\,[x_1y_2]_3^-}\right].$$

Theorem 32 (Orthocentre formula) *If $A_1 \equiv [x_1,y_1]$, $A_2 \equiv [x_2,y_2]$ and $A_3 \equiv [x_3,y_3]$, then the orthocentre O of the triangle $\overline{A_1A_2A_3}$ is*

$$O = \left[\frac{[x_1x_2y_2]_3^- + [y_1y_2^2]_3^-}{[x_1y_2]_3^-}, \frac{[x_1y_1y_2]_3^- + [x_1^2x_2]_3^-}{[x_1y_2]_3^-}\right].$$

We generally use determinants to evaluate anti-symmetric expressions.

10 Parabolic triangles

If all three points lie on the parabola, then we have a **parabolic triangle**.

Theorem 33 *Let A, B and C be three distinct points on a parabola. Let the external points of the sides $\overline{AB}$, $\overline{BC}$ and $\overline{CA}$ be X, Y and Z respectively. Then*

$$a(\overrightarrow{ABC}) : a(\overrightarrow{XYZ}) = -2:1$$

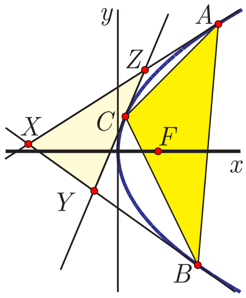

Figure 22.

Proof. Let the points on the parabola be

$$A \equiv [\lambda a^2, 2\lambda a] \quad B \equiv [\lambda b^2, 2\lambda b] \quad C \equiv [\lambda c^2, 2\lambda c].$$

Then by (5) the corresponding external points to the parabola are

$$X \equiv [\lambda ab, \lambda(a+b)] \quad Y \equiv [\lambda bc, \lambda(b+c)] \quad Z \equiv [\lambda ca, \lambda(c+a)].$$

Now by the signed area formula

$$\begin{aligned} a(\overrightarrow{ABC}) &= \frac{1}{2}\begin{vmatrix} \lambda a^2 & 2\lambda a & 1 \\ \lambda b^2 & 2\lambda b & 1 \\ \lambda c^2 & 2\lambda c & 1 \end{vmatrix} \\ &= \lambda^2 \begin{vmatrix} a^2 & a & 1 \\ b^2 & b & 1 \\ c^2 & c & 1 \end{vmatrix} \\ &= \lambda^2 \begin{vmatrix} (a-b)(a+b) & (a-b) & 0 \\ (b-c)(b+c) & (b-c) & 0 \\ c^2 & c & 1 \end{vmatrix} \\ &= \lambda^2 \begin{vmatrix} (a-b)(a+b) & (a-b) \\ (b-c)(b+c) & (b-c) \end{vmatrix} \\ &= \lambda^2 (a-b)(b-c) \begin{vmatrix} a+b & 1 \\ b+c & 1 \end{vmatrix} \\ &= \lambda^2 (a-b)(b-c)(a-c). \end{aligned}$$

Moreover

$$\begin{aligned} a(\overrightarrow{XYZ}) &= \frac{1}{2}\begin{vmatrix} \lambda ab & \lambda(a+b) & 1 \\ \lambda bc & \lambda(b+c) & 1 \\ \lambda ca & \lambda(c+a) & 1 \end{vmatrix} \\ &= \frac{\lambda^2}{2}\begin{vmatrix} ab & a+b & 1 \\ bc & b+c & 1 \\ ca & c+a & 1 \end{vmatrix} \\ &= \frac{\lambda^2}{2}\begin{vmatrix} b(a-c) & a-c & 0 \\ c(b-a) & b-a & 0 \\ ca & c+a & 1 \end{vmatrix} \\ &= \frac{\lambda^2}{2}\begin{vmatrix} b(a-c) & a-c \\ c(b-a) & b-a \end{vmatrix} \\ &= \frac{\lambda^2 (a-c)(b-a)}{2}\begin{vmatrix} b & 1 \\ c & 1 \end{vmatrix} \\ &= \frac{\lambda^2 (a-c)(b-a)(b-c)}{2}. \end{aligned}$$

Hence

$$a(\overrightarrow{ABC}) : a(\overrightarrow{XYZ}) = -2 : 1. \qquad \square$$

Corollary 1 *Let A_1, A_2, ..., A_n be n distinct points on a parabola. Let the external points of the sides $\overline{A_1A_2}$, $\overline{A_2A_3}$, ..., $\overline{A_nA_1}$ be X_1, X_2, ..., X_n respectively. Then from the previous theorem*

$$a(\overrightarrow{A_1A_2\cdots A_n}) : a(\overrightarrow{X_1X_2\cdots X_n}) = -2 : 1.$$

Proof. Let the external points of the chords $\overline{A_1A_3}$, $\overline{A_1A_4}$, ..., $\overline{A_1A_{n-1}}$ be $Y_3, Y_4, \ldots, Y_{n-1}$ respectively. Then

$$\begin{aligned} & a(\overrightarrow{A_1A_2\cdots A_n}) \\ &= a(\overrightarrow{A_1A_2A_3}) + a(\overrightarrow{A_1A_3A_4}) + \cdots + a(\overrightarrow{A_1A_kA_{k+1}}) + \cdots \\ &\quad + a(\overrightarrow{A_1A_{n-1}A_n}) \\ &= -2\left(a(\overrightarrow{X_1X_2Y_3}) + a(\overrightarrow{Y_3X_3Y_4}) + \cdots + a(\overrightarrow{Y_kX_kY_{k+1}}) + \cdots \right. \\ &\quad \left. + a(\overrightarrow{Y_{n-1}X_{n-1}X_n})\right) \\ &= -2\left(a(\overrightarrow{X_1X_2Y_3}) + a(\overrightarrow{X_3Y_4Y_3}) + \cdots + a(\overrightarrow{X_kY_{k+1}Y_k}) + \cdots \right. \\ &\quad \left. + a(\overrightarrow{X_{n-1}X_nY_{n-1}})\right) \\ &= -2\left(a(\overrightarrow{X_1X_2X_3Y_4}) + \cdots + a(\overrightarrow{X_kY_{k+1}Y_k}) + \cdots \right. \\ &\quad \left. + a(\overrightarrow{X_{n-1}X_nY_{n-1}})\right) \\ &= -2\left(a(\overrightarrow{X_1X_2\cdots X_kY_{k+1}}) + \cdots + a(\overrightarrow{X_{n-1}X_nY_{n-1}})\right) \\ &= -2\,a(\overrightarrow{X_1X_2\cdots X_{n-1}Y_n}). \end{aligned}$$

Hence $a(\overrightarrow{A_1A_2\cdots A_n}) : a(\overrightarrow{X_1X_2\cdots X_n}) = -2 : 1.$ $\square$

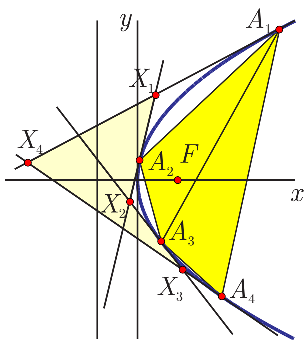

Figure 23.

The following is in [15].

Theorem 34 *Let A, B and C be three distinct points on a parabola. Let the external points of the sides $\overline{AB}$, $\overline{BC}$ and $\overline{CA}$ be X, Y and Z respectively. Then*

$$\frac{Q(X,A)}{Q(A,Z)} = \frac{Q(B,X)}{Q(X,Y)} = \frac{Q(Y,Z)}{Q(Z,C)}.$$

Proof. Suppose the points on the parabola are

$$A \equiv [\lambda a^2, 2\lambda a] \quad B \equiv [\lambda b^2, 2\lambda b] \quad C \equiv [\lambda c^2, 2\lambda c].$$

Then by (5) the corresponding external points to the parabola are

$$X \equiv [\lambda ab, \lambda(a+b)] \quad Y \equiv [\lambda bc, \lambda(b+c)] \quad Z \equiv [\lambda ca, \lambda(c+a)].$$

Now

$$\begin{aligned} Q(X,A) &= (\lambda ab - \lambda a^2)^2 + (\lambda(a+b) - 2\lambda a)^2 \\ &= \lambda^2 a^2 (b-a)^2 + \lambda^2 (b-a)^2 \\ &= \lambda^2 (b-a)^2 (a^2+1) \end{aligned}$$

and

$$\begin{aligned} Q(A,Z) &= (\lambda a^2 - \lambda ca)^2 + (2\lambda a - \lambda(c+a))^2 \\ &= \lambda^2 a^2 (a-c)^2 + \lambda^2 (a-c)^2 \\ &= \lambda^2 (a-c)^2 (a^2+1). \end{aligned}$$

Therefore

$$\frac{Q(X,A)}{Q(A,Z)} = \frac{(b-a)^2}{(a-c)^2}.$$

Similarly

$$\frac{Q(B,X)}{Q(X,Y)} = \frac{(b-a)^2}{(a-c)^2} \qquad \frac{Q(Y,Z)}{Q(Z,C)} = \frac{(b-a)^2}{(a-c)^2}.$$

Hence

$$\frac{Q(X,A)}{Q(A,Z)} = \frac{Q(B,X)}{Q(X,Y)} = \frac{Q(Y,Z)}{Q(Z,C)}. \qquad \square$$

Theorem 35 *Let A, B and C be three distinct points on a parabola. Let the external points of the sides $\overline{AB}$, $\overline{BC}$ and $\overline{CA}$ be X, Y and Z respectively. If the parabola is defined over a field with -3 not a square, then the lines AY, BZ and CX are concurrent. Otherwise if the lines AY, BZ and CX are not concurrent, then they are mutually parallel to each other.*

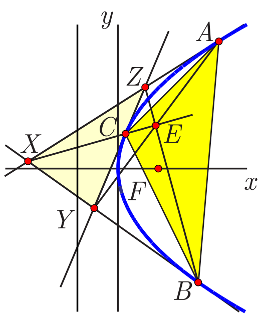

Figure 24.

Proof. Suppose the points on the parabola are

$$A \equiv [\lambda a^2, 2\lambda a] \quad B \equiv [\lambda b^2, 2\lambda b] \quad \text{and} \quad C \equiv [\lambda c^2, 2\lambda c]$$

so that

$$X \equiv [\lambda ab, \lambda(a+b)] \quad Y \equiv [\lambda bc, \lambda(b+c)] \quad Z \equiv [\lambda ca, \lambda(c+a)].$$

Then

$$\begin{aligned} AY &= \left(2a-(b+c) : bc-a^2 : \lambda(a^2(b+c)-2abc)\right) \\ BZ &= \left(2b-(c+a) : ca-b^2 : \lambda(b^2(c+a)-2abc)\right) \\ CX &= \left(2c-(a+b) : ab-c^2 : \lambda(c^2(a+b)-2abc)\right). \end{aligned}$$

Suppose AY, BZ and CX are mutually parallel to each other. Then

$$\begin{aligned} &(2a-(b+c)) \times \left(ca-b^2\right) - (2b-(c+a)) \times \left(bc-a^2\right) \\ &\quad = (b-a)(a^2-ca-ba+b^2+c^2-bc) = 0 \\ &(2a-(b+c)) \times \left(ab-c^2\right) - (2c-(a+b)) \times \left(bc-a^2\right) \\ &\quad = (c-a)(a^2-ca-ba+b^2+c^2-bc) = 0. \end{aligned}$$

Therefore a sufficient condition for the three lines to be mutually parallel is

$$a^2+b^2+c^2-ab-bc-ca = 0. \qquad (9)$$

If we solve for c in (9), then we obtain a quadratic in c. Its discriminant is

$$-3a^2+6ba-3b^2 = -3(a-b)^2.$$

Hence if -3 is a square, then without loss of generality the point C on the parabola can be chosen such that AY, BZ and CX are mutually parallel.
Now suppose that -3 is not a square; then the three lines are not mutually parallel. Therefore

$$\begin{aligned} &\begin{vmatrix} 2a-(b+c) & bc-a^2 & \lambda(a^2(b+c)-2abc) \\ 2b-(c+a) & ca-b^2 & \lambda(b^2(c+a)-2abc) \\ 2c-(a+b) & ab-c^2 & \lambda(c^2(a+b)-2abc) \end{vmatrix} \\ &= \begin{vmatrix} 3(a-b) & -(a-b)(a+b+c) & \lambda(a-b)(ab+bc+ca) \\ 2b-(c+a) & b^2-ca & \lambda(b^2(c+a)-2abc) \\ 2c-(a+b) & c^2-ab & \lambda(c^2(a+b)-2abc) \end{vmatrix} \\ &= \begin{vmatrix} 3(a-b) & -(a-b)(a+b+c) & \lambda(a-b)(ab+bc+ca) \\ 3(b-c) & -(b-c)(a+b+c) & \lambda(b-c)(ab+bc+ca) \\ 2c-(a+b) & c^2-ab & \lambda(c^2(a+b)-2abc) \end{vmatrix} \\ &= (a-b)(b-c)\begin{vmatrix} 3 & -(a+b+c) & \lambda(ab+bc+ca) \\ 3 & -(a+b+c) & \lambda(ab+bc+ca) \\ 2c-(a+b) & c^2-ab & \lambda(c^2(a+b)-2abc) \end{vmatrix} \\ &= 0. \end{aligned}$$

Therefore by the Concurrent lines theorem the lines AY, BZ and CX are concurrent. Hence if -3 is not a square, the lines AY, BZ and CX are always concurrent. Otherwise if the lines are not concurrent then the three lines must be mutually parallel. $\square$

Call the common point of intersection, which is E in Figure 24, the **central point** of the parabolic triangle $\overline{ABC}$.

Theorem 36 *Let A, B and C be three distinct points on a parabola. Let the external points of the sides $\overline{AB}$, $\overline{BC}$ and $\overline{CA}$ be X, Y and Z respectively. Then the centroids of the triangles $\overline{XAB}$, $\overline{YBC}$ and $\overline{ZCA}$ are collinear. Moreover the central point of $\overline{ABC}$ lies on that line.*

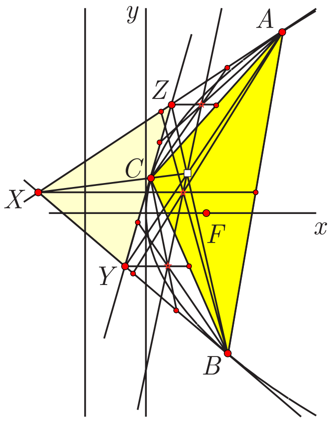

Figure 25.

Proof. Let the points on the parabola be

$$A \equiv [\lambda a^2, 2\lambda a] \quad B \equiv [\lambda b^2, 2\lambda b] \quad \text{and} \quad C \equiv [\lambda c^2, 2\lambda c].$$

Then by (5) the external points to the parabola are

$$X \equiv [\lambda ab, \lambda(a+b)] \quad Y \equiv [\lambda bc, \lambda(b+c)]$$

and $\quad Z \equiv [\lambda ca, \lambda(c+a)]$.

The centroid of triangle $\overline{XAB}$ is

$$\left[\frac{\lambda(a^2+ab+b^2)}{3}, \lambda(a+b)\right].$$

Similarly the centroids of triangle $\overline{YBC}$ and $\overline{ZCA}$ are

$$\left[\frac{\lambda(b^2+bc+c^2)}{3}, \lambda(b+c)\right] \quad \text{and}$$

$$\left[\frac{\lambda(c^2+ca+a^2)}{3}, \lambda(c+a)\right]$$

respectively. Therefore

$$\begin{vmatrix} \lambda(a^2+ab+b^2)/3 & \lambda(a+b) & 1 \\ \lambda(b^2+bc+c^2)/3 & \lambda(b+c) & 1 \\ \lambda(c^2+ca+a^2)/3 & \lambda(c+a) & 1 \end{vmatrix}$$

$$= \frac{\lambda^2}{3}\begin{vmatrix} a^2+ab+b^2 & a+b & 1 \\ b^2+bc+c^2 & b+c & 1 \\ c^2+ca+a^2 & c+a & 1 \end{vmatrix}$$

$$= \frac{\lambda^2}{3}\begin{vmatrix} (a+b+c)(a-c) & a-c & 0 \\ b^2+bc+c^2 & b+c & 1 \\ c^2+ca+a^2 & c+a & 1 \end{vmatrix}$$

$$= \frac{\lambda^2}{3}\begin{vmatrix} (a+b+c)(a-c) & a-c & 0 \\ (a+b+c)(b-a) & b-a & 0 \\ c^2+ca+a^2 & c+a & 1 \end{vmatrix} = 0.$$

The line containing the three centroids is

$$(3 : -(a+b+c) : \lambda(ab+bc+ca)).$$

The coordinate of the central point is

$$\left[\begin{array}{c} \dfrac{\lambda(a^2(b-c)^2+b^2(c-a)^2+c^2(a-b)^2)}{(a-b)^2+(b-c)^2+(c-a)^2}, \\ \dfrac{2\lambda(a(b-c)^2+b(c-a)^2+c(a-b)^2)}{(a-b)^2+(b-c)^2+(c-a)^2} \end{array}\right].$$

Now

$$3\left(\frac{\lambda(a^2(b-c)^2+b^2(c-a)^2+c^2(a-b)^2)}{(a-b)^2+(b-c)^2+(c-a)^2}\right)$$
$$-(a+b+c)\left(\frac{2\lambda(a(b-c)^2+b(c-a)^2+c(a-b)^2)}{(a-b)^2+(b-c)^2+(c-a)^2}\right)$$
$$+\lambda(ab+bc+ca)$$

$$= \frac{\lambda\left(\begin{array}{l} 3(a^2(b-c)^2+b^2(c-a)^2+c^2(a-b)^2) \\ -2(a(b-c)^2+b(c-a)^2+c(a-b)^2)(a+b+c) \\ +(ab+bc+ca)((a-b)^2+(b-c)^2+(c-a)^2) \end{array}\right)}{(a-b)^2+(b-c)^2+(c-a)^2}$$

$$= \frac{\lambda\left(\begin{array}{l} a^2(b-c)^2+b^2(c-a)^2+c^2(a-b)^2 \\ -(ab+ac)(b-c)^2-(ab+bc)(c-a)^2 \\ -(ac+bc)(a-b)^2 \\ +ab(a-b)^2+bc(b-c)^2+ca(c-a)^2 \end{array}\right)}{(a-b)^2+(b-c)^2+(c-a)^2}$$

$$= \frac{\lambda\left(\begin{array}{l} (c-a)(c-b)(a-b)^2 \\ +(a-b)(a-c)(b-c)^2 \\ +(b-c)(b-a)(c-a)^2 \end{array}\right)}{(a-b)^2+(b-c)^2+(c-a)^2}$$

$$= \frac{\lambda(a-b)(b-c)(c-a)(-(a-b)-(b-c)-(c-a))}{(a-b)^2+(b-c)^2+(c-a)^2}$$

$$= 0.$$

Hence the central point lies on the line. □

Theorem 37 *Let A, B and C be three distinct points on a parabola. Let the external points of the sides $\overline{AB}$, $\overline{BC}$ and $\overline{CA}$ be X, Y and Z respectively. Then the orthocentre of triangle $\overline{XYZ}$ lies on the directrix.*

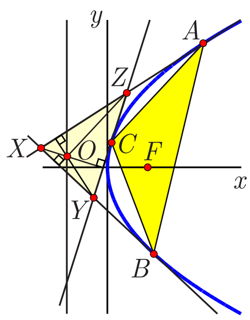

Figure 26.

Proof. Using our standard notation,

$$X \equiv [\lambda ab, \lambda(a+b)] \quad Y \equiv [\lambda bc, \lambda(b+c)]$$

and $\quad Z \equiv [\lambda ca, \lambda(c+a)]$.

Calculating at the abscissa of the orthocentre

$$\begin{aligned}
[x_1x_2y_2]_3^- &= \lambda^3 abc(b-a)(a+b+c) + \lambda^3 abc^2(a-b) \\
&\quad + \lambda^3 abc(a+b)(a-b) \\
&= \lambda^3 abc(a-b)(-(a+b+c)+c+a+b) \\
&= 0 \\
[y_1y_2^2]_3^- &= \begin{vmatrix} \lambda(a+b) & \lambda^2(a+b)^2 & 1 \\ \lambda(b+c) & \lambda^2(b+c)^2 & 1 \\ \lambda(c+a) & \lambda^2(c+a)^2 & 1 \end{vmatrix} \\
&= \lambda^3 \begin{vmatrix} a-c & (a+2b+c)(a-c) & 0 \\ b-a & (a+b+2c)(b-a) & 0 \\ c+a & (c+a)^2 & 1 \end{vmatrix} \\
&= \lambda^3(a-c)(b-a)(c-b) \\
[x_1y_2]_3^- &= \lambda^2(a-b)(b-c)(c-a).
\end{aligned}$$

Therefore

$$\begin{aligned}
\frac{[x_1x_2y_2]_3^- + [y_1y_2^2]_3^-}{[x_1y_2]_3^-} &= \frac{0+\lambda^3(a-c)(b-a)(c-b)}{\lambda^2(a-b)(b-c)(c-a)} \\
&= -\lambda.
\end{aligned}$$

Hence the orthocentre lies on the directrix. □

Theorem 38 *Let A, B and C be three distinct points on a parabola. Let the external points of the sides $\overline{AB}$, $\overline{BC}$ and $\overline{CA}$ be X, Y and Z respectively. Let the orthocentres of the triangles $\overline{XAB}$, $\overline{YBC}$ and $\overline{ZCA}$ be O_X, O_Y and O_Z respectively. Then*

$$a(\overrightarrow{O_XO_YO_Z}) = -a(\overrightarrow{ABC}).$$

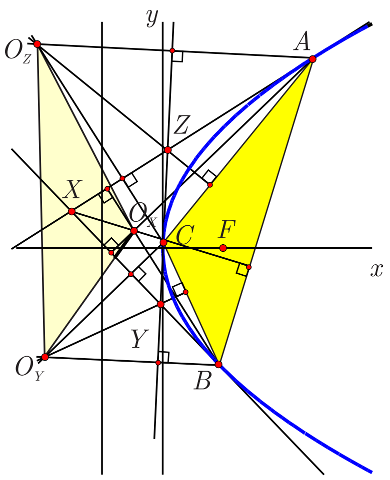

Figure 27.

Proof. Using the standard notation and the orthocentre formula

$$\begin{aligned}
O_X &= [-\lambda(2+ab), \lambda(2+ab)(a+b)] \\
O_Y &= [-\lambda(2+bc), \lambda(2+bc)(b+c)] \\
O_Z &= [-\lambda(2+ca), \lambda(2+ca)(c+a)].
\end{aligned}$$

Therefore by the signed area formula

$$\begin{aligned}
a(\overrightarrow{O_XO_YO_Z}) &= \frac{1}{2}\begin{vmatrix} -\lambda(2+ab) & \lambda(2+ab)(a+b) & 1 \\ -\lambda(2+bc) & \lambda(2+bc)(b+c) & 1 \\ -\lambda(2+ca) & \lambda(2+ca)(c+a) & 1 \end{vmatrix} \\
&= -\frac{\lambda^2}{2}\begin{vmatrix} 2+ab & (2+ab)(a+b) & 1 \\ 2+bc & (2+bc)(b+c) & 1 \\ 2+ca & (2+ca)(c+a) & 1 \end{vmatrix} \\
&= -\frac{\lambda^2}{2}\begin{vmatrix} b(a-c) & (a-c)(b^2+b(a+c)+2) & 0 \\ c(b-a) & (b-a)(c^2+c(b+a)+2) & 0 \\ 2+ca & (2+ca)(c+a) & 1 \end{vmatrix} \\
&= -\frac{\lambda^2(a-c)(b-a)}{2}\begin{vmatrix} b & b^2+b(a+c)+2 \\ c & c^2+c(b+a)+2 \end{vmatrix} \\
&= \lambda^2(a-b)(b-c)(c-a)
\end{aligned}$$

But from the proof of Theorem 34,

$$a(\overrightarrow{ABC}) = \lambda^2(a-b)(b-c)(a-c).$$

Hence $a(\overrightarrow{O_XO_YO_Z}) = -a(\overrightarrow{ABC})$. □

In the remaining theorem, we continue with our established notation. Recall that triangles are **similar** precisely when corresponding spreads are equal.

Theorem 39 *Let the circumcentres of the triangles $\overline{XAB}$, $\overline{YBC}$ and $\overline{ZCA}$ be C_X, C_Y and C_Z respectively. Then triangle $\overline{C_XC_YC_Z}$ is similar to triangle $\overline{XYZ}$.*

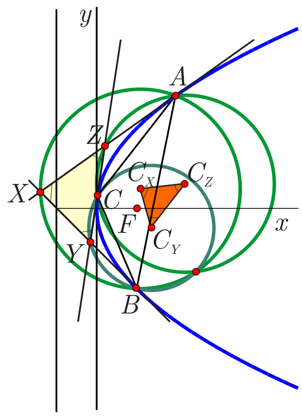

Figure 28.

Proof. (Using a computer) By the circumcenter formula, and using a computer,

$$C_X = \left[\frac{\lambda\left(a^2+2ab+b^2+2\right)}{2}, -\frac{\lambda\,(a+b)\,(ab-1)}{2}\right]$$
$$C_Y = \left[\frac{\lambda\left(b^2+2bc+c^2+2\right)}{2}, -\frac{\lambda\,(b+c)\,(bc-1)}{2}\right]$$
$$C_Z = \left[\frac{\lambda\left(a^2+2ca+c^2+2\right)}{2}, -\frac{\lambda\,(c+a)\,(ca-1)}{2}\right].$$

We may now calculate the spreads

$$s(C_YC_X, C_XC_Z) = \frac{(a-b)^2}{(a^2+1)\,(b^2+1)}$$
$$s(C_ZC_Y, C_YC_X) = \frac{(b-c)^2}{(b^2+1)\,(c^2+1)}$$
$$s(C_XC_Z, C_ZC_Y) = \frac{(a-c)^2}{(c^2+1)\,(a^2+1)}.$$

Similarly in triangle $\overline{ABC}$ we find that

$$s(YX, XZ) = \frac{(a-b)^2}{(a^2+1)\,(b^2+1)}$$
$$s(ZY, YZ) = \frac{(b-c)^2}{(b^2+1)\,(c^2+1)}$$
$$s(XZ, ZY) = \frac{(a-c)^2}{(c^2+1)\,(a^2+1)}.$$

Therefore

$$s(C_YC_X, C_XC_Z) = s(YX, XZ)$$
$$s(C_ZC_Y, C_YC_X) = s(ZY, YZ)$$
$$s(C_XC_Z, C_ZC_Y) = s(XZ, ZY).$$

Thus triangles $\overline{C_XC_YC_Z}$ and $\overline{ABC}$ have identical spreads. Hence triangles $\overline{C_XC_YC_Z}$ and $\overline{ABC}$ are similar. □

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https://doi.org/10.31896/k.22.5
Original scientific paper
15. 11. 2018.

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The Three Reflections Theorem Revisited

The Three Reflections Theorem Revisited

ABSTRACT

It is well-known that, in a Euclidean plane, the product of three reflections is again a reflection, iff their axes pass through a common point. For this "Three reflections Theorem" (3RT) also non-Euclidean versions exist, see e.g. [4]. This article presents affine versions of it, considering a triplet of skew reflections with axes through a common point. It turns out that the essence of all those cases of 3RT is that the three pairs (axis, reflection direction) of the given (skew) reflections can be observed as an involutoric projectivity. For the Euclidean case and its non-Euclidean counterparts this property is automatically fulfilled.

From the projective geometry point of view a (skew) reflection is nothing but a harmonic homology. In the affine situation a reflection is an indirect involutoric transformation, while "direct" or "indirect" makes no sense in projective planes. A harmonic homology allows an interpretation both, as an axial reflection and as a point reflection. Nevertheless, one might study products of three harmonic homologies, which result in a harmonic homology again. Some special mutual positions of axes and centres of the given homologies lead to elations or even to the identity, too.

A consequence of the presented results are further generalisations of the 3RT, e.g. in planes with Minkowski metric, affine or projective 3-space, or in circle geometries.

Key words: three reflections theorem, axial reflection, harmonic homology, involutoric projectivity

MSC2010: 51Mxx

Nov pogled na teorem tri simetrije

SAŽETAK

U euklidskoj ravnini poznato je da je produkt tri simetrije ponovo simetrija ako i samo ako se njihove osi sijeku u jednoj zajedničkoj točki. Također poznat je i neeuklidski analogon "teorema tri simetrije" (3RT), vidi npr. [4]. U ovom članku predstavljene su afine verzije tog teorema tako da se proučavaju tri mimosmjerne simetrije kojima se osi sijeku u jednoj točki. Pokazat će se da je važno, u svim verzijama 3RT-a, da se tri para (os, smjer simetrije) danih (mimosmjernih)simetrija mogu proučavati kao involutivni projektivitet. Za euklidski i neeuklidski slučaj ovo svojstvo je automatski ispunjeno.

Sa stajališta projektivne geometrije (mimosmjerna) simetrija je harmonička homologija. U afinoj geometriji simetrija je indirektna involutivna transformacija, dok u projektivnoj geometriji nema smisla govoriti o "direktnoj" i "indirektnoj" transformaciji. Harmonička homologija dopušta interpretaciju i kao osnu simetriju i kao centralnu simetriju. Ipak, može se proučavati produkt triju harmoničkih homologija koji je ponovno harmonička homologija. Nekim posebnim međusobnim položajima centara i osi danih homologija može se dobiti elacija ili čak identitet.

Posljedica danih rezultata su daljnje generalizacije 3RT-a, npr. u ravninama s Minkowski metrikom, afinim ili projektivnim 3-dimenzionalnim prostorima ili u geometrijama kružnice.

Ključne riječi: teorem tri simetrije, osna simetrija, harmonička homologija, involutivni projektivitet

1 Introduction

According to F. Bachmann [2] reflections can be observed as the basic transformations of a geometry, and the set of these transformations defines at least a sub-geometry of "classical" geometries. Thereby a general transformation of such a sub-geometry is the (finite) product of reflections, and so the question arises, under which conditions is such a product again a reflection. For the product of three reflections this leads to the conditions described as "Three Reflections Theorem" (3RT) in a Euclidean or (non-Euclidean) Cayley-Klein planes, see e.g. [4], [5].

In the following chapter we will state the known facts about the 3RT, providing the tools for further generalisations. Chapter 3 shortly refers to the non-Euclidean case and to circle geometries, while the following chapters present seemingly new generalisations of the 3RT in affine and projective planes and spaces.

2 Basic facts about the 3RT

We start with the Euclidean plane Π as place of action. A (line) reflection $\sigma : \Pi \to \Pi$ is an indirect involutoric congruence transformation of Π; the set of fixed points is a line, the "axis" a of σ, the set of fixed lines consists of a and a pencil of parallels orthogonal to a. For an arbitrary congruence transformation yield the following two theorems:

Theorem 1 *Each direct congruence transformation is either the identity ι, a translation τ, or a rotation ρ. Each indirect congruence transformation is either a slide reflection λ, or a (line) reflection σ.*

Theorem 2 *Each direct resp. indirect congruence transformation is the product of maximal 2 resp. 3 (line) reflections.*

Special cases of direct congruence transformations are (1) the identity ι, and (2) the rotation ρ having rotation angle π. Such a rotation is called a "half turn" and it is a point reflection at the same time with a pencil of fixed lines through the (single) fixed centre of the half turn ρ.

Remark 1 *Given two reflections σ_1, σ_2 with axes a_1, a_2 forming an angle $\measuredangle a_1a_2 =: \alpha$, then the product $\sigma_1\sigma_2 = \rho$ is a rotation with rotation angle $\delta = 2\alpha$ and exactly one fixed point. Reversely, the reflections σ_1, σ_2 to a given rotation ρ are not uniquely determined. From this property it can be deduced that the product $\sigma_1\sigma_2 = \rho$ is a halfturn $\Leftrightarrow \measuredangle a_1a_2 =: \frac{\pi}{2}$.*

For indirect congruence transformations reflections are the only special cases. Each slide reflection λ is product of three reflections σ_i, which can be chosen such that two have parallel axes a_1, a_2 and the third a_3 is orthogonal to the former two and acting as the "slide axis". Thus the question arises about the conditions for the a_i leading to a product *reflection* $\sigma_1\sigma_2\sigma_3$ instead of just a general slide reflection. The results are well-known:

Theorem 3 (the classical 3RT) $\sigma_1\sigma_2\sigma_3 =: \sigma_4$ *is a reflection $\Longleftrightarrow a_1 \parallel a_2 \parallel a_3$, or $a_1 \cap a_2 \cap a_3 = \{A\} \ldots$ common point.*

As a *key idea for a proof*, which will be used also further on, we assume $\sigma_1\sigma_2\sigma_3 =: \sigma_4$ to be a reflection. $\Rightarrow$ $\sigma_1\sigma_2\sigma_3\sigma_4 = \sigma_4\sigma_4 = id \Rightarrow (\sigma_1\sigma_2)(\sigma_3\sigma_4) = id \Rightarrow (\sigma_1\sigma_2)$ and $(\sigma_3\sigma_4)$ are inverse rotations or inverse translations. In other words, $\sigma_3\sigma_4$ must be just another description for $\sigma_2\sigma_1$. As a consequence, the angles $\measuredangle a_1a_2$ and $\measuredangle a_3a_4$ of their axes α_i must be equal and the $\cap a_i = \{A\}$ resp. $\mathrm{dist}(a_1, a_2) = \mathrm{dist}(a_3, a_4)$ for parallel a_i.

3 Non-Euclidean and circle-geometric versions of the 3RT

a) As this chapter still concerns known facts, we restrict ourselves to presenting some examples starting with the hyperbolic plane as a place of action. We use the Klein-model, i.e. the full projective plane endowed with an absolute hyperbolic polarity according to e.g. N. Wildberger [7]. A hyperbolic reflection σ is a harmonic homology of the (regular) absolute conic ω. This means that the axis a of σ is not tangent to ω. Here again the product $\sigma_1\sigma_2\sigma_3 =: \sigma_4$ is a *reflection* $\Leftrightarrow a_1 \cap a_2 \cap a_3 = \{A\}$, see Figure 1. We omit the discussion of special cases.

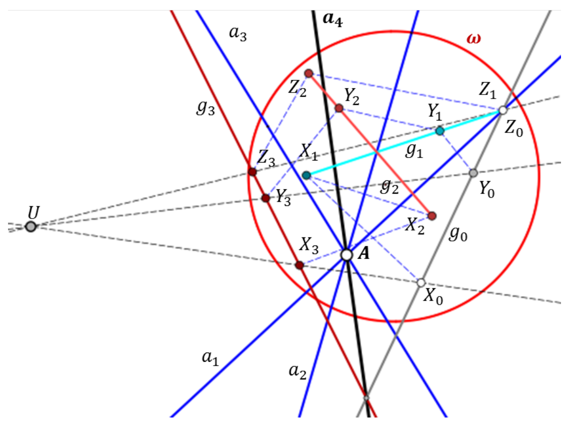

Figure 1: *Visualisation of the hyperbolic version of the 3RT.*

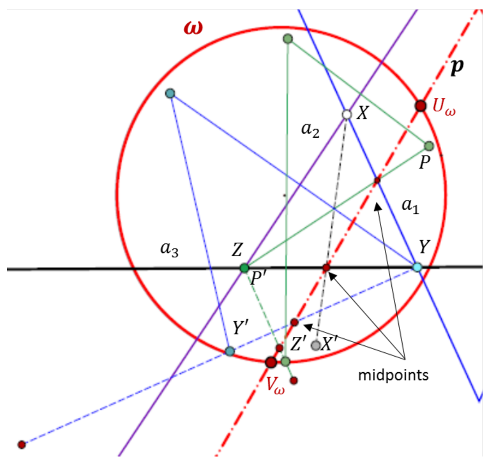

Figure 2: *The product of three hyperbolic reflections induces a projectivity in the absolute conic ω.*

The *general case* then means that the three given axes a_i form a trilateral. Figure 2 shows the case of an inner trilateral $(a_1a_2a_3)$ of ω: The product $\sigma_1\sigma_2\sigma_3$ induces a (hyperbolic) projectivity π at ω with projectivity axis p and fixed points U_ω, V_ω.
The construction of the projectivity axis p turns out to be an analogue to the construction of the Euclidean slide-reflection axis (Figure 3): p passes through midpoints of segments formed by pairs of homologuos points X, $X' = X^{\sigma_1\sigma_2\sigma_3}$.

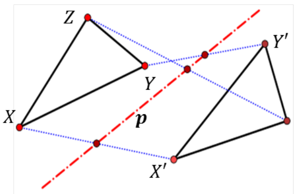

Figure 3: *Construction of the slide-reflection axis p in the Euclidean plane.*

b) As an example for a circle geometry place of action we will use the Euclidean Möbius case. "Reflection" σ now represents an *inversion* at a Möbius circle a.
A "Möbius three reflections theorem" then reads as follows:

Theorem 4 (Möbius 3RT) $\sigma_1\sigma_2\sigma_3 =: \sigma_4$ *is an inversion* $\Longleftrightarrow$ a_1, a_2, $a_3 \in$ *common pencil of Möbius circles.*

As before, $\sigma_1\sigma_2\sigma_3\sigma_4 = \sigma_4\sigma_4 = id \Leftrightarrow \sigma_3\sigma_4 = \sigma_2\sigma_1$, inverse to $\sigma_1\sigma_2$, which is a "Möbius rotation". And, as in the Euclidean case, the rotation angle is twice the (real or imaginary) intersection angle between the two fixed circles a_1, a_2. Figure 4, 5 and 6 visualise such Möbius rotations generated by two inversions.
Recently E. Molnar [4] treated also non-Euclidean circle geometries according to F. Bachmann's point of view, namely generating a geometry via reflections, c.f. [2].
Projective geometric generalisations of the concept inversion seem to trace back to Thomas Archer Hirst (1830 - 1892), even though there are hardly any references to be found:

Definition 1 *A "Hirst inversion" ι is an involutoric mapping of the n-dimensional projective space P^n where a centre point O and a polarity $\pi : P^n \to P^n$ is given, such that (1) O, X, X^ι are collinear, and (2) X, X^ι are conjugate points in π, $(X \neq O)$.*

Remark 2 *This version of defining a Hirst inversion was presented by H. Brauner (1928 - 1990) in his unfortunately unpublished "Lectures on Geometry". How to arrange three Hirst inversions such that their product again is a Hirst inversion seems to be an open problem.*

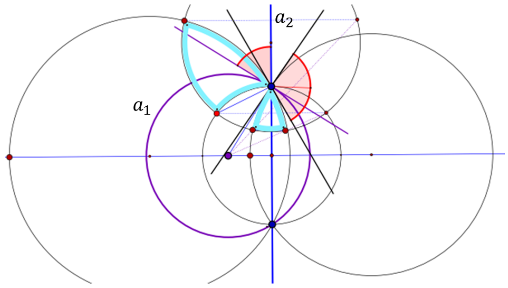

Figure 4: *a_1, a_2 span an elliptic pencil of Möbius circles. The rotation angle of the product $\sigma_1\sigma_2$ is twice the angle $\measuredangle a_1a_2$.*

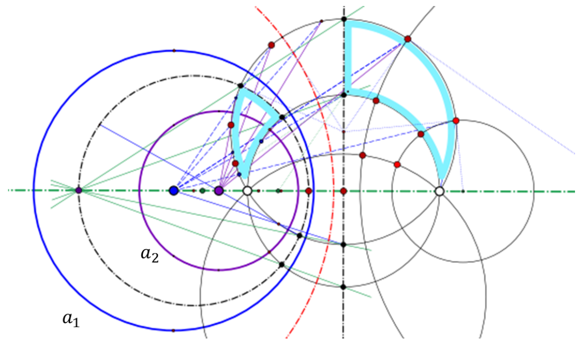

Figure 5: *a_1, a_2 span a hyperbolic pencil of Möbius circles. The rotation angle of the product $\sigma_1\sigma_2$ is imaginary, but still twice the angle $\measuredangle a_1a_2$.*

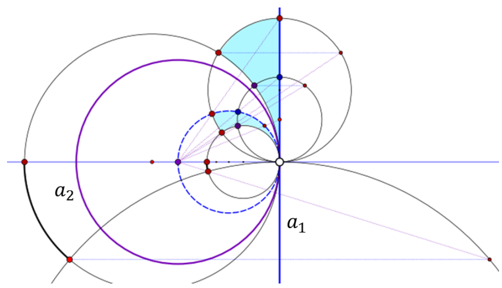

Figure 6: *The limit rotation case: a_1, a_2 span a parabolic pencil of Möbius circles.*

4 The affine 3RT

Place of action is now an affine plane. Let its coordinate field be a commutative field $\mathcal{F}$ with $char\mathcal{F} \neq 2,3$. As we deal with (general) reflections, we consider affine transformations η of $eSL(2,\mathcal{F})$, i.e. they have coordinate representations by matrices with $\det = \pm 1$.

Remark 3 *A (skew) reflection σ is an involutoric perspective affine transformation with axis a and a pencil of parallel fixed lines $\{b\ldots\}$ not containing a. Obviously, "skew" without a concept of orthogonality does not make sense. Again, we distinguish direct- and indirect-affine transformations $\eta \in eSL(2,\mathcal{F})$.*

We summarize the well-known facts

Theorem 5 *Each direct transformation $\eta \in eSL(2,\mathcal{F})$ is either (1) the identity, (2) a translation, (3) a shear (transvection), or (4) an affine rotation. Each indirect transformation $\eta \in eSL(2,\mathcal{F})$ is either (5) a shear-reflection, or (6) a (line) reflection.*

Theorem 6 *Each $\eta \in eSL(2,\mathcal{F})$ is the product of maximal 3 (line) reflections. Each coordinate representation matrix of η can therefore be factorised by matrices of three reflections σ_1, σ_2, σ_3.*

Figure 7 shows the constructions of reflection axes a_i and reflection-directions b_i to given direct-affine resp. indirect-affine triangles T, T' having equal areas, while Figure 8 visualises the construction of a reflection σ_1 and a shear δ to two indirect-affine triangles.

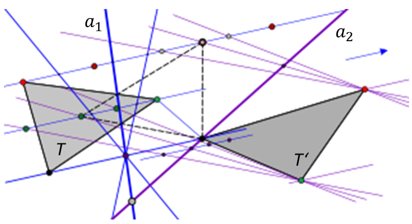

Figure 7: *Construction of two reflections σ_1, σ_2 to direct-affine triangles T, T' such that $\sigma_1\sigma_2 : T \to T'$.*

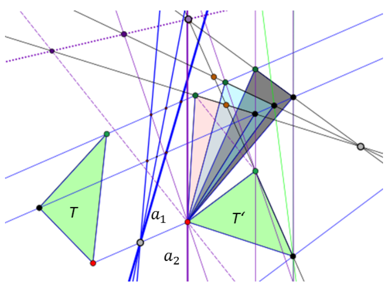

Figure 8: *Construction of a reflection σ and a shear δ to indirect-affine triangles T, T' such that $\sigma\delta : T \to T'$.*

In the first, the direct case, one should find the second fixed point of a projectivity with known first fixed point. In the indirect case variation of σ such that δ becomes a shear leads to a perspectivity. In both cases the construction deals only with linear graphic operations.

Now we ask, when the product of three affine reflections σ_1, σ_2, σ_3, is a reflection $\sigma_1\sigma_2\sigma_3 =: \sigma_4$, too. According the key idea $\sigma_1\sigma_2\sigma_3\sigma_4 = \sigma_4\sigma_4 = id$ the first two factors, $\sigma_1\sigma_2$, and the last two ones, $\sigma_3\sigma_4$, must define inverse affine transformations.

General case: σ_1, σ_2 such that there exists exactly one proper fixed point A, that means, $\sigma_1\sigma_2$ is an "affine rotation". (Note that for a reflection $\sigma(a,b)$ the axis a and direction of the reflection b are different.) Furthermore, we consider three reflections σ_i, $i = 1,2,3$, with axes a_i and reflection-directions b_i such that all six directions are different.
As a necessary condition for $\sigma_1\sigma_2\sigma_3 =: \sigma_4$ being a reflection, too, we get that a_3 must pass through A. The pairs (a_1,b_1), (a_2,b_2) define an involutoric projectivity μ and (a_3,b_3), (a_4,b_4) must define the same μ. This leads to the sufficient condition that all three pairs (a_1,b_1), (a_2,b_2), (a_3,b_3) are pairs of an involutoric projectivity μ. This gives

Theorem 7 *Necessary and sufficient condition for an "affine 3RT" (in the general case) is that the given three reflections σ_1, σ_2, σ_3 have axes a_i and reflection-directions b_i fulfilling the conditions $a_1 \cap a_2 \cap a_3 = \{A\} \wedge (a_1,b_1)$, (a_2,b_2), (a_3,b_3) are pairs of an involution μ.*

In planar Cayley-Klein geometries the sufficient condition "involution" μ is automatically fulfilled and therefore there is no need to mention it.

Special case: σ_1, σ_2 have exactly one proper fixed point A and $a_1 \in \{b_2\} \wedge a_2 \in \{b_1\}$. That means, $\sigma_1\sigma_2$ is a point reflection and the pairs (a_1,b_1), (a_2,b_2) are identical and do not define an involution μ. In this case we could start with $\sigma_4\sigma_1\sigma_2\sigma_3 = \sigma_4\sigma_4 = id$ and have $\sigma_2\sigma_3$ as a proper affine rotation defining an involution μ. Moreover, $\sigma_3\sigma_4$ is the inverse point reflection to $\sigma_1\sigma_2$.
We omit the discussion of further cases.

5 A 3RT in normed planes

Real affine planes can also be endowed with a *norm* based on a convex, centrally symmetric "unit circle" c. Such a plane is called the *Minkowski plane* Π_c and there exist several possibilities to define an orthogonality in Π_c. Most common is the Birkhoff's left-orthogonality, which is non-symmetric. For Minkowski geometry see e.g. [1], [3], [6] and [8].

Definition 2 *A Minkowski (line) reflection is an involutoric affine reflection, whereby the reflection-direction b is left-orthogonal to the axis a.*

Note that a Minkowski reflection σ is, in general, not norm preserving, as the unit circle c and its image c^σ are, in general, not translatoric congruent. Nevertheless, one could study products of three Minkowski reflections:

Theorem 8 (Minkowski-3RT) *The product of three Minkowski reflections σ_1, σ_2, σ_3 is a Minkowski reflection $\sigma_4 \iff \sigma_1$, σ_2, σ_3 fulfill the affine 3RT conditions and $\sigma_1\sigma_2\sigma_3 =: \sigma_4$ fulfills the left-orthogonality condition.*

For a general unit circle c and two given Minkowski reflections σ_1, σ_2 there will, in general, not exist a σ_3 such that $\sigma_1, \sigma_2, \sigma_3$ fulfill the affine 3RT conditions. For an arbitrary unit circle c there exists at least one pair of "conjugat" diameters such that left-orthogonality is symmetric, see [6]. In case c has two such pairs, the first one can act as a_1, a_2 and the second one as a_3, a_4, similar to the special case described in the former chapter. Figure 9 shows such a case with a smooth unit circle c.

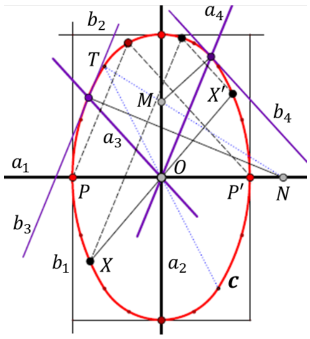

Figure 9: *The Minkowski unit circle c consisting of circular bi-arcs possesses a discrete number of conjugate diameters, which can be used as axes for a valid Minkowski-3RT.*

Special cases can occur for affine regular $2n$-gons as unit circles c, see Figure 10.

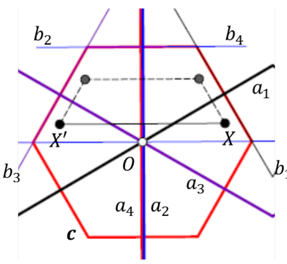

Figure 10: *For an affine-regular hexagon as Minkowski unit circle c and Minkowski reflections σ_1, σ_2, σ_3 with symmetry axes of c as axes a_i the product reflection σ_4 coincide with σ_2.*

To conclude we can find cases of normed planes, where a Minkowski-3RT is valid for specially chosen Minkowski reflections $\sigma_1, \sigma_2, \sigma_3$.

6 The 3RT in affine 3-spaces

Place of action is now a three-dimensional affine space Π^3 with a commutative coordinate field $\mathcal{F}$ and $char\mathcal{F} \neq 2,3$. As we deal with (plane) reflections, we consider affine transformations $\eta \in eSL(3,\mathcal{F})$, i.e. they have coordinate representations by matrices with $\det = \pm 1$.

If $\mathcal{F} \cong \mathbb{R}$ and Π^3 is a Euclidean 3-space, then from Chapters 3 and 4 follows immediately

Theorem 9 *The necessary and sufficient condition for the 3RT in a Euclidean 3-space is that σ_1, σ_2, σ_3 have axis planes a_1, a_2, a_3 which span a pencil of planes.*

For an affine reflection σ in a general affine 3-space Π^3 the reflection-direction b is not parallel to the fixed plane a. As a consequence of Theorem 5 the following theorem, which describes the general case, is obvious:

Theorem 10 *Necessary and sufficient conditions for an affine 3RT in an affine 3-space Π^3 are: The three reflections σ_1, σ_2, σ_3 have axes a_i and reflection-directions b_i fulfilling*

(1) $a_1 \cap a_2 \cap a_3 =: a \ldots$ axis of a pencil of planes,

(2) b_1, b_2, b_3 are parallel to a plane $\gamma \wedge \gamma \nparallel a$,

(3) $\gamma \cap a_i =: c_i$, (c_1,b_1), (c_2,b_2), (c_3,b_3) are pairs of an involution μ.

For some special cases of the given reflections σ_1, σ_2, σ_3 the Theorems 7 and 8 have to be modified. We omit a complete discussion of such cases.
One could also extend the original question, when the product of three reflections σ_1, σ_2, σ_3 is a reflection to a product being a special affine transformation. For example, in the Euclidean 3-space, the reflections at pairwise orthogonal planes a_i lead to a point reflection. We will meet such cases in Chapter 8.
The general direct-affine transformation $\eta \in eSL(3,\mathcal{F})$ is the product of 4 reflections, while indirect-affine transformations can be factorised by 3 reflections. The constructive treatment of these factorisations is a problem of its own.

7 Projective geometric interpretation of the planar 3RT: the 3HHT

Place of action is a projective plane Π. Let its coordinate field be a commutative field $\mathcal{F}$, $char\mathcal{F} \neq 2,3$, for example $\mathcal{F} \cong \mathbb{R}$. "Reflections" σ_i now represents harmonic homologies of $PGL(\Pi,\mathcal{F})$ with axis a_i and centre B_i. Note that the concepts "direct" or "indirect" do not make sense anymore. Harmonic homologies are point reflections and line reflections at the same time.
Applying the key idea that from the assumption $\sigma_1\sigma_2\sigma_3 =: \sigma_4$ is a harmonic homology it follows $\sigma_1\sigma_2\sigma_3\sigma_4 = \sigma_4\sigma_4 = id$. This means that $\sigma_1\sigma_2$ and $\sigma_3\sigma_4$ must be inverse collineations, and therefore we will, at first, study the product of two harmonic homologies:
We start with the general case of $\sigma_1(a_1,B_1)$, $\sigma_2(a_2,B_2)$ represented by Figure 11. The line s connecting the centres B_i of σ_i is a fixed line of the product collineation $\sigma_1\sigma_2 =: \kappa$,

the intersection point S of the axes a_i is a fixed point of κ. The other (real or imaginary) fixed points R, T are fixed points of the involution μ defined by the two pairs $(B_1, A_1 := s \cap a_1)$ and $(B_2, A_2 := s \cap a_2)$. By mapping consecutively a point X to X' and $Y := X'$ to $Y' \ldots$ we obtain a series of points of a conic. This justifies to call κ a "projective rotation".

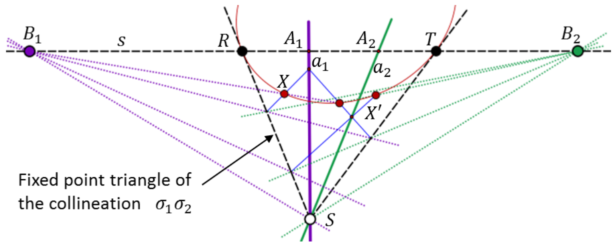

Figure 11: *Product collineation of two harmonic homologies*

The mentioned *general case* describes the classical situation for the Euclidean and affine 3RT, if s is interpreted as the ideal line of the place of action. Therefore it follows

Theorem 11 (3HHT) *Let two harmonic homologies σ_1, σ_2 be in general position. The necessary and sufficient conditions that the product with a third harmonic homology σ_3 is a harmonic homology σ_4 again, is*

(1) $B_3 \in s := B_1B_2$ and $\cap a_i = \{S\}$, $i = 1,2,3$, where B_i are centers and a_i axes of σ_i,

(2) (A_1, B_1), (A_2, B_2), (A_3, B_3) are pairs of an involution μ in s, $(A_i := a_i \cap s)$.

Among the many special cases of mutual positions of $\{(B_1, a_1), (B_2, a_2), (B_3, a_3)\}$ we mention the following:

a) σ_1, σ_2 with $B_1 \in a_2 \wedge B_2 \in a_1$.
In this case, the product $\sigma_1\sigma_2 = \sigma$ is already a harmonic homology with centre $S = a_1 \cap a_2$ and axis $s = B_1B_2$. If we choose $\sigma_3 = \sigma$, then the product $\sigma_1\sigma_2\sigma_3 = \sigma_4 = \iota$ is the identity. For $B_3 \in B_1B_2$ and $a_3 \ni a_1 \cap a_2$ condition (1) is the only condition for $\sigma_1\sigma_2\sigma_3 = \sigma_4$ being a harmonic homology.

b) σ_1, σ_2 with $a_1 = a_2$ and B_1, B_2 arbitrary.
In this case a_3 must coincide with $a_1 = a_2$ and we have a common fixed axis, while B_3 can be chosen arbitrarily, see fig. 12. This is the classical situation for the product of three point-reflections in the Euclidean plane and also in the affine plane. Obviously, the following extension to higher dimensions holds too:

Theorem 12 *Three harmonic homologies σ_1, σ_2, σ_3 in a projective n-space with coinciding hyperplanes a_i in a projective n-space have a harmonic homology $\sigma_4 = \sigma_1\sigma_2\sigma_3$ as their product.*

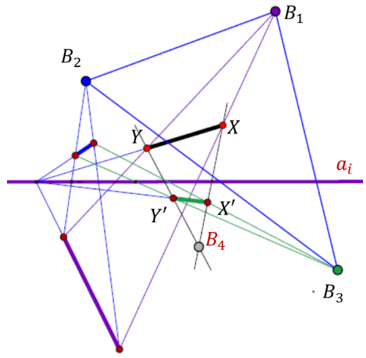

Figure 12: *The product of three harmonic homologies with coinciding axes is a harmonic homology.*

The dual situation, namely coinciding centres B_i and arbitrarily chosen axes a_i leads to the dual version of Theorem 10.

8 Products of harmonic axial collineations and axial reflections

Extending the Three Reflections Theorem to spaces of higher dimensions, as e.g. formulated as Theorem 10, induces an additional idea: Let us generalise σ_i to harmonic *axial collineations* in n-dimensional projective spaces Π^n. We assume the fixed spaces a_i, b_i of σ_i to be skew and complementary subspaces of Π^n, i.e. dim $a_i = n - d_i - 1$, dim $b_i = d_i$. We even might combine axial collineations σ_i, σ_j with dim $b_i \neq$ dim b_j. But already for dimension $n = 4$ we would not get an even remotely comprehensible set of cases and subcases to deal with. Therefore, we restrict the place of action to a projective 3-space Π^3 and its coordinate field to $\mathbb{R} \subset \mathbb{C}$. Furthermore, we will only treat the case $d_i = d_j = 1$, i.e. the pair of axes a, b of a harmonic axial collineation σ that are skew lines. As such a collineation is involutoric it is also called an "axial involution".

As in the former chapters, we pose the question, under which conditions the product $\sigma_1\sigma_2\sigma_3 =: \sigma_4$ turns out to be an axial involution. Assuming $\sigma_1\sigma_2\sigma_3 =: \sigma_4$ to be a reflection it leads to inverse collineations $\sigma_1\sigma_2$ and $\sigma_3\sigma_4$. With this in mind, we first have to study the product κ of two axial involutions σ_1, σ_2.

We start with the general case $\sigma_1(a_1, b_1)$, $\sigma_2(a_2, b_2)$ with $\{a_1, b_1, a_2, b_2\}$ spanning a 2D-set of lines, such a set is called a *line congruence*), see Figure 13:

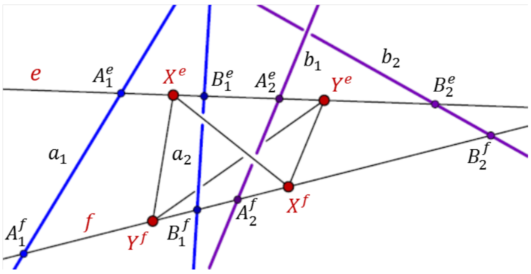

Figure 13: *Sketch visualizing two harmonic axial collineations and the fixed point tetrahedron of their product collineation.*

There are two (real, imaginary or coinciding) lines e, f meeting all four lines a_1, b_1, a_2, b_2. Let us consider the case, when e, f are real and distinct. The construction of such lines is known as the H. E. Timerding problem. Thereby, one intersects the quadric Φ defined by three lines a_1, b_1, a_2 with the additional line b_2. The generators Φ through the intersection points and not belonging to the same regulus of Φ, as a_1, b_1, a_2, are the lines e, f intersecting all four given lines.
The lines e, f are fixed lines of κ; we label their intersection points with the axes a_i, b_i by A_i^e, A_i^f and B_i^e, B_i^f. The pairs (A_1^e, B_1^e) and (A_2^e, B_2^e) define an involutoric projectivity μ^e in, the fixed points (over $\mathbb{C}$) X^e, Y^e which are fixed points of κ. The analogue procedure for the line f results in the remaining fixed points (over $\mathbb{C}$) X^f, Y^f of κ. By mapping an admissible point $P \mapsto P^{\sigma_1\sigma_2}$ we finally receive a well-defined collineation κ.
The conditions to be fulfilled by a third harmonic axial collineation $\sigma_3(a_3, b_3)$ such that $\sigma_1\sigma_2\sigma_3 =: \sigma_4$ is again an axial collineation are then obvious:

Theorem 13 (axial 3HHT) *The product $\sigma_1\sigma_2\sigma_3 =: \sigma_4$ of three generally positioned harmonic axial collineations σ_1, σ_2, σ_3 is again a harmonic axial collineation, if and only if their axes a_i, b_i belong to a line congruence (i.e. they span a 2D-set of lines) with axes e, f (over $\mathbb{C}$) and the intersection pairs (A_i^e, B_i^e) and (A_i^f, B_i^f) with these axes corresponds to involutions μ^e, μ^f on e and f.*

Note that the product collineation κ of two axial involutions σ_1, σ_2 generates a subgroup $\{\kappa^n, n \in \mathbb{Z}\} \subset PGL(\Pi^3, \mathbb{R} \subset \mathbb{C})$, namely the one keeping the quadrics of a pencil of quadrics fixed. This is an analogue to the planar case of projective rotations, as we mentioned in Chapter 7.
Among the many cases of mutual positions of $\sigma_1(a_1, b_1)$, $\sigma_2(a_2, b_2)$ the one with $\{a_1, b_1, a_2, b_2\}$ spanning a 1D-set of lines, (namely a regulus on a quadric Φ) shall at least be mentioned. In this case the set of fixed lines is the complementary regulus and the product $\sigma_1\sigma_2 =: \kappa$ already is an axial collineation. A third harmonic axial collineation σ_3 then must have axes belonging to the same regulus as a_1, b_1, a_2, b_2 such that $\sigma_1\sigma_2\sigma_3 =: \sigma_4$ is again a harmonic axial collineation.
Another very special case of mutual positions of $\sigma_1(a_1, b_1)$, $\sigma_2(a_2, b_2)$ would be that lines a_1, b_1, a_2, b_2 form a skew quadrilateral. Interpreting the plane $b_1 \vee b_2$ as the ideal plane of the Euclidean 3-space and a_1, a_2 as (intersecting) orthogonal lines, we might speak of σ_1, σ_2 as *axial reflections*. The product $\sigma_1\sigma_2 =: \kappa$ is then an axial reflection at an axis orthogonal to $a_1 \vee a_2$ and passing through $a_1 \cap a_2$. Choosing κ as the third axial reflection gives then the identity as the product $\sigma_1\sigma_2\sigma_3$.
Finally, we consider the case of axial reflections σ_1, σ_2, σ_3 whereby a_i are skew edges of a cube. In this case the product $\sigma_1\sigma_2\sigma_3$ turns out to be a translation τ, see Figure 14.

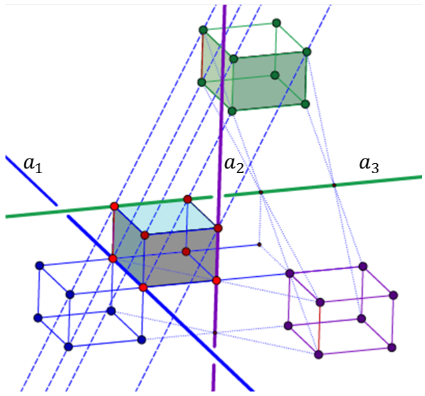

Figure 14: *The product of three axial reflections at pairwise orthogonal axes is a translation.*

Conclusion

The Three Reflections Theorem belongs to basic and well-known geometric facts. But as is shown in the chapters above, there are still surprising results to gain, when we generalise the place of action. Leading idea is Bachmann's point of view of generating a geometry via reflections, even when it in most cases defines only an important sub-geometry of "classical" geometries. There is a unifying treatment for all mentioned generalisations: we study products of two reflections first and choose the third such that it, together with a fourth, gives the inverse product. This makes it possible to omit explicit calculations.

Acknowledgement

The author thanks Prof. E. Jurkin and the reviewer(s) for valuable help and hints.

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