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Curves of Brocard Points in Triangle Pencils in Isotropic Plane

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ABSTRACT

In this paper we consider a triangle pencil in an isotropic plane consisting of the triangles that have the same circumscribed circle. We study the locus of their Brocard points, two curves of order 4.

Key words: isotropic plane, triangle pencil, Brocard points

MSC2010: 51N25

Krivulje Brocardovih točaka u pramenovima trokuta u izotropnoj ravnini

SAŽETAK

U radu se promatra pramen trokuta sa zajedničkom opisanom kružnicom. Pokazuje se da Brocardove točke tih trokuta leže na dvije krivulje 4. reda.

Ključne riječi: izotropna ravnina, pramen trokuta, Brocardove točke

1 Introduction

In [1] the author gave a historical overview and presented many results regarding the Brocard points of polygons in the Euclidean plane. The Brocard points of the triangles in the isotropic plane were introduced and studied in [3], [7] and [8], while such points for harmonic quadrangles were observed in [5] and [6].

In this paper we study the pencil of triangles having the same circumscribed circle and determine the locus of their Brocard points. In a way this paper is a sequel of [2] where a similar study for curves of centroids, Gergonne points and symmedian centers in triangle pencils in the isotropic plane was given.

Let us start by recalling some basic facts about the isotropic plane. It is a real projective plane where the metric is induced by a real line f and a real point F incident with it. All straight lines through the absolute point F are called isotropic lines, and all points incident with the absolute line f are called isotropic points. Two lines are parallel if they are incident with the same isotropic point, and two points are parallel if they lie on the same isotropic line. In the affine model of the isotropic plane where the coordinates of the points are defined by $x=\frac{x_1}{x_0}$, $y=\frac{x_2}{x_0}$, the absolute line has the equation $x_0=0$ and the absolute point has the coordinates $(0,0,1)$. For two non-parallel points $A=(x_A,y_A)$ and $B=(x_B,y_B)$ a distance is defined by $d(A,B)=x_B-x_A$, and for two non-parallel lines p and q, given by the equations $y=k_px+l_p$ and $y=k_qx+l_q$, an angle is defined by $\angle(p,q)=k_q-k_p$, [7]. As a circle is defined as a conic touching the absolute line at the absolute point, it has an equation of a form $y=ax^2+bx+c$, $a,b,c\in\mathbb{R}$.

2 Brocard points

Let a triangle ABC having the circumscribed circle k with equation $y=x^2$ be given (see [4]). The triangle vertices are of the form:

$$A(a,a^2),\quad B(b,b^2),\quad C(c,c^2) \tag{1}$$

and its sides have the equations

$$\begin{aligned} AB &\quad \dots \quad y=(a+b)x-ab \\ BC &\quad \dots \quad y=(b+c)x-bc \\ CA &\quad \dots \quad y=(a+c)x-ac. \end{aligned} \tag{2}$$

The tangent lines to k at the points A, B and C are given by equations:

$$\begin{array}{lll} t_A & \dots & y = 2ax - a^2 \\ t_B & \dots & y = 2bx - b^2 \\ t_C & \dots & y = 2cx - c^2. \end{array} \tag{3}$$

Theorem 1 *Let ABC be a triangle and let the lines a', b', c' be incident with the vertices A, B, C and form equal angles with the sides AB, BC, CA, respectively. For the triangle $A'B'C'$, where $A' = c' \cap a'$, $B' = a' \cap b'$ and $C' = b' \cap c'$, the following equalities hold:*

$$\begin{aligned} \angle(CA, AB) &= \angle(C'A', A'B') \\ \angle(AB, BC) &= \angle(A'B', B'C') \\ \angle(BC, CA) &= \angle(B'C', C'A') \end{aligned} \tag{4}$$

and

$$\frac{d(A',B')}{d(A,B)} = \frac{d(B',C')}{d(B,C)} = \frac{d(C',A')}{d(C,A)}. \tag{5}$$

Proof. Let the angle from the theorem be denoted by h, i.e.

$\angle(a', AB) = \angle(b', BC) = \angle(c', CA) = h.$

Then the lines a', b', c' are given by:

$$\begin{array}{lll} a' & \dots & y = (a+b-h)x + a(h-b) \\ b' & \dots & y = (b+c-h)x + b(h-c) \\ c' & \dots & y = (c+a-h)x + c(h-a) \end{array} \tag{6}$$

and their intersections are

$$\begin{array}{l} A'\left(a - \frac{a-c}{b-c}h,\quad a^2 - \frac{(a+b)(a-c)}{b-c}h + \frac{a-c}{b-c}h^2\right) \\ B'\left(b - \frac{b-a}{c-a}h,\quad b^2 - \frac{(b+c)(b-a)}{c-a}h + \frac{b-a}{c-a}h^2\right) \\ C'\left(c - \frac{c-b}{a-b}h,\quad c^2 - \frac{(c+a)(c-b)}{a-b}h + \frac{c-b}{a-b}h^2\right). \end{array} \tag{7}$$

It follows from (2) and (6) that:

$\angle(CA, AB) = (a+b) - (c+a) = b - c$

and

$\angle(C'A', A'B') = \angle(c', a') = (a+b-h) - (c+a-h) = b - c.$

Therefore, $\angle(CA, AB) = \angle(C'A', A'B')$. The other two equalities of (4) can be proved analogously. From (7) we get:

$$\begin{aligned} d(A',B') &= b - \frac{b-a}{c-a}h - a + \frac{a-c}{b-c}h \\ &= (b-a) + h \cdot \frac{ab+bc+ac-a^2-b^2-c^2}{(b-c)(c-a)} \\ &= (b-a)\left[1 - h \cdot \frac{ab+bc+ac-a^2-b^2-c^2}{(a-b)(b-c)(c-a)}\right]. \end{aligned}$$

Thus,

$$\frac{d(A',B')}{d(A,B)} = 1 - h \cdot \frac{ab+bc+ca-a^2-b^2-c^2}{(a-b)(b-c)(c-a)}. \tag{8}$$

Similarly we get that the ratios $\frac{d(B',C')}{d(B,C)}$ and $\frac{d(C',A')}{d(C,A)}$ take the same value. □

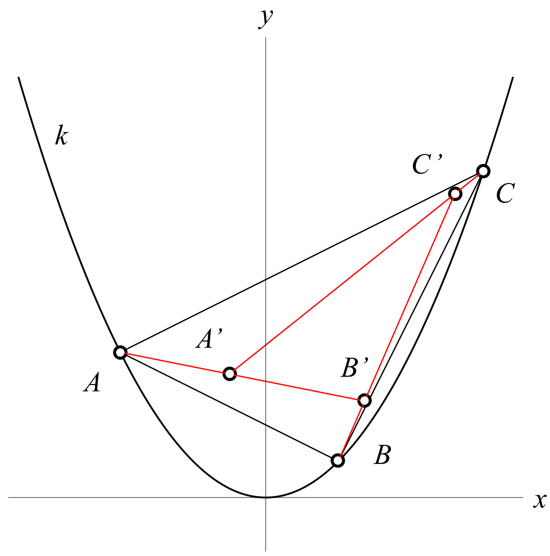

Figure 1: *Visualization of Theorem 1*

It follows immediately from (8):

Corollary 1 *Let ABC be a triangle and let the lines a', b', c' be incident with the vertices A, B, C and form equal angles h with the sides AB, BC, CA, respectively. The lines lines a', b', c' are concorrent if and only iff*

$$h = \frac{(a-b)(b-c)(c-a)}{ab+bc+ca-a^2-b^2-c^2}. \tag{9}$$

The angle h from Corollary 1 is called the Brocard angle, and the point P_1 incident with the lines a', b', c', is called the *first Brocard point* of the triangle ABC.
From (7) we get the coordinates of P_1:

$$\begin{aligned} x &= \frac{a+b+c}{3} - \frac{h}{3}\left(\frac{a-c}{b-c} + \frac{b-a}{c-a} + \frac{c-b}{a-b}\right) \\ y &= \frac{a^2+b^2+c^2}{3} + \frac{h^2}{3}\left(\frac{a-c}{b-c} + \frac{b-a}{c-a} + \frac{c-b}{a-b}\right) \\ &\quad - \frac{h}{3}\left(\frac{(a+b)(a-c)}{b-c} + \frac{(b+c)(b-a)}{c-a} + \frac{(c+a)(c-b)}{a-b}\right), \end{aligned} \tag{10}$$

where h is given by (9).

Similarly, the *second Brocard point* P_2 is defined as the point such that its connection lines with the vertices A, B, C form the equal angles with the sides AC, CB, and BA, respectively. These angles equal $-h$.

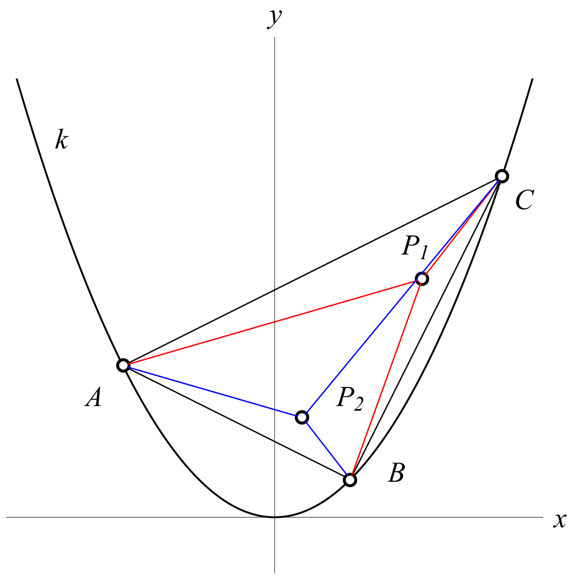

Figure 2: *The Brocard points P_1 and P_2 of the triangle ABC*

3 Triangle pencils

In order to get the pencil of the triangles, we will keep the vertices A and B fixed and move vertex C along the circumscribed circle k. Now the expressions (10) present the parametric equation of the locus of the first Brocard points, a curve k_1. By eliminating c we get an implicit equation of k_1:

$$x^4-2ax^3-x^2y+(2a^2+2ab+b^2)x^2-2bxy+y^2$$
$$-2ab(a+b)x+a(2b-a)y+a^2b^2=0. \quad (11)$$

It is of degree 4, so k_1 is a curve of order 4. Its only intersection point with the absolute line is the absolute point. It intersects the circle k in two basic points A and B, both with intersection multiplicity 2. It can be easily checked that A is a cusp of k_1 with tangent line AB, while B is a regular point at which k_1 touches k. The points A and B are Brocard points of two degenerated triangles of the pencil obtained when the third vertex C coincide with A and B, respectively. This observation can be summerized in:

Theorem 2 *Let the points A and B on the circle k be given. The curve of the first Brocard points of all triangles ABC having the same circumscribed circle k is a curve of order 4. It has a cusp in the point A and touches k at the point B.*

Analogously, it can be shown that the equation of that locus of the second Brocard points is

$$x^4-2bx^3-x^2y+(a^2+2ab+2b^2)x^2-2axy+y^2$$
$$-2ab(a+b)x+2aby-b^2y+a^2b^2=0 \quad (12)$$

and that the following theorem holds:

Theorem 3 *Let the points A and B on the circle k be given. The curve of the second Brocard points of all triangles ABC having the same circumscribed circle k is a curve of order 4. It has a cusp in the point B and touches k at the point A.*

Two Brocard curves k_1 and k_2 given by (11) and (12) intersect in points $\left(\frac{a+b}{2}, \frac{5a^2+5b^2-2ab\pm\sqrt{5}(a-b)^2}{8}\right)$ parallel to the midpoint $M_{AB}\left(\frac{a+b}{2}, \frac{a^2+b^2}{2}\right)$ of the points A and B.

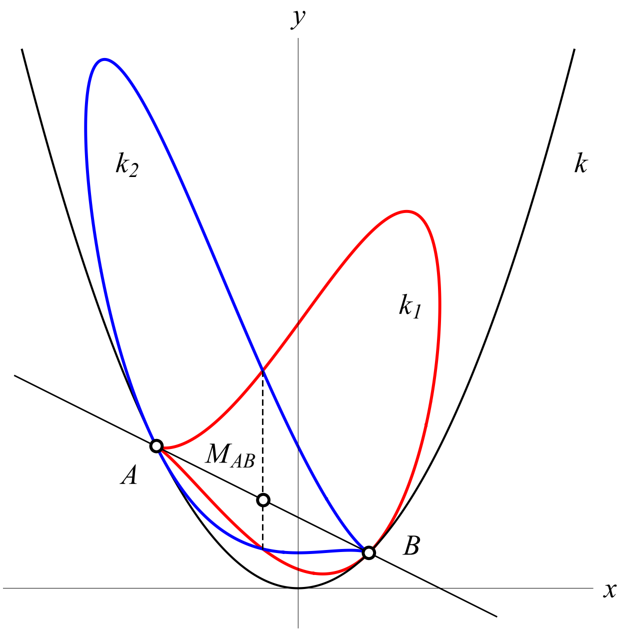

Figure 3: *The curves k_1 and k_2 of the first and second Brocard points for a pencil of triangles with the same circumscribed circle k*

Now, we will study the Brocard curves of the pencil of tangential triangles. The tangential triangle $A_tB_tC_t$ of a given triangle ABC is a triangle formed by the tangent lines to the circumscribed circle k of the triangle ABC at its vertices. The equations of the tangent lines are given by (3) and they intersect in the points $A_t=\left(\frac{b+c}{2}, bc\right)$, $B_t=\left(\frac{a+c}{2}, ac\right)$ and $C_t=\left(\frac{a+b}{2}, ab\right)$, parallel to the midpoints of the sides BC, CA and AB, respectively.
Keeping the points A and B fixed and moving C on the circle k, we obtain the pencil of tangential triangles $A_tB_tC_t$. The triangles of this pencil have the same inscribed circle, the circle k. They have one fixed vertex C_t, and two fixed sides t_A and t_B. Repeating the procedure from above we

calculate the locus of the first Brocard points of the pencil of tangential triangles:

$$\begin{aligned}&32ax^5-16x^4y-16ab(5a+3b)x^4+24(a+b)x^3y+4x^2y^2\\&+8(10a^3+13a^2b+3ab^2+b^3)x^3-4(3a^2+8ab+6b^2)x^2y\\&-2(2a-b)xy^2-y^3-4(10a^4+21a^3b+11a^2b^2+ab^3+3b^4)x^2\\&+2(a^3+7a^2b+7ab^2+7b^3)xy+(a^2+2ab-2b^2)y^2\\&+2a^3(5a^2+15ab+13b^2)x-b(2a^3+4a^2b+2ab^2+3b^3)y\\&-a^6-4a^5b-5a^4b^2-2a^3b^3-b^6=0. \qquad (13)\end{aligned}$$

Thus, the curve of the first Brocard points k_{t1} is a curve of order 5. It intersects the absolute line at the absolute point with the intersection multiplicity 4, and at the isotropic point of the line t_A. It has a singular point at C_t since every line through C_t intersects t_A in the point C_t counted three times and two further points. This fact can be proved by putting $y=m(x-\frac{a+b}{2})+ab$, $m\in\mathbb{R}$, into (13), which than becomes an equation in x with a triple root $x=\frac{a+b}{2}$. Only in the special case when $y=2bx-b^2$, the equation (13) takes the form $(a-b)(a+b-2x)^5=0$ and $x=\frac{a+b}{2}$ is its fivefold root. Thus, all tangents to k_{t1} at C_t coincide with t_B, Figure 4.

The similar study can be done for the curve of the second Brocard points k_{t2} and the analogous results would be obtained. Therefore, we can conclude our observation with the following:

Theorem 4 *The curves of Brocard points of all the tangential triangles in the pencil of triangles having the same circumscribed circle are the curves of order 5.*

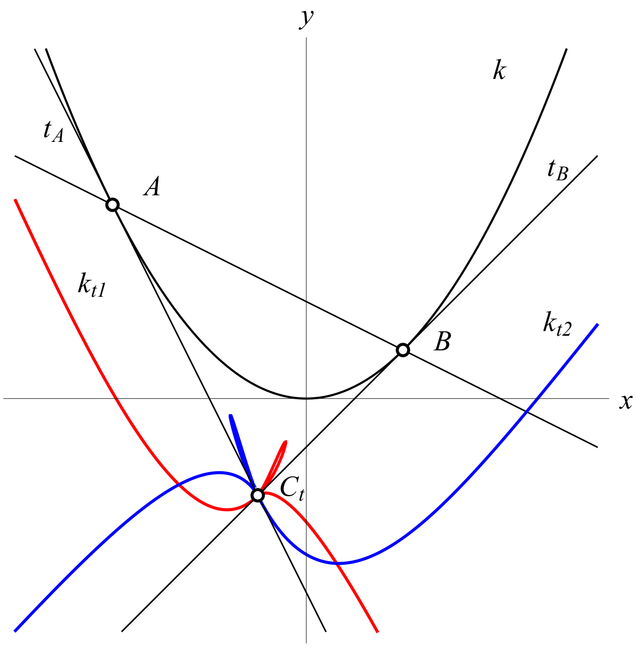

Figure 4: *The Brocard curves k_{t1} and k_{t2} in the pencil of tangential triangles of the triangles with the same circumscribed circle k*

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