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STRUČNI RADOVI

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SZILVIA BÉLA

Stitching B-Spline Curves Symbolically

Stitching B-spline Curves Symbolically

ABSTRACT

We present an algorithm for stitching B-spline curves, which is different from the generally used least square method. Our aim is to find a symbolic solution for unifying the control polygons of arcs separately described as 4th degree B-spline curves. We show the effect of interpolation conditions and fairing functions as well.

Key words: B-spline curve, B-spline surface, merging, interpolation, fairing

MSC2010: 65D17, 65D05, 65D07, 68U05, 68U07

Simboličko spajanje B-splajn krivulja

SAŽETAK

Predstavljamo algoritam za spajanje B-splajn krivulja, koji se razlikuje od općenito upotrebljavane metode najmanjih kvadrata. Naš cilj je naći simboličko rješenje za ujedinjavanje kontrolnih poligona lukova koji se svaki zasebno opisuju kao B-splajn krivulje 4. stupnja. Također pokazujemo utjecaj uvjeta interpolacije i postizanja glatkih funkcija.

Ključne riječi: B-splajn krivulja, B-splajn ploha, integriranje, interpolacija, postizanje glatkoće

1 Introduction

Stitching or merging B-spline curves is a frequently used technique in geometric modeling, and is usually implemented in CAD-systems. These algorithms are basically numerical interpolations using the least squares method. The problem, how to replace two or more curves which are generated separately and defined as B-spline curves, has well functioning numerical solutions, therefore, relatively few papers have been published about this topic. In [6] and [3] methods for approximate merging of B-spline curves and surfaces are given. In [4] one of the symbolical algorithms is described, which extends B-spline curves by adding more interpolation points one by one at the end of the curve. In [5] the construction of a covering surface is shown for unifying more B-spline surfaces.

We approach the stitching problem from a geometrical point of view, and represent a symbolical solution to compute the control points of the new curve from the control points of the two given curve segments and appropriate interpolation conditions. This symbolical solution is stable, it can be used generally for any two given curves. The error of the interpolation depends on the curvatures of the input curves. Larger difference in their curvatures raises the error. In order to reduce the error, two of the new control points are adjusted by fairing conditions using the concrete numerical data. This computation requires minimization of quadratic functions leading to solve linear equations. In this way we avoid non-linear optimization problems. Applying fairing functions for modifying the shape and the properties of curves and surfaces is a standard technique. In [7], [8] and [9] constructions of B-spline surfaces with boundary conditions are presented using fairing functions. Finally, merging of B-spline surface patches are shown applying the developed curve stitching method for their parameter curves.

2 Symbolical solution for stitching two B-spline curve segments

In our symbolical solution for stitching two given curves we assume that they are represented by B-spline segments of degree 4 with uniform periodic knot vectors. The one-parameter vector function of such a curve is

$$\mathbf{r}(t) = \begin{pmatrix} t^4 & t^3 & t^2 & t & 1 \end{pmatrix} \cdot \mathbf{M} \cdot \begin{pmatrix} \mathbf{p}_0 \\ \mathbf{p}_1 \\ \mathbf{p}_2 \\ \mathbf{p}_3 \\ \mathbf{p}_4 \end{pmatrix}, 0 \le t \le 1,$$

where

$$\mathbf{M} = \frac{1}{24} \begin{pmatrix} 1 & -4 & 6 & -4 & 1 \\ -4 & 12 & -12 & 4 & 0 \\ 6 & -6 & -6 & 6 & 0 \\ -4 & -12 & 12 & 4 & 0 \\ 1 & 11 & 11 & 1 & 0 \end{pmatrix}.$$

* Supported by a joint project between the TU Berlin and the BUTE.

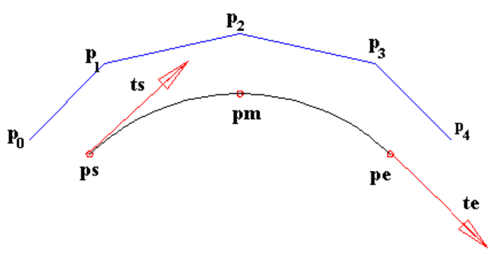

Figure 1: *Input data and control points of one curve segment*

We recall the symbolical solution of the interpolation problem ([10]), where the input data are the interpolation points **ps**, **pm**, **pe**, and the derivatives at the endpoints **ts** and **te** (Fig. 1). The output is the 5 control points $\mathbf{p}_i$, $i = 0,\dots,4$ computed from the conditions

$\mathbf{r}(0) = \mathbf{ps}$, $\mathbf{r}(0.5) = \mathbf{pm}$, $\mathbf{r}(1) = \mathbf{pe}$, $\dot{\mathbf{r}}(0) = \mathbf{ts}$, $\dot{\mathbf{r}}(1) = \mathbf{te}$.

The control points are expressed by the input data as the solution of this system of linear equations.

$$\begin{pmatrix} \mathbf{p}_0 \\ \mathbf{p}_1 \\ \mathbf{p}_2 \\ \mathbf{p}_3 \\ \mathbf{p}_4 \end{pmatrix} = \begin{pmatrix} -30.1667\mathbf{pe} - 46.1667\mathbf{ps} + 77.3333\mathbf{pm} + 6.33333\mathbf{te} - 16.3333\mathbf{ts} \\ 7.83333\mathbf{pe} + 11.8333\mathbf{ps} - 18.6667\mathbf{pm} - 1.66667\mathbf{te} + 2.66667\mathbf{ts} \\ -6.16667\mathbf{pe} - 6.16667\mathbf{ps} + 13.3333\mathbf{pm} + 1.33333\mathbf{te} - 1.33333\mathbf{ts} \\ 11.8333\mathbf{pe} + 7.83333\mathbf{ps} - 18.6667\mathbf{pm} - 2.66667\mathbf{te} + 1.66667\mathbf{ts} \\ -46.1667\mathbf{pe} - 30.1667\mathbf{ps} + 77.3333\mathbf{pm} + 16.3333\mathbf{te} - 6.33333\mathbf{ts} \end{pmatrix}$$

In order to demonstrate the behaviour of this symbolic interpolation method we approximated a circular arc $\mathbf{c}(t)$ with central angle $\leq \pi/3$ interpolated by the curve $\mathbf{r}(t)$. The numerical error measured by $\int_0^1 (\mathbf{c}(t) - \mathbf{r}(t))^2 dt$ is less than 10^{-28}, i.e approximately zero.

We use this experience for stitching two joining B-spline curve segments. In that algorithm we will use also similar interpolation data and B-spline functions of degree 4.

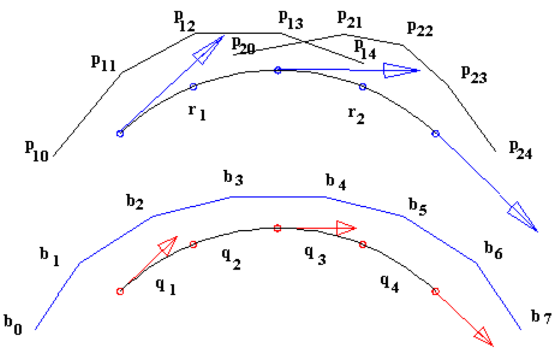

Figure 2: *Merging two curves into 4 B-spline segments*

We assume that the two input segments are given by B-spline functions, one by $\mathbf{r}_1(t)$ with control points $\mathbf{p}_{1j}$ and the other by $\mathbf{r}_2(t)$ with control points $\mathbf{p}_{2j}$, $(j = 0,\dots,4)$.

We generate the resulting B-spline curve with 4 segments $\mathbf{q}_i(t)$, $0 \leq t \leq 1$, $(i = 1,\dots,4)$ determined by 8 control points $\mathbf{b}_j$, $(j = 0,\dots,7)$.

The interpolation conditions are 5 points + 3 tangent vectors (Fig. 2).

$$\mathbf{q}_1(0) = \mathbf{r}_1(0),\ \mathbf{q}_2(0) = \mathbf{r}_1(0.5),\ \mathbf{q}_2(1) = \mathbf{r}_1(1),$$
$$\mathbf{q}_3(1) = \mathbf{r}_2(0.5),\ \mathbf{q}_4(1) = \mathbf{r}_2(1)$$
$$\dot{\mathbf{q}}_1(0) = \dot{\mathbf{r}}_1(0),\ \dot{\mathbf{q}}_2(1) = \dot{\mathbf{r}}_1(1),\ \dot{\mathbf{q}}_4(1) = \dot{\mathbf{r}}_2(1)$$

These 8 equations are linear in the unknown control points of the new B-spline curve. The solution of the system results in the required control points $\mathbf{b}_j$, $(j = 0,\dots,7)$ expressed as linear combinations of the given control points $\mathbf{p}_{1i}$, $\mathbf{p}_{2i}$, $(i = 0,\dots,4)$.

Especially,

$$\begin{aligned}
\mathbf{b}_0 = {} & 1.07083\mathbf{p}_{10} + 2.0166\mathbf{p}_{11} - 4.2305\mathbf{p}_{12} + 3.7694\mathbf{p}_{13} + 1.4069\mathbf{p}_{14} \\
& -0.0090\mathbf{p}_{20} - 0.6500\mathbf{p}_{21} - 1.8236\mathbf{p}_{22} - 0.5416\mathbf{p}_{23} - 0.0090\mathbf{p}_{24} \\
\mathbf{b}_1 = {} & -0.01527\mathbf{p}_{10} + 0.5944\mathbf{p}_{11} + 1.0083\mathbf{p}_{12} - 0.9638\mathbf{p}_{13} - 0.3236\mathbf{p}_{14} \\
& +0.0020\mathbf{p}_{20} + 0.1500\mathbf{p}_{21} + 0.4208\mathbf{p}_{22} + 0.1250\mathbf{p}_{23} + 0.0020\mathbf{p}_{24} \\
\mathbf{b}_2 = {} & 0.0090\mathbf{p}_{10} + 0.2055\mathbf{p}_{11} + 0.2930\mathbf{p}_{12} + 0.7444\mathbf{p}_{13} + 0.2145\mathbf{p}_{14} \\
& -0.0013\mathbf{p}_{20} - 0.1000\mathbf{p}_{21} - 0.2805\mathbf{p}_{22} - 0.0833\mathbf{p}_{23} - 0.0013\mathbf{p}_{24} \\
\mathbf{b}_3 = {} & -0.0020\mathbf{p}_{10} + 0.1833\mathbf{p}_{11} + 0.9152\mathbf{p}_{12} - 0.3555\mathbf{p}_{13} - 0.2076\mathbf{p}_{14} \\
& +0.0013\mathbf{p}_{20} + 0.1000\mathbf{p}_{21} + 0.2805\mathbf{p}_{22} + 0.0833\mathbf{p}_{23} + 0.0013\mathbf{p}_{24} \\
\mathbf{b}_4 = {} & 0.0013\mathbf{p}_{10} - 0.1222\mathbf{p}_{11} + 0.0750\mathbf{p}_{12} + 1.4361\mathbf{p}_{13} + 0.3097\mathbf{p}_{14} \\
& -0.0020\mathbf{p}_{20} - 0.1500\mathbf{p}_{21} - 0.4208\mathbf{p}_{22} - 0.1250\mathbf{p}_{23} - 0.0020\mathbf{p}_{24} \\
\mathbf{b}_5 = {} & -0.0013\mathbf{p}_{10} + 0.1222\mathbf{p}_{11} - 0.1861\mathbf{p}_{12} - 1.6305\mathbf{p}_{13} - 0.3375\mathbf{p}_{14} \\
& +0.0090\mathbf{p}_{20} + 0.6500\mathbf{p}_{21} + 1.8236\mathbf{p}_{22} + 0.5416\mathbf{p}_{23} + 0.0090\mathbf{p}_{24} \\
\mathbf{b}_6 = {} & 0.0020\mathbf{p}_{10} - 0.1833\mathbf{p}_{11} + 0.3069\mathbf{p}_{12} + 2.4944\mathbf{p}_{13} + 0.5131\mathbf{p}_{14} \\
& -0.01527\mathbf{p}_{20} - 0.8500\mathbf{p}_{21} - 1.3361\mathbf{p}_{22} + 0.0833\mathbf{p}_{23} - 0.01527\mathbf{p}_{24} \\
\mathbf{b}_7 = {} & -0.0090\mathbf{p}_{10} + 0.7944\mathbf{p}_{11} - 1.4041\mathbf{p}_{12} - 10.9388\mathbf{p}_{13} - 2.2423\mathbf{p}_{14} \\
& +0.0708\mathbf{p}_{20} + 3.3500\mathbf{p}_{21} + 6.0583\mathbf{p}_{22} + 4.2500\mathbf{p}_{23} + 1.0708\mathbf{p}_{24}
\end{aligned}$$

The range of magnitudes of the coefficients show that the solution is stable. The corresponding vector equation of the unified B-spline curve is

$$\mathbf{q}_i(t) = \begin{pmatrix} t^4 & t^3 & t^2 & t & 1 \end{pmatrix} \cdot \mathbf{M} \cdot \begin{pmatrix} \mathbf{b}_{i+j-1} \\ \mathbf{b}_{i+j} \\ \mathbf{b}_{i+j+1} \\ \mathbf{b}_{i+j+2} \\ \mathbf{b}_{i+j+3} \end{pmatrix}, 0 \leq t \leq 1,$$

where $i = 1,\dots,4$ (4 segments), $j = 0,\dots,4$ (each with 5 control points).

The examples show that the interpolation error depends on the variation of the curvatures of the given input curves. This error is measured by the integrated sum of the quadratic difference between the corresponding points of the given and the computed new curves, while each segment is parametrized on the $[0,1]$ interval. That is, the error

$$\text{error} = \sum_{\textit{all segments}} \int_0^1 (\mathbf{r}_{ik}(t) - \mathbf{q}_j(t))^2 \, dt,$$

$$(i = 1,2,\ k = 1,2,\ j = 1\dots 4).$$

Fig. 2 shows the segmentation of the curves, where each of the given curves $\mathbf{r}_1(t)$ and $\mathbf{r}_2(t)$ is divided into two parts. In Fig. 3 the result of stitching two circular arcs of equal curvatures is shown with the computed control points. The interpolation points are marked by circles. The new B-spline curve interpolates the given arcs practically with zero (10^{-26}) error. If the curvatures of the two arcs are different, the error is larger (Fig. 4 and Fig. 5, the given curves are drawn lighter, the interpolation points are marked with circles, the interpolation derivatives are not shown). Moreover, the resulting curve shows a wavy shape due to the low number of interpolation conditions.
The interpolation error can be reduced by prescribing more interpolation conditions. The shape of the curve can be improved by fairing (smoothing) conditions.

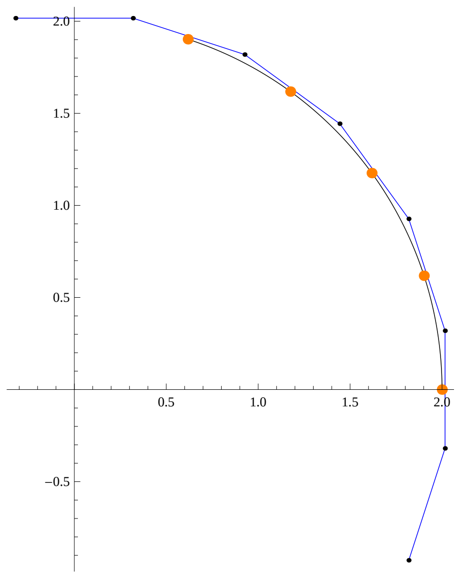

Figure 3: *Merged circular arcs, error* ≈ 0

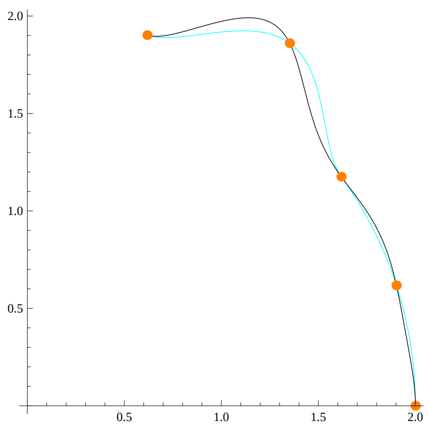

Figure 4: *Merged curves with different curvatures, error*$= 0,0066$

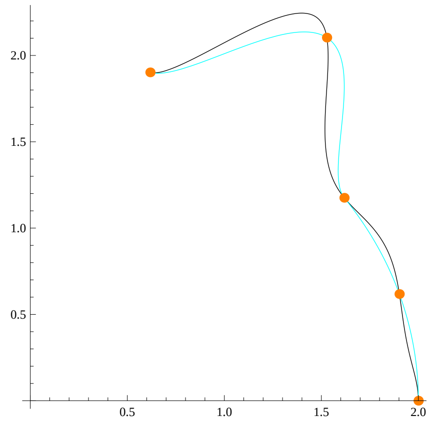

Figure 5: *Merged curves with more different curvatures, error*$= 0,026$

3 The effect of fairing conditions

The solution, where 8 control points are determined from 2×5 given control points, result in a uniquely determined B-spline curve with 4 segments. In order to apply fairing conditions free control points are necessary. Therefore, the prescribed 8 interpolation conditions have to be relaxed. In our investigation we have deleted two interpolation points (the midpoints of the input curves), and have chosen two variable control points $\mathbf{b}_3$ and $\mathbf{b}_4$ for modifying the shape of the resulting curve. In this case 3 points and 3 derivatives are prescribed,

$$\mathbf{q}_1(0) = \mathbf{r}_1(0),\ \mathbf{q}_2(1) = \mathbf{r}_1(1),\ \mathbf{q}_4(1) = \mathbf{r}_2(1)$$
$$\dot{\mathbf{q}}_1(0) = \dot{\mathbf{r}}_1(0),\ \dot{\mathbf{q}}_2(1) = \dot{\mathbf{r}}_1(1),\ \dot{\mathbf{q}}_2(1) = \dot{\mathbf{r}}_2(1)$$

We consider the same integrated sum of the quadratic differences between the given and required B-spline curve segments, which measures the interpolation error, but now it contains two free control points, and is considered as target function to be minimized.

$$F(\mathbf{b}_3, \mathbf{b}_4) = \sum_{\text{all segments}} \int_0^1 (\mathbf{r}_{ik}(t) - \mathbf{q}_j(t))^2\, dt,$$

$$(i = 1,2,\ k = 1,2,\ j = 1\ldots 4).$$

This function is quadratic in the variables. Therefore, the minimization leads to a system of linear equations. The minimal value measures the interpolation error.

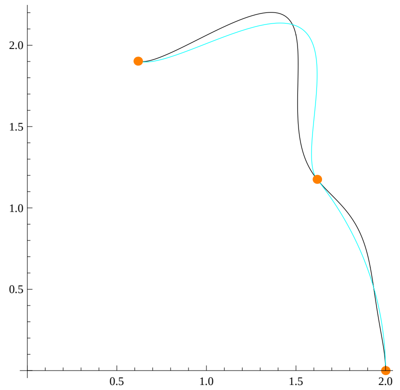

Figure 6: *The error*$= 0,024$

Though the error has been reduced, but the shape is still wavy. In order to get smoother curve, we add to the target function two additional terms. One is for minimizing the difference of the derivatives between the given and required curves, the other for minimizing the variation of the second derivative of the middle curve segments $\mathbf{q}_2(t)$ and $\mathbf{q}_3(t)$, where the curvatures of the given curves show larger difference.
The extended target function is

$$\begin{gathered}\Sigma_{all\,segments}[\int_0^1(\mathbf{r}_{ik}(t)-\mathbf{q}_j(t))^2\,dt\\ +0,2\cdot\int_0^1(\dot{\mathbf{r}}_{ik}(t)-\dot{\mathbf{q}}_j(t))^2]\,dt\\ +0,1\cdot\Sigma_{j=2}^{3}\int_0^1\ddot{\mathbf{q}}_j(t)^2\,dt\end{gathered}$$

The minimization of this target function results in a smoother curve and larger error. The coefficients 0,2 and 0,1 are chosen by experiments. If the weight of the third term is larger, the upper bump in the new merged curve disappears and the error is growing (Fig. 7). It is obvious that smoothing requires more interpolation conditions.

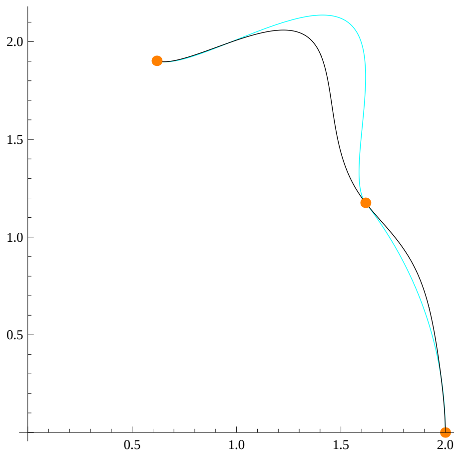

Figure 7: *After fairing the error*$= 0,048$

4 Improving the solution

More interpolation conditions lead to more control points, therefore, the resulting curve will consist of more curve segments. After several experiments our solution will have 8 curve segments with 12 new control points (Fig. 8). Accordingly, the input curves have to be segmented each into 4 parts.

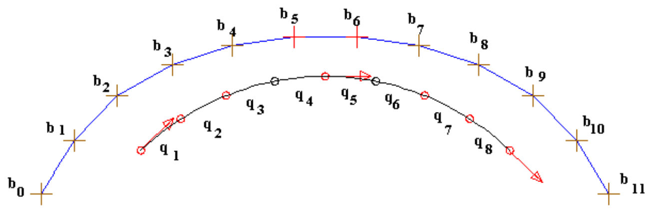

Figure 8: *Segmentation of the merged curve*

First, all the 12 new control points are computed from 12 interpolation conditions, which are 7 interpolation points and 5 tangent vectors. The interpolation points are points of the input curves corresponding to the starting point of the curve segment $\mathbf{q}_1(t)$ and to the end points of the 1., 2., 4., 6., 7., 8. curve segments (Fig. 8). The first derivatives are prescribed at the two end points, at the midpoints and at the joining point of the given curves. These conditions expressed with the B-spline vector functions are linear in the unknown control points $\mathbf{b}_i$, $i = 0,\ldots,11$. The solution is expressed by linear combinations of the given control points $\mathbf{p}_{1j}$ and $\mathbf{p}_{2j}$, $j = 0,\ldots,4$. This symbolical solution is shown in Fig. 9. The error has been succesfully reduced from 0,026 to 0,0035.

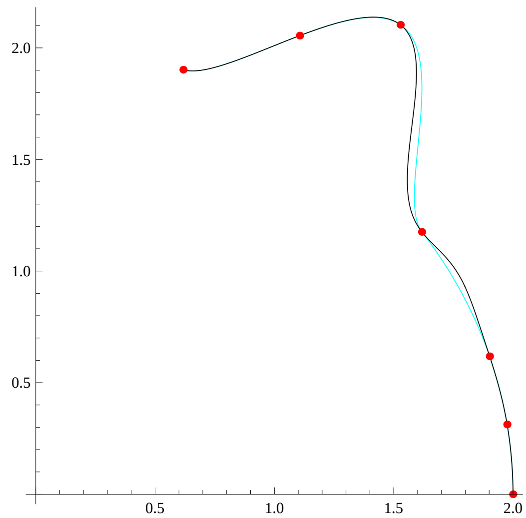

Figure 9: *Merged curve computed by the symbolical solution. The error*$= 0,0035$.

If the given curves do not join, but there is a gap between them, the interpolation point is the midpoint between the two end points of the given curves and similarly, the interpolation tangent vector at this point is the middle value of the two end tangents. In this way a B-spline curve can be determined which replaces parts of the two given curves and connect them smoothly (Fig. 10).

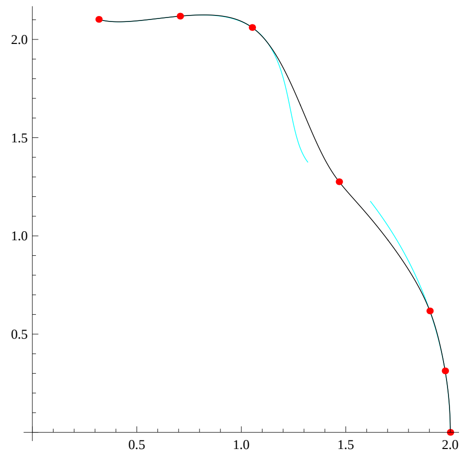

Figure 10: *Stitching two curves with a gap*

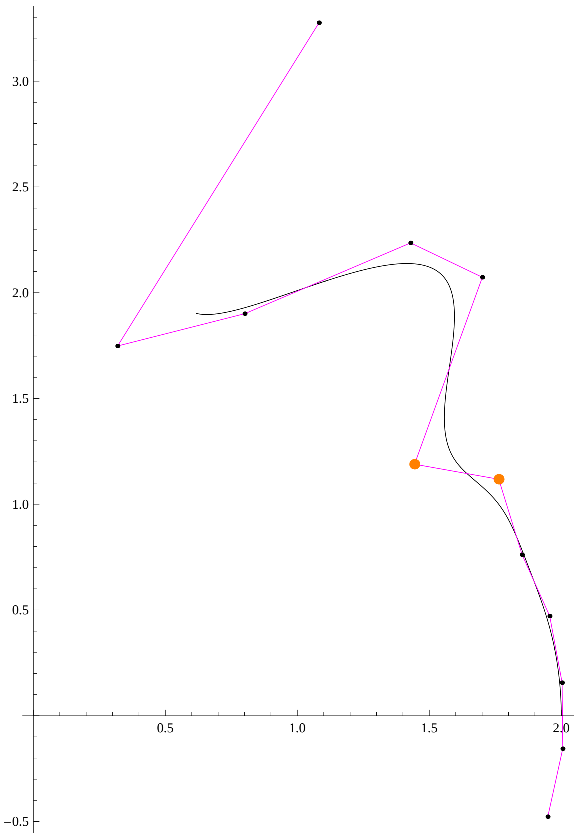

Figure 11: *The control polygon with two variable control points*

The shape of the solution can be improved by applying fairing conditions. Our investigations have shown that two variable control points provide satisfactory solutions. Fig. 11 shows the control polygon of the merged B-spline curve with 10 precomputed and 2 free control points. The interpolation conditions are now 7 points and 3 tangent vectors, and the fairing condition is given by the same target function as in Section 2, but with 8 curve segments. The solution of minimization results in a slightly smoother curve. The interpolation error slightly increased in this case from 0,0035 to 0,004. The picture of the curve looks like in Fig. 9, the difference is not visible.

The symbolical solution (without fairing) leads to smoother curve and smaller error, if the variation of the curvatures of the given curve is smaller. On the base of this experience we have applied it for stitching B-spline patches.

5 Stitching two B-spline patches

We assume that the surface patches are represented by two-parameter vector functions of 4×4 degree with periodical uniform knot vectors. The matrix form is

$$\mathbf{r}(u,v) = (u^4\ u^3\ u^2\ u\ 1)\cdot \mathbf{M}\cdot \mathbf{B}\cdot \mathbf{M}^T\cdot (v^4\ v^3\ v^2\ v\ 1)^T,$$

$$(u,v)\in[0,1]\times[0,1]$$

and

$$\mathbf{M} = \frac{1}{24}\begin{pmatrix} 1 & -4 & 6 & -4 & 1\\ -4 & 12 & -12 & 4 & 0\\ 6 & -6 & -6 & 6 & 0\\ -4 & -12 & 12 & 4 & 0\\ 1 & 11 & 11 & 1 & 0 \end{pmatrix}.$$

The geometric data are the points of the control net denoted by

$$\mathbf{B} = [\mathbf{b}[i,j]]\,, \quad i=0,\dots 4, \quad j=0,\dots 4.$$

In the computation of merging two given B-spline patches we apply the symbolical solution shown for merging two B-spline curve segments. Each given control net consists of 5×5 control points. The new control net of 5×12 control points are computed row by row by the same scheme applied for curves. The resulting surface has 1×8 patches joining with third order continuity, if there are no multiple control points and knot values.

In Fig. 12 two B-spline surface patches are shown defined separately. In Fig. 13 the merged surface is shown. The interpolation error has been computed by numerical integration of the squared differences between the points of the given and the resulting surfaces at the same parameter values. This estimated error is 0,0032.

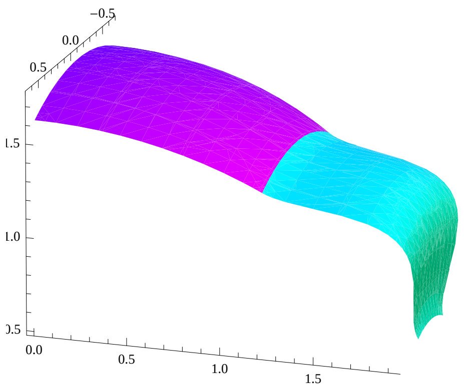

Figure 12: *Two given surface patches*

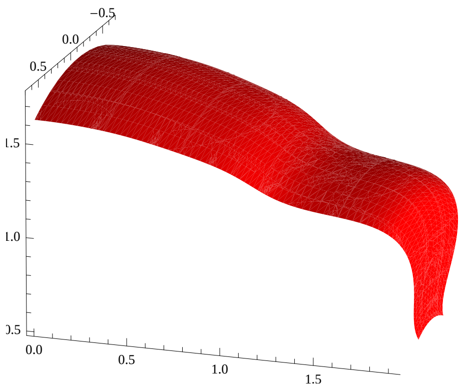

Figure 13: *The merged surface*

Stitching of separately defined B-spline patches ensures higher order continuity along the joining curves than known constructions. In several applications surfaces are determined by local geometric data ([10], [11]). This is the case for example in surface manufacturing in a neighborhood of a processing tool, however, a smooth resulting surface is required. A series in a stripe of separately generated surface patches are shown in [12].

6 Conclusions

We have presented stitching algorithms for two given B-spline curve segments. Our final symbolical solution generates a B-spline curve with 8 segments independently from the numerical input data. This continuous curve approximates the two separately defined (even not joining) curve segments. We have proposed an additional fairing method to improve the shape of the resulting curve. We have also analyzed the interpolation error in many different cases. The proposed algorithm gave the most satisfactory result. This has been applied for merging two B-spline surface patches.

Our aim is to extend this method for stitching more B-spline curves.

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Lamoenian Circles of the Collinear Arbelos

Lamoenian Circles of the Collinear Arbelos

ABSTRACT

We give an infinite sets of circles which generate Archimedean circles of a collinear arbelos.

Key words: arbelos, collinear arbelos, radical circle, Lamoenian circle

MSC2010: 51M04, 51M15, 51N20

Lamoenove kružnice kolinearnog arbelosa

SAŽETAK

Pokazujemo beskonačne skupove kružnica koje generiraju Arhimedove kružnice kolinearnog arbelosa.

Ključne riječi: arbelos, kolinearni arbelos, potencijalna kružnica, Lamoenova kružnica

1 Introduction

For a point O on the segment AB, let α, β and γ be circles with diameters AO, BO and AB respectively. Each of the areas surrounded by the three circles is called an arbelos. The radical axis of the circles α and β divides each of the arbeloi into two curvilinear triangles with congruent incircles (see the lower part of Figure 1). Circles congruent to those circles are said to be Archimedean.

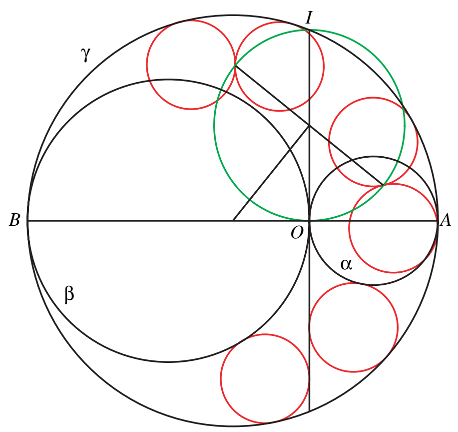

Figure 1: *A circle generating Archimedean circles with* γ

For a point T and a circle δ, if two congruent circles of radius r touching at T also touch δ at points different from T, we say T generates circles of radius r with δ, and the two circles are said to be generated by T with δ. If the generated circles are Archimedean, we say T generates Archimedean circles with δ. Frank Power seems to be the earliest discoverer of this kind Archimedean circles: The farthest points on α and β from AB generate Archimedean circles with γ [6].

Let I be one of the points of intersection of γ and the radical axis of α and β. Floor van Lamoen has found that the endpoints of the diameter of the circle with diameter IO perpendicular to the line joining the centers of this circle and γ generate Archimedean circles with γ [2] (see the upper part of Figure 1). We say a circle $\mathcal{C}$ generates circles of radius r with δ, if the endpoints of a diameter of $\mathcal{C}$ generate circles of radius r with δ. Circles generating Archimedean circles with γ are said to be Lamoenian. In this article we consider those circles in a general way.

2 The collinear arbelos

In this section we consider a generalized arbelos. For two points P and Q in the plane, (PQ) and $P(Q)$ denote the circle with diameter PQ and the circle with center P passing through Q respectively. For a circle δ, O_δ denotes its center. For two points P and Q on the line AB, let $\alpha = (AP)$, $\beta = (BQ)$ and $\gamma = (AB)$. Let O be the point of intersection of AB and the radical axis of the circles α and β and let $u = |AB|$, $s = |AQ|/2$ and $t = |BP|/2$. Unless otherwise stated, we use a rectangular coordinate system with origin O such that the points A, B and P have coordinates $(a,0)$, $(b,0)$ and $(p,0)$ respectively with $a - b = u$. The configuration (α, β, γ) is called a collinear arbelos if the four points lie in the order (i) B, Q, P, A or (ii) B, P, Q, A, or (iii) P, B, A, Q. In each of the cases the configurations are explicitly denoted by $(BQPA)$, $(BPQA)$ and $(PBAQ)$ respectively. In the case $P = Q = O$, (α,β,γ) gives an ordinary arbelos, and (α,β,γ) is called a tangent arbelos. Archimedean circles of the ordinary arbelos are generalized to the collinear arbelos (α,β,γ) as circles of radius $st/(s+t)$, which we denote by r_A [3]. Circles of radius r_A are also called Archimedean circles of (α,β,γ). The radius is also expressed by

$$r_\mathrm{A} = \frac{|AO||BP|}{2u} = \frac{a|p-b|}{2u}. \tag{1}$$

3 Lamoenian circles of the collinear arbelos

A circle generating circles of radius r_A with γ is also said to be Lamoenian for the collinear arbelos (α,β,γ). In this section we give a condition that a circle is Lamoenian. For a circle δ of radius r and a point T, let us define

$$\mathrm{r}(T,\delta) = \frac{|r^2 - |TO_\delta|^2|}{2r},$$

which equals the radius of the generated circles by T with δ by the Pythagorean theorem.

Theorem 1 *Let δ be a circle of radius r and let J, H be points with J lying on δ. The circle (HJ) generates circles of radius s with δ if and only if*

$$|HO_\delta|^2 = r(r \pm 4s). \tag{2}$$

In this event, the following statements are true.

(i) *If a points K lies on the circle $O_\delta(H)$, the circle (KJ) generates circles of radius s with δ.*

(ii) *The point $O_{(HJ)}$ lies on the circle of radius $r/2$ with center $O_{(HO_\delta)}$.*

Proof. Let $h = |HO_\delta|$ (see Figure 2). We use a rectangular coordinate system with origin O_δ such that the coordinates of H is $(h,0)$ in this proof. Let (f,g) be the coordinates of the point $O_{(HJ)}$, and let T be one of the endpoints of the diameter of (HJ) perpendicular to $O_\delta O_{(HJ)}$. Then $\overrightarrow{O_{(HJ)}T} = k(-g,f)$ and $\overrightarrow{O_\delta T} = (f-kg, g+kf)$ for a real number k. From $|O_{(HJ)}T| = |O_{(HJ)}H|$, $(-kg)^2+(kf)^2 = (f-h)^2+g^2$, which implies

$$k^2 = \frac{(f-h)^2+g^2}{f^2+g^2}. \tag{3}$$

The circle (HJ) generates circles of radius s with δ if and only if

$$\mathrm{r}(T,\delta) = \frac{|r^2 - ((f-kg)^2+(g+kf)^2)|}{2r} = s.$$

Since (3) holds, the last equation is equivalent to

$$\frac{1}{4}h^2 + \left(f - \frac{h}{2}\right)^2 + g^2 = \frac{1}{2}r(r \pm 2s),$$

where the plus (resp. minus) sigh should be taken when T lies outside (resp. inside) of δ. If (v,w) are the coordinates of the point J, $(v+h)/2 = f$ and $w/2 = g$. Therefore the last equation is equivalent to

$$\frac{1}{4}h^2 + \frac{1}{4}r^2 = \frac{1}{2}r(r \pm 2s),$$

which is also equivalent to (2). The part (i) obviously holds. The center of (HJ) is the image of J by the dilation with center H and scale factor $1/2$. This proves (ii). □

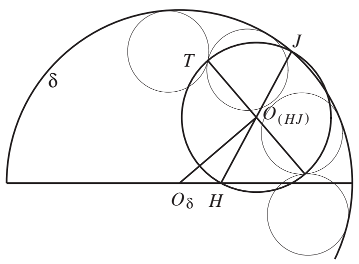

Figure 2

Let ε be the circle with center O_γ belonging to the pencil of circles determined by α and β for the collinear arbelos (α,β,γ). We call ε the radical circle of (α,β,γ). The circle is considered in [4] and [5] for $(BQPA)$ and $(BPQA)$. If α and β have a point in common, ε passes through the point. For $(BQPA)$ let V be the point of tangency of one of the tangents of α from O (see Figure 3). Then $|OV|^2 = ap$. If $|OO_\gamma|^2 > ap$, a tangent from O_γ to the circle $O(V)$ can be drawn. Then ε passes through the point of tangency. If $|OO_\gamma|^2 = ap$, ε is the point circle O_γ, which coincides with one of the limiting points of the pencil. If $|OO_\gamma|^2 < |ap|$, ε does not exist. Let e be the radius of ε. For $(BQPA)$, $e^2 = |OO_\gamma|^2 - ap$ by the Pythagorean theorem. For $(BPQA)$ and $(PBAQ)$, $e^2 = |OO_\gamma|^2 + |ap|$ (see Figure 4). In any case

$$e^2 = |OO_\gamma|^2 - ap. \tag{4}$$

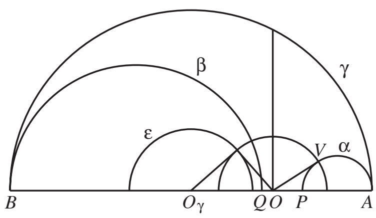

Figure 3: *The case $|O_\gamma O|^2 > |ap|$ for $(BQPA)$*

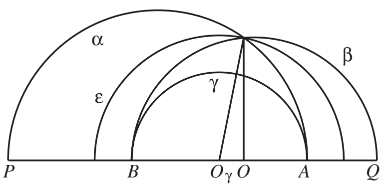

Figure 4: $(PBAQ)$

Theorem 2 *For a collinear arbelos (α,β,γ) with radical circle ε, if points J and H lie on γ and ε respectively, then the circle (HJ) is Lamoenian.*

Proof. For $(BPQA)$ and $(BQPA)$, $r_{\rm A} = a(p-b)/(2u)$ by (1). Therefore by (4),

$$\frac{u}{2}\left(\frac{u}{2}-4r_{\rm A}\right)=\frac{(a-b)^2}{4}-a(p-b)=\frac{(a+b)^2}{4}-ap=e^2.$$

Similarly for $(PBAQ)$, we get

$$\frac{u}{2}\left(\frac{u}{2}+4r_{\rm A}\right)=e^2.$$

Hence the theorem is proved by Theorem 1. □

4 Quartet of circles

In this section we show that a Lamoenian circle given by Theorem 2 is a member of a set of four Lamoenian circles. All the suffixes are reduced modulo 4 in this section. Let J_0 be a point on a circle δ, and let H be a point which does not lie on δ (see Figures 5, 6). Let R_0R_1 be the diameter of the circle (HJ_0) perpendicular to the line $O_\delta O_{(HJ_0)}$ and let R_0 and R_1 generate circles of radius s with δ. Let J_1 be the point of intersection of the line J_0R_1 and δ, and let R_2 be the point such that $HR_1J_1R_2$ is a rectangle. Then the circle (HJ_1) also generates circles of radius s with δ by Theorem 1. While R_1 generates circles of radius s with δ. Therefore R_2 also generates circles of radius s with δ. Similarly we construct the points J_2 and J_3 on δ and the points R_3 and R_4 such that J_2 and J_3 lie on the lines J_1R_2 and J_2R_3 respectively and $HR_2J_2R_3$ and $HR_3J_3R_4$ are rectangles. Then R_3 generates circles of radius s with δ and R_4 coincides with R_0. Now we get the points J_i on δ and R_i $(i=0,1,2,3)$ such that R_iR_{i+1} is the diameter of (HJ_i), R_i generates circles of radius s with δ, $J_0J_1J_2J_3$ is a rectangle, R_i lies on the line J_iJ_{i-1}. The four circles (HJ_i) $(i=0,1,2,3)$ are called a quartet on δ, and H and $J_0J_1J_2J_3$ are called the base point and the rectangle of the quartet respectively.

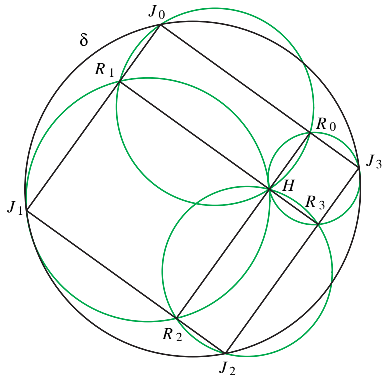

Figure 5: *H lies inside of δ*

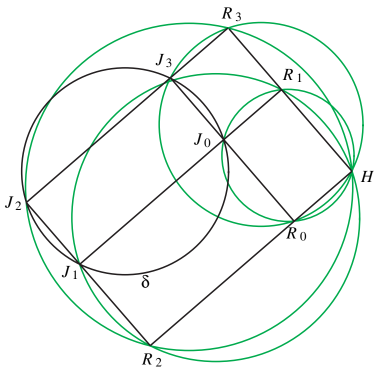

Figure 6: *H lies outside of δ*

By the definition of R_i, R_0, R_2, H are collinear, also R_1, R_3, H are collinear, and the two lines are perpendicular. Let $l_i = |HR_i|$. Then $|HJ_0|^2+|HJ_2|^2 = l_0^2+l_1^2+l_2^2+l_3^2 = |HJ_1|^2+|HJ_3|^2$. Therefore $|HJ_0|^2+|HJ_2|^2 = |HJ_1|^2+|HJ_3|^2$ holds.

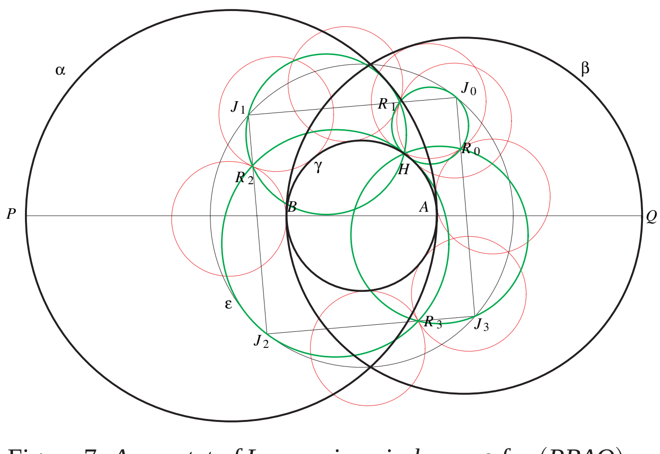

Figure 7: *A quartet of Lamoenian circles on ε for* $(PBAQ)$

For a collinear arbelos (α,β,γ) with radical circle ε, if the two points H and J_0 lie on ε and γ respectively, we can construct a quartet (HJ_i) $(i=0,1,2,3)$ on γ consisting of Lamoenian circles by Theorem 2. Also if H and J_0 lie on γ and ε respectively, we can construct a quartet (HJ_i) $(i=0,1,2,3)$ on ε consisting of Lamoenian circles (see Figure 7).

Theorem 3 *For a quartet (HJ_i) $(i=0,1,2,3)$ on a circle δ, the rectangle is a square if and only if (HJ_i) touches δ for some i. In this event, (HJ_{i+2}) also touches δ, and (HJ_{i-1}) and (HJ_{i+1}) are congruent and intersect at O_δ.*

Proof: If (HJ_0) touches δ, $R_0J_0R_1$ is an isosceles right triangle, since $|O_\delta R_0| = |O_\delta R_1|$. This implies that $J_3J_0J_1$ is also an isosceles right triangle, i.e., $J_0J_1J_2J_3$ is a square. Conversely let us assume $J_0J_1J_2J_3$ is a square. We assume that (HJ_i) does not touch δ for $i=0,1,2,3$. The sides or the extended sides of the square and the circle $O_\delta(R_0)$ intersect at eight points, four of which are R_0, R_1, R_2, R_3. If $|J_iR_i| = |J_iR_{i+1}|$, (HJ_i) touches δ. Therefore $|J_iR_i| \neq |J_iR_{i+1}|$ for $i=0,1,2,3$. This can happen only when R_1, R_2, R_3, R_4 lie inside of δ (see Figures 8 and 9). Hence $|J_0R_0| = |J_1R_1| = |J_2R_2| = |J_3R_3| \neq |J_0R_1| = |J_1R_2| = |J_2R_3| = |J_3R_0|$. Therefore the four rectangles $HR_iJ_iR_{i+1}$ $(i=0,1,2,3)$ are congruent. Then they must be squares, since H is their common vertex. But this implies $|J_iR_i| = |J_iR_{i+1}|$, a contradiction. Hence (HJ_i) touches δ for some i. Then H lies on J_iJ_{i+2}. Therefore (HJ_{i+2}) also touches δ. While $J_{i-1}J_{i+1}$ and HO_δ are perpendicular and intersect at O_δ. Therefore (HJ_{i-1}) and (HJ_{i+1}) are congruent and pass through O_δ. $\square$

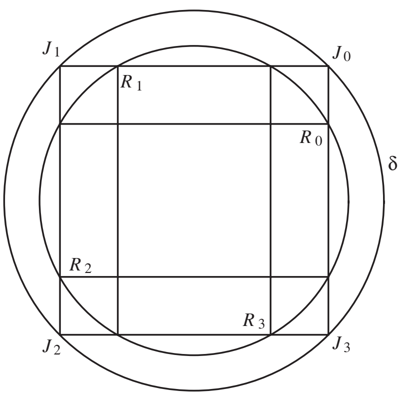

Figure 8

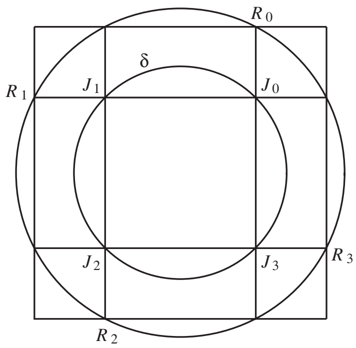

Figure 9

5 Special cases

We conclude this article by considering the tangent arbelos (α,β,γ) with $O=P=Q$. Since $\varepsilon = O_\gamma(O)$, Power's result mentioned in the introduction is restated as both α and β are Lamoenian. Figure 10 shows a quartet on γ with base point O with $J_0 = A$, in which α and β are members of the quartet. Figure 11 shows a quartet on ε with base point A with $J_0 = O$. In this figure α and the reflected image of β in O_γ are members of the quartet. In each of the cases, the rectangle is a square.

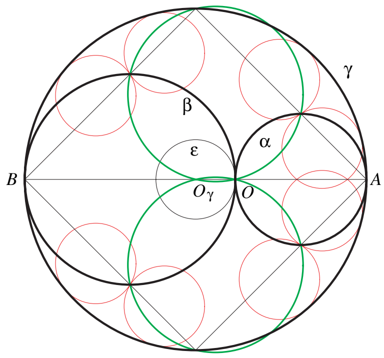

Figure 10: *A quartet on γ with base point O*

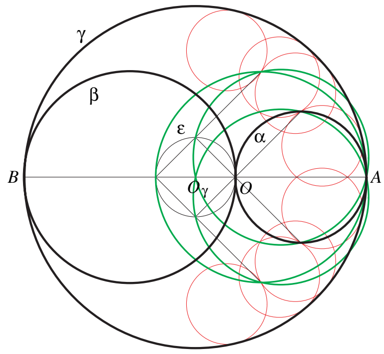

Figure 11: *A quartet on ε with base point A*

Let $\mathcal{L}$ be the radical axis of α and β. Quang Tuan Bui has found that the points of intersection of the circles (AO_β) and (BO_α) lie on $\mathcal{L}$ and generate Archimedean circles with γ for the tangent arbelos (α, β, γ) [1]. Let R_1 be one of the points of intersection, and let the line parallel to AB passing through R_1 intersect γ at a point K, where K lies on the same side of $\mathcal{L}$ as A. Figure 12 shows a quartet on γ with base point O with $J_0 = K$. In this figure R_0 and R_2 lie on AB while R_3 lies on $\mathcal{L}$. Figure 13 shows a quartet on ε with base point K with $J_0 = O$. In this figure, R_1J_0 touches ε at O. Therefore $J_1 = J_0 = O$, i.e., the rectangle degenerates into a segment, and the quartet consists of two different Lamoenian circles.

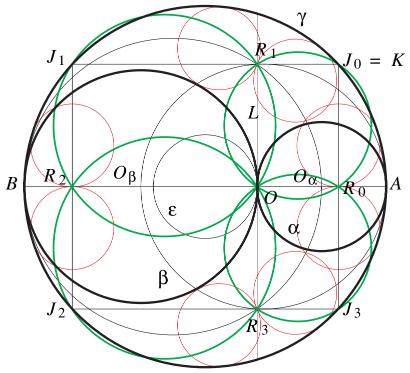

Figure 12: *A quartet on γ with base point O*

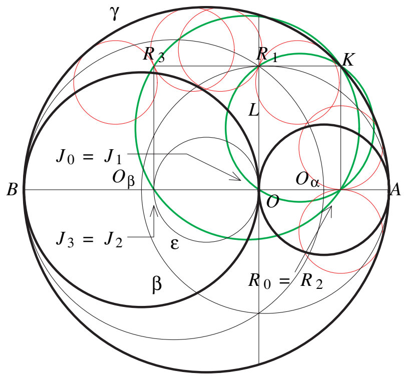

Figure 13: *A quartet on ε with base point K*

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The Parabola in Universal Hyperbolic Geometry I

The Parabola in Universal Hyperbolic Geometry I

ABSTRACT

We introduce a novel definition of a parabola into the framework of universal hyperbolic geometry, show many analogs with the Euclidean theory, and also some remarkable new features. The main technique is to establish parabolic standard coordinates in which the parabola has the form $xz = y^2$. Highlights include the discovery of the twin parabola and the connection with sydpoints, many unexpected concurrences and collinearities, a construction for the evolute, and the determination of (up to) four points on the parabola whose normals meet.

Key words: universal hyperbolic geometry, parabola

MSC2010: 51M10, 14N99, 51E99

Parabola u univerzalnoj hiperboličkoj geometriji I

SAŽETAK

Uvodimo novu definiciju parabole u okvir univerzalne hiperboličke geometrije, pokazujemo mnoge analogone s euklidskom geometrijom, ali i neka izvanredna nova svojstva. Osnovna je tehnika uspostavljanje paraboličnih standardnih koordinata u kojima parabola ima jednadžbu oblika $xz = y^2$. Ističemo otkriće parabole blizanke, vezu sa sidtočkama, mnoge neočekivane konkurentnosti i kolinearnosti, konstrukciju evolute te određivanje (do najviše) četiriju točaka parabole u kojima normale parabole prolaze jednom točkom.

Ključne riječi: univerzalna hiperbolička geometrija, parabola

1 Introduction

This paper begins the study of the *parabola* in *universal hyperbolic geometry (UHG)*. The framework is that of [16], [17], [18], [19] and [20]; a completely algebraic and more general formulation of hyperbolic geometry which extends to general fields (not of characteristic two), and also unifies elliptic and hyperbolic geometries. We will see that this investigation opens up many new phenomenon, and hints again at the inexhaustible beauty of conic sections!

In Euclidean geometry, the parabola plays several distinguished roles. It is the graph resulting from a quadratic function $f(x) = a + bx + cx^2$, and so familiar as the second degree Taylor expansion of a general function. The parabola is also a conic section in the spirit of Apollonius, obtained by slicing a cone with a plane which is parallel to one of the generators of the cone. In affine geometry the parabola is the distinguished conic which is tangent to the line at infinity. In everyday life, the parabola occurs in reflecting mirrors and automobile head lamps, in satellite dishes and radio telescopes, and in the trajectories of comets.

Of course the ancient Greeks also studied the familiar metrical formulation of a parabola: it is the locus of a point which remains equidistant from a fixed point F, called the *focus*, and a fixed line f, called the *directrix*. (We have a good reason for using the same letters for both concepts, with only case separating them). Such a conic $\mathcal{P}$ has a line of symmetry: the *axis* a through F perpendicular to f. It also has a distinguished point V called the *vertex*, which is the only point of the parabola lying on the axis a, aside from the point at infinity. The vertex V is the midpoint between the focus F and the **base point** $B \equiv af$.

For such a classical parabola $\mathcal{P}$ hundreds of facts are known, see [1], [4], [5], [8], [10], [13], [14]; quite a few of them going back to Archimedes and Apollonius, others added in more recent centuries. Of particular importance are theorems that relate to an arbitrary point P on the conic and its tangent line p. In particular the construction of p itself is important: there are two common ways of doing this. One is to take the foot T of the altitude from P to the directrix f, and connect P to the midpoint M of $\overline{TF}$; so that $p = PM$. Another is to take the perpendicular line t to PF through F, and find its meet S with the directrix; this gives $p = PS$. The point S is equidistant from T and F, and the circle $\mathcal{S}$ with center S through F is tangent to both the lines PF and PT.

A related and useful fact is that a chord PN is a focal chord—meaning that it passes through F—precisely when the meet of the two tangents at P and N lies on the directrix f, and in this case the two tangents are perpendicular. These facts are illustrated in Figure 1. Another result, which figures often in calculus, is that if P and Q are arbitrary points on the parabola with Z the meet of their tangents p and q, and T,U and W are the feet of the altitudes from P,Q and Z to the directrix, then W is the midpoint of $\overline{TU}$.

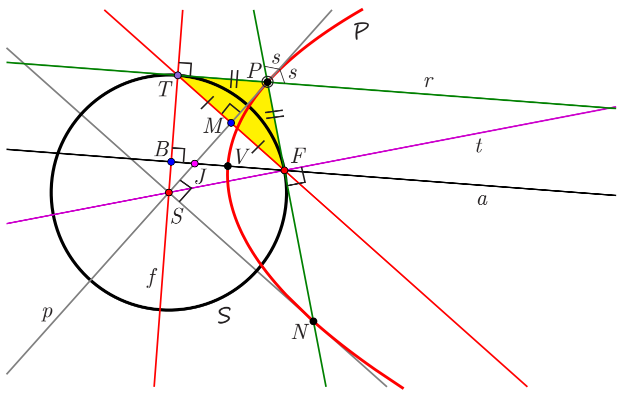

Figure 1: *The Euclidean Parabola*

So when we investigate hyperbolic geometry, some natural questions are: what is the analog of a parabola in this context, what properties of the Euclidean case carry over in this setting, and what additional properties might the hyperbolic parabola have that do not hold in the Euclidean case? These issues have been studied by several authors, such as [2], [15], [9].

In this paper we answer these questions in a new and more general way, using the wider framework of UHG, and allowing the beginnings of a much deeper investigation. There is a very natural analog of a parabola in this hyperbolic setting, and many, but certainly not all, properties of the Euclidean parabola hold or have reasonable analogs for it. But there are many interesting aspects which have no Euclidean counterpart, such as the existence of a dual or twin parabola, and an intimate connection with the theory of sydpoints, as laid out in [20].

The outline of the paper is as follows. We first give a very brief review of universal hyperbolic geometry, where the algebraic notions of *quadrance* and *spread* replace the more traditional transcendental measurements of *distance* and *angle*. We then define the parabola in the hyperbolic setting (we often refer simply to the *hyperbolic parabola*), give a dynamic geometry package construction for it, introduce some basic points associated to it, and use some of these and the Fundamental theorem of Projective Geometry to define *standard coordinates*, in which the parabola has the convenient equation $xz = y^2$. This allows a simple parametrization for the curve, as well as pleasant explicit formulas for many interesting points, lines, conics and higher degree curves associated to it.

In our study of the basic points and lines associated with the parabola $\mathcal{P}_0$, concrete and explicit formulae are key objectives, because they allow us a firm foundation for deeper investigations. The main thrust of the paper is then to show how the hyperbolic parabola shares many similarities with the Euclidean parabola. The highlights include the duality leading to the twin parabola, a straightedge construction of the evolute of the parabola, and a conic construction of four points on the parabola whose normals pass through a fixed point (in the Euclidean case there are at most three points with this property).

This paper is the first of a series on the hyperbolic parabola. In future papers we will show that there are many new and completely unexpected aspects of the hyperbolic parabola; it is a very rich topic indeed.

1.1 A brief review of universal hyperbolic geometry

We work over a fixed field, not of characteristic two, and give a formulation of universal hyperbolic geometry valid with a general symmetric bilinear form—this generality will be important for us when we introduce parabolic standard coordinates. This is only a quick introduction; the reader may consult [17], [18], [19], [20] for more details.

A **(projective) point** is a proportion $a = [x : y : z]$ in square brackets, or equivalently a projective row vector $a = \begin{bmatrix} x & y & z \end{bmatrix}$ (unchanged if multiplied by a non-zero number). A **(projective) line** is a proportion $L = \langle l : m : n \rangle$ in pointed brackets, or equivalently a projective column vector

$$L = \begin{bmatrix} l \\ m \\ n \end{bmatrix}.$$

The **incidence** between the point $a = [x : y : z]$ and the line $L = \langle l : m : n \rangle$ is given by the relation $aL = lx + my + nz = 0$. The **join** of points is defined by

$$\begin{aligned} a_1a_2 &\equiv [x_1 : y_1 : z_1] \times [x_2 : y_2 : z_2] \\ &\equiv \langle y_1z_2 - y_2z_1 : z_1x_2 - z_2x_1 : x_1y_2 - x_2y_1 \rangle \end{aligned} \quad (1)$$

while the **meet** L_1L_2 of lines $L_1 \equiv \langle l_1 : m_1 : n_1 \rangle$ and $L_2 \equiv \langle l_2 : m_2 : n_2 \rangle$ is similarly defined by

$$\begin{aligned} L_1L_2 &\equiv \langle l_1 : m_1 : n_1 \rangle \times \langle l_2 : m_2 : n_2 \rangle \\ &\equiv [m_1n_2 - m_2n_1 : n_1l_2 - n_2l_1 : l_1m_2 - l_2m_1]. \end{aligned} \quad (2)$$

Collinearity of three points a_1, a_2, a_3 will here be represented by the abbreviation $[[a_1a_2a_3]]$, and similarly the concurrency of three lines L_1, L_2, L_3 will be abbreviated $[[L_1L_2L_3]]$. These are determinantal conditions.

The metrical structure is given by a (non-degenerate) 3×3 projective symmetric matrix $\mathbf{C}$ and its adjugate $\mathbf{D}$ (where bold signifies a projective matrix- determined only up to a non-zero multiple). The points a_1 and a_2 are **perpendicular** precisely when $a_1\mathbf{C}a_2^T=0$, written $a_1\perp a_2$, while lines L_1 and L_2 are **perpendicular** precisely when $L_1^T\mathbf{D}L_2=0$, written $L_1\perp L_2$. The point a and the line L are **dual** precisely when $L=a^\perp\equiv\mathbf{C}a^T$, or equivalently $a=L^\perp\equiv L^T\mathbf{D}$, so that points are perpendicular precisely when one is incident with the dual of the other, and similarly for two lines. A point a is **null** precisely when it is perpendicular to itself, that is, when $a\mathbf{C}a^T=0$, while a line L is **null** precisely when it is perpendicular to itself, that is, when $L^T\mathbf{D}L=0$. The null points determine the **null conic**, sometimes also called the *absolute*.

Universal Hyperbolic geometry in the Cayley Klein model arises from the special case

$$\mathbf{C}=\mathbf{D}=\mathbf{J}\equiv\begin{bmatrix}1&0&0\\0&1&0\\0&0&-1\end{bmatrix}. \tag{3}$$

In this framework the point $a=[x:y:z]$ is null precisely when $x^2+y^2-z^2=0$, and dually the line $L=\langle l:m:n\rangle$ is null precisely when $l^2+m^2-n^2=0$. So we can picture the null circle in affine coordinates $X\equiv x/z$ and $Y\equiv y/z$ as the (blue) circle $X^2+Y^2=1$. The **quadrance** q between points and the **spread** S between lines are then given by essentially the same formulas:

$$\begin{aligned} &q\left([x_1:y_1:z_1],[x_2:y_2:z_2]\right)\\ &\quad=1-\frac{(x_1x_2+y_1y_2-z_1z_2)^2}{\left(x_1^2+y_1^2-z_1^2\right)\left(x_2^2+y_2^2-z_2^2\right)}\\ &S\left(\langle l_1:m_1:n_1\rangle,\langle l_2:m_2:m_2\rangle\right)\\ &\quad=1-\frac{(l_1l_2+m_1m_2-n_1n_2)^2}{\left(l_1^2+m_1^2-n_1^2\right)\left(l_2^2+m_2^2-n_2^2\right)}. \end{aligned} \tag{4}$$

The figures in this paper are generated in this model, with however the outside of the null circle playing just as big a role as the inside–this takes some getting used to for the classical hyperbolic geometer! In addition, it will be necessary for us to adopt a more general and flexible approach to deal with projective changes of coordinates, which will be needed to study the parabola in what we call standard coordinates.

So more generally, the bilinear forms determined by a general 3×3 projective symmetric matrix $\mathbf{C}$ and its adjugate $\mathbf{D}$ can be used to define the dual notions of (**projective**) **quadrance** $q(a_1,a_2)$ between points a_1 and a_2, and (**projective**) **spread** $S(L_1,L_2)$ between lines L_1 and L_2 as

$$q(a_1,a_2)\equiv 1-\frac{\left(a_1\mathbf{C}a_2^T\right)^2}{\left(a_1\mathbf{C}a_1^T\right)\left(a_2\mathbf{C}a_2^T\right)}\qquad\text{and}$$

$$S(L_1,L_2)\equiv 1-\frac{\left(L_1^T\mathbf{D}L_2\right)^2}{\left(L_1^T\mathbf{D}L_1\right)\left(L_2^T\mathbf{D}L_2\right)}. \tag{5}$$

While the numerators and denominators of these expressions depend on choices of representative vectors and matrices for $a_1,a_2,\mathbf{C},L_1,L_2$ and $\mathbf{D}$, (which are by definition defined only up to scalars), the overall expressions are well-defined projectively.

It follows that $q(a,a)=0$ and $S(L,L)=0$, while $q(a_1,a_2)=1$ precisely when $a_1\perp a_2$, and dually $S(L_1,L_2)=1$ precisely when $L_1\perp L_2$. Also quadrance and spread are naturally dual:

$$S\left(a_1^\perp,a_2^\perp\right)=q(a_1,a_2).$$

In [16], it was shown that both these metrical notions can also be reformulated projectively and rationally using suitable cross ratios (and no transcendental functions!) To connect with the more familiar distance between points $d(a_1,a_2)$, and angle between lines $\theta(L_1,L_2)$ in the Klein projective model: when we restrict to points and lines inside the null circle,

$$q(a_1,a_2)=-\sinh^2(d(a_1,a_2))\qquad\text{and}$$

$$S(L_1,L_2)=\sin^2(\theta(L_1,L_2)).$$

For a triangle $\overline{a_1a_2a_3}$ with associated trilateral $\overline{L_1L_2L_3}$, we define $q_1\equiv q(a_2,a_3)$, $q_2\equiv q(a_1,a_3)$ and $q_3\equiv q(a_1,a_2)$, and $S_1\equiv S(L_2,L_3)$, $S_2\equiv S(L_1,L_3)$ and $S_3\equiv S(L_1,L_2)$. The main trigonometric laws in the subject can be restated in terms of these quantities (see UHG I [17]).

2 The parabola and its construction

In this section we introduce definitions and some basic results for a *parabola* in *universal hyperbolic geometry*. We will work and illustrate the theory in the familiar Cayley-Klein setting with our null circle/absolute the unit circle in the plane. The situation is in some sense richer than in the Euclidean setting because of *duality*: whenever we define an important point x, its dual line $X=x^\perp$ is also likely to be important, and vice versa. We remind the reader that we will consistently employ small letters for points and capital letters for lines, with the convention that if x_i is a point, then $X_i=x_i^\perp$ is the corresponding dual line and conversely.

So what is a parabola in the hyperbolic setting? As already discussed in [9], the definition is not obvious: there are several different possible ways of trying to generalize the Euclidean theory. Recall that if a is a point and L is a line, then the quadrance $q(a,L)$ is defined to be the quadrance between a and the foot t of the altitude line from a to L.

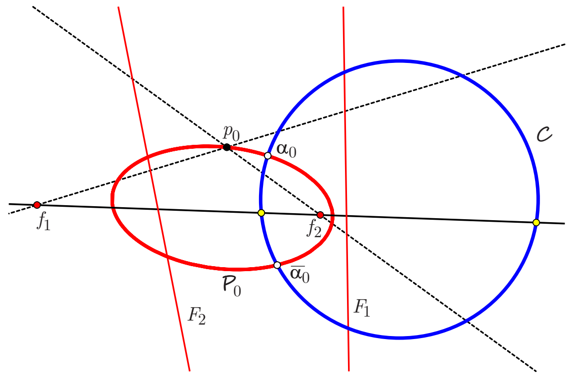

Figure 2: *A parabola $\mathcal{P}_0$ with foci f_1 and f_2*

Definition 1 *Suppose that f_1 and f_2 are two non-perpendicular points such that $f_1 f_2$ is a non-null line. The **parabola** $\mathcal{P}_0$ with **foci** f_1 and f_2 is the locus of a point p_0 satisfying*

$$q(f_1,p_0)+q(p_0,f_2)=1. \tag{6}$$

*The lines $F_1 \equiv f_1^{\perp}$ and $F_2 \equiv f_2^{\perp}$ are the **directrices** of the parabola $\mathcal{P}_0$.*

This definition is likely surprising to the classical geometer. In Euclidean geometry, such a relation defines a *circle*, so at this point it is not clear what justification we have for our definition of a parabola. The following connects our theory with the more traditional approach in [11] and [7].

Theorem 1 (Parabola focus directrix) *The point p_0 satisfies (6) precisely when either of the following hold:*

$$q(f_1,p_0)=q(p_0,F_2) \qquad \text{or} \qquad q(f_2,p_0)=q(p_0,F_1).$$

Proof. If $(f_1 p_0)F_1 \equiv t_1$ and $(f_2 p_0)F_2 \equiv t_2$ are the feet of the altitudes from a point p_0 on the parabola $\mathcal{P}_0$ with foci f_1 and f_2 to the directrices F_1 and F_2, then f_1 and t_1 are perpendicular points, as are f_2 and t_2. It follows that $q(f_1,p_0)+q(p_0,t_1)=1$ and $q(f_2,p_0)+q(p_0,t_2)=1$. But then (6) is equivalent to $q(f_1,p_0)=q(p_0,F_2)$ or to $q(f_2,p_0)=q(p_0,F_1)$. □

In this way we recover the ancient Greek metrical definition of the parabola, but we note now that there are *two* foci-directrix pairs: (f_1,F_2) and (f_2,F_1). This is a main feature of the hyperbolic theory of the parabola: a fundamental symmetry between the two foci-directrix pairs.
The reason for the index 0 on the point p_0 and the parabola $\mathcal{P}_0$ will become clearer when we introduce the twin parabola $\mathcal{P}^0$. We observe that the foci f_1 and f_2 do not lie on the parabola $\mathcal{P}_0$, since for example if f_1 lies on $\mathcal{P}_0$, then $q(f_1,f_1)+q(f_2,f_1)=1$, which would imply that $q(f_1,f_2)=1$, contradicting that the assumption of non-perpendicularity of f_1 and f_2. In Figure 2 we see an example of a parabola $\mathcal{P}_0$, in red, with foci f_1 and f_2, and directrices F_1 and F_2, also in red.

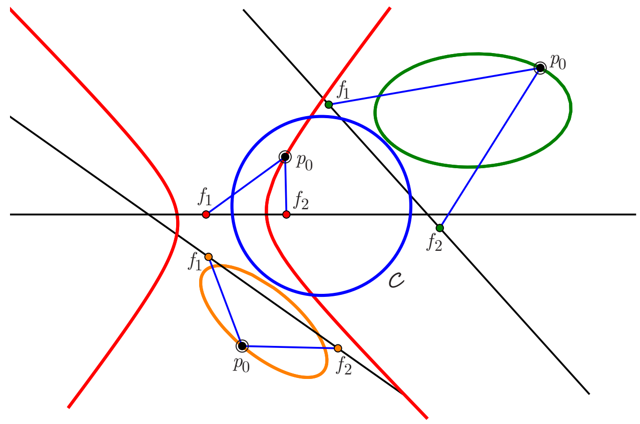

Figure 3: *Various examples of parabolas*

In Figure 3 we see some different examples of parabolas over the rational numbers, at least approximately. When the foci f_1 and f_2 are both interior points of the null circle $\mathcal{C}$, there is no point p satisfying the condition $q(p,f_1)+q(p,f_2)=1$, since the quadrance between any two interior points is always negative, and the quadrance between an interior point and an exterior point is greater than or equal to 1. This paper deals with non-empty parabolas, by extending the field if necessary, as we shall see.

Theorem 2 (Parabola conic) *The parabola $\mathcal{P}_0$ with foci f_1 and f_2 is a conic.*

Proof. Suppose that $f_1=[x_1:y_1:z_1]$ and $f_2=[x_2:y_2:z_2]$. Then the point $p=[x:y:z]$ lies on $\mathcal{P}_0$ precisely when

$$\left(1-\frac{(xx_1+yy_1-zz_1)^2}{(x^2+y^2-z^2)\left(x_1^2+y_1^2-z_1^2\right)}\right) + \left(1-\frac{(xx_2+yy_2-zz_2)^2}{(x^2+y^2-z^2)\left(x_2^2+y_2^2-z_2^2\right)}\right)=1$$

which yields the quadratic equation

$$\begin{aligned}&\left(x^2+y^2-z^2\right)\left(x_1^2+y_1^2-z_1^2\right)\left(x_2^2+y_2^2-z_2^2\right)\\ &\quad=(xx_1+yy_1-zz_1)^2\left(x_2^2+y_2^2-z_2^2\right)\\ &\qquad+(xx_2+yy_2-zz_2)^2\left(x_1^2+y_1^2-z_1^2\right).\end{aligned}$$ □

2.1 Basic definitions

We now define some basic points and lines associated to a parabola $\mathcal{P}_0$ with foci f_1 and f_2, and directrices $F_1 \equiv f_1^{\perp}$

and $F_2 \equiv f_2^\perp$. The **axis** of the parabola $\mathcal{P}_0$ is the line $A \equiv f_1 f_2$. The **axis point** of $\mathcal{P}_0$ is the dual point $a \equiv A^\perp$. By assumption the axis A is a non-null line, so that a does not lie on A.
If the axis A has null points, we shall call these the **axis null points** of $\mathcal{P}_0$, and denote them by η_1 and η_2, in no particular order. The axis point and line will generally be in black in our diagrams, while the axis null points will be in yellow.

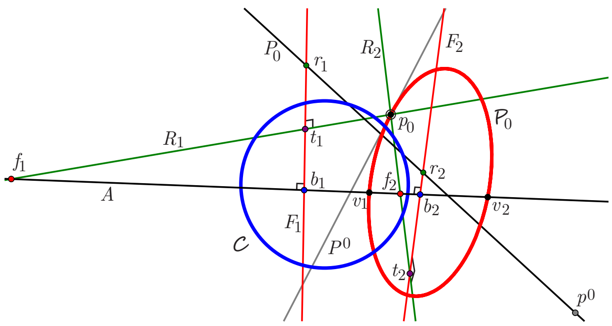

Figure 4: *Dual and tangent lines, twin point and focal lines*

Theorem 3 *The axis $A = f_1 f_2$ of a hyperbolic parabola $\mathcal{P}_0$ is a line of symmetry, and its dual point a is a center.*

Proof. We denote the reflection of an arbitrary point p_0 lying on $\mathcal{P}_0$ in the axis line A by $r_A(p_0) = \overline{p_0}$. Then we need to prove that $\overline{p_0}$ also lies on $\mathcal{P}_0$. Recall that the hyperbolic reflection in a line (or equivalently the reflection in the dual point of that line) is an isometry, so for any two points a and b,

$$q(a,b) = q(r_A(a), r_A(b)).$$

Thus, since f_1, f_2 are fixed by r_A (they lie on A),

$$\begin{aligned} 1 &= q(f_1, p_0) + q(f_2, p_0) \\ &= q(r_A(p_0), r_A(f_1)) + q(r_A(p_0), r_A(f_1)) \\ &= q(\overline{p_0}, f_1) + q(\overline{p_0}, f_2). \end{aligned}$$

This shows that $\overline{p_0}$ lies on the parabola $\mathcal{P}_0$. Since reflecting p_0 in A is the same as reflecting p_0 in a, the point a is also the center of the parabola. □

The **base points** of $\mathcal{P}_0$ are the points $b_1 \equiv AF_1$ and $b_2 \equiv AF_2$. The dual lines $B_1 \equiv a f_1$ and $B_2 \equiv a f_2$ are the **base lines** of $\mathcal{P}_0$. Both base points and base lines will be shown in blue in our diagrams.
The **vertices** v_1 and v_2 are the points, if they exist, where the parabola meets the axis; they are in no particular order. The duals of the vertices are the **vertex lines** $V_1 \equiv v_1^\perp$ and $V_2 \equiv v_2^\perp$. The vertices and vertex lines will be shown in black.

A generic point on $\mathcal{P}_0$ will be denoted p_0, and its **dual line** denoted P_0. Both are shown in black in our diagrams, with often a small circle drawn around p_0 to highlight it. The **tangent line** to $\mathcal{P}_0$ at p_0 will be denoted P^0, and its dual point p^0 will be called the **twin point** of p_0. Both p^0 and P^0 will be shown in grey.
The **focal lines of** p_0 are $R_1 \equiv p_0 f_1$ and $R_2 \equiv p_0 f_2$, and the **altitude base points of** p_0 are $t_1 \equiv R_1 F_1$ and $t_2 \equiv R_2 F_2$. The duals of the focal lines are the **focal points** $r_1 \equiv R_1^\perp$ and $r_2 \equiv R_2^\perp$ **of** p_0. The duals of the focal base points are the **altitude base lines** $T_1 \equiv t_1^\perp$ and $T_2 \equiv t_2^\perp$ **of** p_0. The focal lines and points will be shown in green in our diagrams. Figure 4 shows these various basic points and lines associated to the parabola $\mathcal{P}_0$.

2.2 Construction with a dynamic geometry program

It is helpful to have a construction of a hyperbolic parabola that can be used with a dynamic geometry package, such as Geometer's Sketchpad, GeoGebra, C.a.R., Cinderella, Cabri etc., used to create loci. For this it is helpful to refresh our minds about the construction of the Euclidean parabola, because a similar technique applies to construct a hyperbolic parabola. We also mention some related facts that will have analogs in the hyperbolic setting.
Firstly, we choose a point F (focus**)**, and a line f (directrix**)**, not passing through F. Draw the perpendicular line a (axis) to F through f. Using an arbitrary point T on the directrix f, construct the midpoint M of the side $\overline{TF}$, and draw the perpendicular line p to TF through M. Finally, the intersection of the altitude r to f through T and the line p is a point P on the parabola $\mathcal{P}$, which is then the locus of the point P as T moves on f, as in Figure 1.

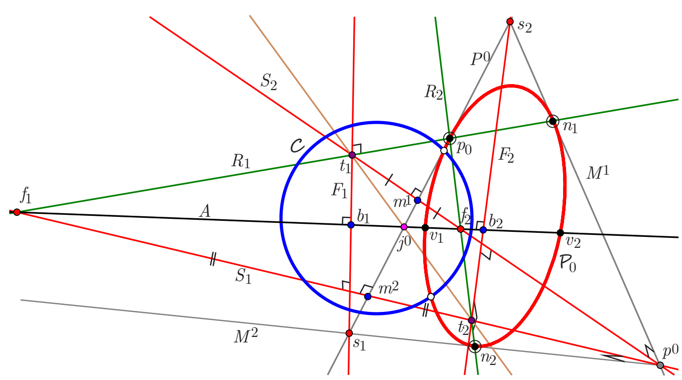

Figure 5: *Construction of a hyperbolic parabola $\mathcal{P}_0$*

To construct a hyperbolic parabola $\mathcal{P}_0$ from a pair of foci f_1 and f_2 with axis A, we proceed as in the Euclidean case, but we must be aware that the existence of midpoints is more subtle–they may not exist, and when they do, there are generally two of them! The situation is illustrated in Figure 5; choose a point t_1 on the directrix $F_1 \equiv f_1^\perp$ with the property that the side $\overline{t_1 f_2}$ has midpoints, call them m^1 and p^0, with corresponding midlines $M^1 = (m^1)^\perp$ and $P^0 = (p^0)^\perp$. One way of choosing such a point t_1 is to first choose an arbitrary point a_1 on F_1 and then reflect $b_1 \equiv F_1 A$ in a_1 to obtain t_1. In the triangle $\overline{b_1 t_1 f_2}$, two sides now have midpoints, so by Menelaus' theorem ([17]) the third side $\overline{t_1 f_2}$ will also have midpoints.

Now construct the meets $p_0 \equiv P^0 R_1$ and $n_1 \equiv M^1 R_1$, where $R_1 = t_1 f_1$. Then p_0 and n_1 will both be points on the parabola $\mathcal{P}_0$. The Figure also shows the symmetry available here: it is equally possible to choose a point t_2 on the other directrix $F_2 \equiv f_2^\perp$ with the property that the side $\overline{t_2 f_1}$ has midpoints, call them m^2 and p^0, with corresponding midlines $M^2 = (m^2)^\perp$ and $P^0 = (p^0)^\perp$. In that case the points $p_0 \equiv P^0 R_2$ and $n_2 \equiv M^2 R_2$, where $R_2 = t_2 f_2$, lie on the parabola $\mathcal{P}_0$. In Figure 5, the two points t_1 and t_2 are related by the fact that $t_1 t_2$ meets the axis A at the same point j^0 as does P^0; this accounts for the fact that $\overline{t_1 f_2}$ and $\overline{t_2 f_1}$ have a common midpoint p^0.

The justification for this construction will be given later, after we establish a suitable framework for coordinates and derive formulas for all the relevant points.

2.3 Dual conics and the connection with sydpoints

The theory of the hyperbolic parabola connects strongly with the *notion of sydpoints* as developed in [20].

The reason is that the sydpoints f^1 and f^2 of the side $\overline{f_1 f_2}$, should they exist (and our assumptions on our field will guarantee that they do) are naturally determined by the geometry of $\mathcal{P}_0$, and then they become the foci for the *twin parabola* $\mathcal{P}^0$ (in orange in our diagrams), which turns out to be the dual of the conic $\mathcal{P}_0$ with respect to the null circle $\mathcal{C}$. The sydpoint symmetry between the sides $\overline{f_1 f_2}$ and $\overline{f^1 f^2}$ is key to understanding many aspects of these conics. Although we will be studying the twin parabola more in the next paper in this series, it will be useful to be aware of it, as it explains some of our notational conventions.

In Figure 6, we see the parabola $\mathcal{P}_0$ with foci f_1, f_2 and a point p_0 on it, as well as the twin parabola $\mathcal{P}^0$ with foci f^1, f^2 and the twin point p^0 on it, which is the dual of the tangent P^0 to $\mathcal{P}_0$ at p_0. Reciprocally the dual of p_0 is the tangent to $\mathcal{P}^0$ at p^0. Note carefully that the tangents to both the parabola $\mathcal{P}_0$ and the null circle $\mathcal{C}$ at their common meets, namely the null points α_0 and $\overline{\alpha_0}$, pass through the foci of the twin parabola $\mathcal{P}^0$. Dually, note that the tangents to both the parabola $\mathcal{P}^0$ and the null circle $\mathcal{C}$ at their common meets, namely the null points δ_0 and $\overline{\delta_0}$, pass through the foci of $\mathcal{P}_0$. This Figure also shows the twin directrices F^1 and F^2, and the twin base points b^1 and b^2.

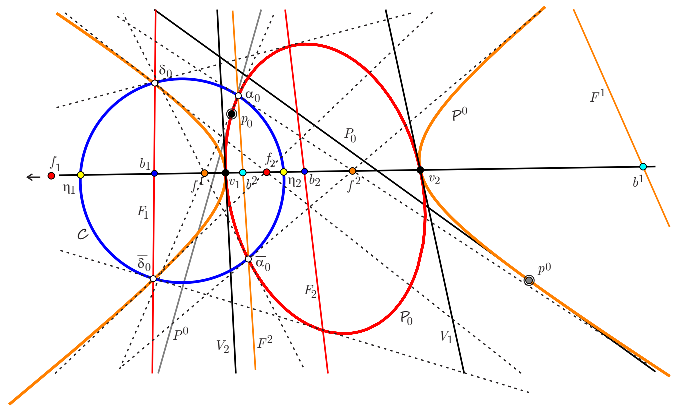

Figure 6: *The parabola $\mathcal{P}_0$ and its twin $\mathcal{P}^0$*

3 Standard Coordinates and duality

3.1 The four basis null points

In order to bring a systematic treatment to the study of the hyperbolic parabola $\mathcal{P}_0$, we need an appropriate coordinate system to bring $\mathcal{P}_0$ into as simple a form as possible. Although there is a great deal of choice for such an attempt, the one that we present here is the simplest and most elegant we could find; in it the beauty of the parabolic theory is reflected in an elegance and coherence in the corresponding formulae.

The key point is that aside from the two foci f_1 and f_2 which we used to define the parabola, there are four other points which naturally lie on the parabola and which can be used effectively as a basis for projective coordinates: the two vertices v_1 and v_2, together with two null points α_0 and $\overline{\alpha_0}$ which are symmetrically placed with respect to the axis.

We need to say some words about the existence of four such points. A priori there is no guarantee that the axis A meets the parabola; it will do so when the corresponding quadratic equation formed by meeting the line with the conic has a solution. The existence of the vertices is then an *assumption* that we may justify by adjoining an algebraic square root, if required, to our field.

We will use the four points v_1, v_2, α_0 and $\overline{\alpha_0}$, no three which are collinear, as a basis of a new projective coordinate system.

Theorem 4 (Parabola vertices) *If there is a non-null point v_1 lying both on the axis A and the parabola $\mathcal{P}_0$, then the perpendicular point $v_2 \equiv v_1^\perp A$ also lies on both the axis and the parabola, and these then are the only two points with this property.*

Proof. Suppose that v_1 lies on the axis $A \equiv f_1 f_2$ and on the parabola. Then if v_1 is not a null point,

$$q(f_1, v_1) + q(v_1, f_2) = 1.$$

Define $v_2 \equiv v_1^{\perp} A$, so that $q(v_1, v_2) = 1$. Now recall that if a, b and c are collinear points with $q(a, b) = 1$, then $q(a, c) + q(c, b) = 1$. So $q(v_1, f_1) + q(f_1, v_2) = 1$ and $q(v_1, f_2) + q(f_2, v_2) = 1$. Combining all three equations we see that $q(f_1, v_2) + q(v_2, f_2) = 1$, showing that v_2 also lies on the parabola. Since a line meets a conic at most at two points, there can be no other points on the axis and on $\mathcal{P}_0$. $\square$

We can see from Figure 3 that a parabola need not necessarily meet its axis. However any given line will meet a given conic if we are allowed to augment the field to an appropriate quadratic extension. So by possibly extending our field, *we will henceforth assume that our parabola $\mathcal{P}_0$ meets the axis $A = f_1 f_2$*. By the above theorem, it then meets this axis in exactly two points, which we call the **vertices** of the parabola, and denote by v_1 and v_2.

What about the existence of null points on $\mathcal{P}_0$? The meet of any two conics might have from zero to four points.

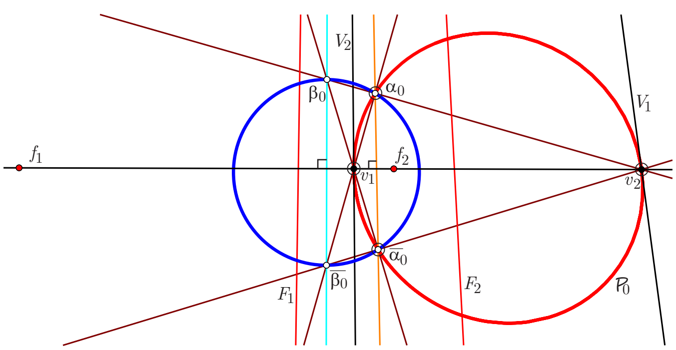

Figure 7: *The four basis points v_1, v_2, α_0 and $\overline{\alpha_0}$*

The parabola $\mathcal{P}_0$ with foci f_1 and f_2 need not meet the null conic $\mathcal{C}$. However for most examples, especially those of interest to a classical geometer working in the Klein model in the interior of the unit disk, we do have such an intersection—at least approximately over the rational numbers. So by possibly extending our field to a quartic extension, *we will henceforth assume that our parabola P_0 passes through at least one null point α_0*. By the assumption in the previous theorem such a null point α_0 cannot lie on the axis, so if we reflect it in the axis we get a second null point $\overline{\alpha_0} \equiv r_a(\alpha_0)$ which also lies on $\mathcal{P}_0$, since $\mathcal{P}_0$ is invariant under r_a. Clearly no three of the four **basis points** v_1, v_2, α_0 and $\overline{\alpha_0}$ are collinear, since they all lie on the parabola.

3.2 The Fundamental theorem and standard coordinates

We now invoke the *Fundamental Theorem of Projective Geometry*, which allows us to make a unique projective change of coordinates so that the four basis points become

$$v_1 = [0:0:1] \qquad v_2 = [1:0:0]$$
$$\alpha_0 = [1:1:1] \qquad \overline{\alpha_0} = [1:-1:1].$$

It follows that

$$A = v_1 v_2 = [0:0:1] \times [1:0:0] = \langle 0:1:0 \rangle.$$

These new coordinates will be called **standard coordinates** for the parabola $\mathcal{P}_0$, or **parabolic standard coordinates**. Note carefully that the introduction of such new coordinates will necessarily change the form of the quadrance and spread!

We now define, as in Figure 7, the points obtained by reflecting α_0 and $\overline{\alpha_0}$ in v_2: namely

$$\beta_0 \equiv r_{v_2}(\alpha_0) \quad \text{and} \quad \overline{\beta_0} \equiv r_{v_2}(\overline{\alpha_0}).$$

Because reflection is an isometry, these are also null points. Our notation with the overbar is something we will employ consistently: α_0 and $\overline{\alpha_0}$ are reflections in the point a, or equivalently in the dual line A, and so similarly for β_0 and $\overline{\beta_0}$.

Theorem 5 (β points) *We have $\beta_0 = (\alpha_0 v_2)(\overline{\alpha_0} v_1)$ and $\overline{\beta_0} = (\overline{\alpha_0} v_2)(\alpha_0 v_1)$. Furthermore in the new coordinate system $\beta_0 = [-1:1:1]$ and $\overline{\beta_0} = [-1:-1:1]$.*

Proof. The quadrangle of null points $\alpha_0 \overline{\alpha_0} \beta_0 \overline{\beta_0}$ has one diagonal point v_2, obviously from the definition of β_0 and $\overline{\beta_0}$. It has another diagonal point a, because both $\alpha_0 \overline{\alpha_0}$ and $\beta_0 \overline{\beta_0}$ pass it; the first by construction and the second because it is obtained from the first by reflection in v_2, which lies on $A = a^{\perp}$. So the third diagonal point is the dual of $a v_2$, which is v_1 by the previous theorem. It follows that $\beta_0 = (\alpha_0 v_2)(\overline{\alpha_0} v_1)$ and $\overline{\beta_0} = (\overline{\alpha_0} v_2)(\alpha_0 v_1)$. Now we can calculate that

$$\begin{aligned}
\beta_0 &= ([1:1:1] \times [1:0:0]) \times ([1:-1:1] \times [0:0:1]) \\
&= \langle 0:1:-1 \rangle \times \langle 1:1:0 \rangle = [-1:1:1] \\
\overline{\beta_0} &= ([1:-1:1] \times [1:0:0]) \times ([1:1:1] \times [0:0:1]) \\
&= \langle 0:1:1 \rangle \times \langle 1:-1:0 \rangle = [-1:-1:1].
\end{aligned}$$
$\square$

When we apply a general projective transformation of the projective plane to get the four points v_1, v_2, α_0 and $\overline{\alpha_0}$ into standard position, the metrical structure will change. While we started with the symmetric matrix J for the form, the new symmetric matrix is of the form $\mathbf{C} = MJM^T$ for some invertible matrix M. However this matrix $\mathbf{C}$ is not arbitrary; since we require that the four points lie on the parabola $\mathcal{P}_0$. We now arrive at the crucial result which sets up our coordinate system, and is the basis for all subsequent calculations. This is the fact that the new matrix $\mathbf{C}$, and its adjugate $\mathbf{D}$, have a particularly simple form, depending on a single parameter α which subsequently appears in almost all our formulas.

Theorem 6 (Parabola standard coordinates) *The symmetric bilinear form in standard coordinates is given by $v_1 \odot v_2 = v_1 \mathbf{C} v_2^T$ where*

$$\mathbf{C} = \begin{bmatrix} \alpha^2 & 0 & 0 \\ 0 & 1-\alpha^2 & 0 \\ 0 & 0 & -1 \end{bmatrix} \quad \text{and}$$

$$\mathbf{D} = \operatorname{adj}(\mathbf{C}) = \begin{bmatrix} \alpha^2 - 1 & 0 & 0 \\ 0 & -\alpha^2 & 0 \\ 0 & 0 & \alpha^2\left(1-\alpha^2\right) \end{bmatrix} \tag{7}$$

for some number α. In terms of α, the parabola $\mathcal{P}_0$ has equation $xz - y^2 = 0$ and its foci are

$$f_1 = [\alpha+1 : 0 : \alpha(\alpha-1)] \quad \text{and} \quad f_2 = [1-\alpha : 0 : \alpha(\alpha+1)].$$

Proof. Suppose that our new bilinear form in standard coordinates is given by $v_1 \odot v_2 = v_1 \mathbf{C} v_2^T$ where

$$\mathbf{C} = \begin{bmatrix} a & d & f \\ d & b & g \\ f & g & c \end{bmatrix} \quad \text{and}$$

$$\mathbf{D} = \operatorname{adj}(\mathbf{C}) = \begin{bmatrix} bc-g^2 & fg-cd & dg-bf \\ fg-cd & ac-f^2 & df-ag \\ dg-bf & df-ag & ab-d^2 \end{bmatrix}.$$

The fact that the four points $\alpha_0 = [1:1:1]$, $\overline{\alpha_0} = [1:-1:1]$, $\beta_0 = [-1:1:1]$ and $\overline{\beta_0} = [-1:-1:1]$ must all be null points means

$$\alpha_0 \mathbf{C} \alpha_0^T = \overline{\alpha_0} \mathbf{C} \left(\overline{\alpha_0}\right)^T = \beta_0 \mathbf{C} \beta_0^T = \overline{\beta_0} \mathbf{C} \left(\overline{\beta_0}\right)^T = 0.$$

These conditions lead to the following linear system of equations involving the entries of $\mathbf{C}$:

$$a+b+c+2d+2f+2g=0$$
$$a+b+c-2d+2f-2g=0$$
$$a+b+c-2d-2f+2g=0$$
$$a+b+c+2d-2f-2g=0.$$

From this we deduce that $d=f=g=0$, and $a=-(b+c)$. So the matrices have the form, up to scaling, of:

$$\mathbf{C} = \begin{bmatrix} a & 0 & 0 \\ 0 & 1-a & 0 \\ 0 & 0 & -1 \end{bmatrix} \quad \text{and}$$

$$\mathbf{D} = \begin{bmatrix} a-1 & 0 & 0 \\ 0 & -a & 0 \\ 0 & 0 & -a(a-1) \end{bmatrix}.$$

But there is also the condition that $\mathcal{P}_0$ is a parabola with foci f_1 and f_2, passing through all four basis points $v_1 = [0:0:1], v_2 = [1:0:0], \alpha_0 = [1:1:1]$ and $\overline{\alpha_0} = [1:-1:1]$. Since the foci lie on the axis $A = v_1 v_2$, we can write $f_1 = [m_1 : 0 : 1]$ and $f_2 = [m_2 : 0 : 1]$ for some m_1, m_2. Then recall that the quadrance and spread are determined by the projective matrices $\mathbf{C}$ and $\mathbf{D}$ by the rules (5).
We then compute

$$\begin{aligned} q(f_1, f_2) &= 1 - \frac{(am_1m_2-1)^2}{\left(am_1^2-1\right)\left(am_2^2-1\right)} \\ &= -a\frac{(m_1-m_2)^2}{\left(am_2^2-1\right)\left(am_1^2-1\right)}. \end{aligned}$$

Since f_1 and f_2 are by assumption not perpendicular,

$$am_1m_2 - 1 \neq 0. \tag{8}$$

Also v_1 and v_2 lie on $\mathcal{P}_0$, so that

$$\begin{aligned} & q([m_1:0:1],[0:0:1]) + q([m_2:0:1],[0:0:1]) - 1 \\ & \quad = \frac{(am_1m_2-1)(am_1m_2+1)}{\left(am_2^2-1\right)\left(am_1^2-1\right)} = 0 \qquad \text{and} \\ & q([m_1:0:1],[1:0:0]) + q([m_2:0:1],[1:0:0]) - 1 \\ & \quad = -\frac{(am_1m_2-1)(am_1m_2+1)}{\left(am_1^2-1\right)\left(am_2^2-1\right)} = 0. \end{aligned}$$

Both these conditions, given (8), are equivalent to the relation

$$am_1m_2 + 1 = 0 \tag{9}$$

which we henceforth assume, implying that we may write

$$m_1 = m \quad \text{and} \quad m_2 = -\frac{1}{am}$$

for some non-zero number m.
In addition we must ensure that α_0 and $\overline{\alpha_0}$ lie on $\mathcal{P}_0$, but since these are both null points, the quadrances $q(f_1, \alpha_0)$ and $q(f_2, \alpha_0)$ etc. are undefined, and we must rather work with the general equation of the parabola. This is

$$\begin{aligned} & q([m:0:1],[x:y:z]) + q\left(\left[-\frac{1}{am}:0:1\right],[x:y:z]\right) - 1 \\ & \quad = \frac{4amxz - y^2(a-1)\left(am^2-1\right)}{\left(am^2-1\right)\left(ax^2-ay^2+y^2-z^2\right)} = 0 \end{aligned}$$

which shows the equation of the parabola to be

$$4amxz - y^2(a-1)\left(am^2-1\right) = 0. \tag{10}$$

Now the condition that $\alpha_0 = [1:1:1]$ and $\overline{\alpha_0} = [1:-1:1]$ lie on $\mathcal{P}_0$ is that

$$\begin{aligned} 4am - (a-1)\left(am^2-1\right) &= a(1-a)m^2 + 4am + (a-1) \\ &= 0. \end{aligned} \tag{11}$$

Given that we started out with the existence of f_1 and f_2 assumed, we see that the discriminant of this quadratic equation

$$(4a)^2 - 4a(1-a)(a-1) = 4a(a+1)^2$$

must be a square. But this occurs precisely when a is a square, say

$$a = \alpha^2.$$

In this case the quadratic equation (11) has the form $\alpha^2(1-\alpha^2)m^2 + 4\alpha^2 m + (\alpha^2 - 1) = 0$ with solutions

$$m = m_1 = \frac{1+\alpha}{\alpha(\alpha-1)} \quad \text{and} \quad m_2 = \frac{1-\alpha}{\alpha(\alpha+1)}.$$

Combining these with (10), the identity

$$4\alpha^2 \frac{(\alpha+1)}{\alpha(\alpha-1)} xz - y^2(\alpha^2-1)\left(\alpha^2\left(\frac{1+\alpha}{\alpha(\alpha-1)}\right)^2 - 1\right) = \frac{4(xz-y^2)\alpha(\alpha+1)}{\alpha-1} = 0$$

shows that the equation of the parabola pleasantly simplifies to be

$$xz - y^2 = 0. \tag{12}$$

The foci may now be expressed as

$$f_1 = [m_1 : 0 : 1] = [\alpha+1 : 0 : \alpha(\alpha-1)] \quad \text{and}$$
$$f_2 = [m_2 : 0 : 1] = [1-\alpha : 0 : \alpha(\alpha+1)]. \qquad \square$$

Notice that

$$\det \begin{bmatrix} \alpha^2 & 0 & 0 \\ 0 & 1-\alpha^2 & 0 \\ 0 & 0 & -1 \end{bmatrix} = \alpha^2(\alpha-1)(\alpha+1) \neq 0$$

so $\alpha \neq 0, \pm 1$, since $\mathbf{C}$ is an invertible projective matrix.

The following Figure shows a view in the standard coordinate plane, where $[x:y:1]$ is represented by the affine point $[x,y]$. This corresponds roughly to a value of $\alpha = 0.3$. While it is both interesting and instructive to see different views of such a standard coordinate plane, this is somewhat unfamiliar to the classical geometer, so we will stick mostly to the Universal Hyperbolic Geometry model for our diagrams, where the unit circle always appears in blue as the unit circle $x^2 + y^2 = 1$.

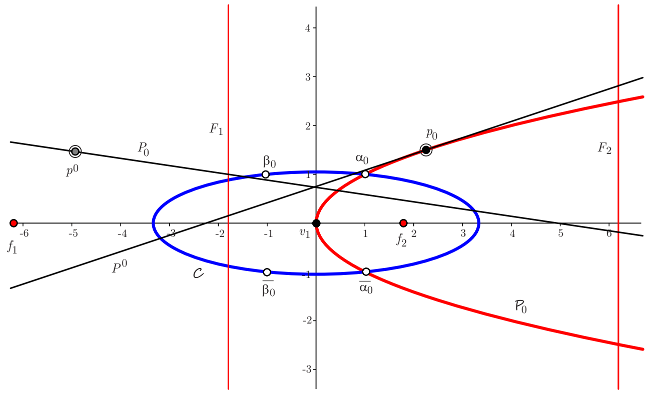

Figure 8: *A standard coordinate view of a parabola*

Theorem 7 (Parabola quadrance) *The* ***quadrance*** *of the parabola is*

$$q_{\mathcal{P}_0} \equiv q(f_1, f_2) = \frac{(\alpha^2+1)^2}{4\alpha^2}.$$

Proof. We compute that

$$q_{\mathcal{P}_0} = q([\alpha+1 : 0 : \alpha(\alpha-1)], [1-\alpha : 0 : \alpha(\alpha+1)])$$
$$= \frac{1}{4\alpha^2}(\alpha-1)^2(\alpha+1)^2 + 1 = \frac{(\alpha^2+1)^2}{4\alpha^2}. \qquad \square$$

We note that $q_{\mathcal{P}_0}$ is a square. This is a reflection of the fact that the assumption of the existence of vertices implies that the sides $\overline{f_1 b_2}$ and $\overline{f_2 b_1}$ have midpoints, see the Midpoint theorem [17].

The condition for points and lines to be null, in other words the equation for the null circle, is the following in standard coordinates.

Theorem 8 (Null points/ lines) *The point $p = [x:y:z]$ in standard coordinates is a null point precisely when*

$$\alpha^2 x^2 + (1-\alpha^2)y^2 - z^2 = 0.$$

The line $L = \langle l:m:n \rangle$ is a null line precisely when

$$(1-\alpha^2)l^2 + \alpha^2 m^2 + \alpha^2(\alpha^2-1)n^2 = 0.$$

Proof. These follow by using (7) to expand the respective conditions

$$[x:y:z]\,\mathbf{C}\,[x:y:z]^T = 0 \quad \text{and}$$
$$\langle l:m:n \rangle^T \mathbf{D} \langle l:m:n \rangle = 0. \qquad \square$$

3.3 Quadrance and spread in standard coordinates

We can now give explicit formulas for quadrance and spread in standard coordinates.

Theorem 9 (Quadrance formula) *The quadrance between the points $p_1 = [x_1 : y_1 : z_1]$ and $p_2 = [x_2 : y_2 : z_2]$ in parabolic standard coordinates is*

$$q(p_1,p_2) = -\frac{(x_1y_2-x_2y_1)^2\alpha^4+\left((x_1z_2-x_2z_1)^2-(y_1z_2-y_2z_1)^2-(x_1y_2-x_2y_1)^2\right)\alpha^2+(y_1z_2-y_2z_1)^2}{\left(\alpha^2x_1^2-y_1^2(\alpha^2-1)-z_1^2\right)\left(\alpha^2x_2^2-y_2^2(\alpha^2-1)-z_2^2\right)}.$$

Proof. From (4) and formula (7) for $\mathbf{C}$,

$$[x_1,y_1,z_1]\,\mathbf{C}\,[x_2,y_2,z_2]^T = \alpha^2x_1x_2 - y_1y_2\left(\alpha^2-1\right) - z_1z_2.$$

The formula follows using an identity calculation. □

Theorem 10 (Spread formula) *The spread between $L_1 = \langle l_1 : m_1 : n_1\rangle$ and $L_2 = \langle l_2 : m_2 : n_2\rangle$ is*

$$S(L_1,L_2) = \frac{\left((l_1n_2-l_2n_1)^2-(m_1n_2-m_2n_1)^2\right)\alpha^2+\left((l_1m_2-l_2m_1)^2-(l_2n_1-l_1n_2)^2\right)}{\left(l_1^2(\alpha^2-1)-\alpha^2m_1^2-\alpha^2n_1^2(\alpha^2-1)\right)\left(l_2^2(\alpha^2-1)-\alpha^2m_2^2-\alpha^2n_2^2(\alpha^2-1)\right)}.$$

Proof. From (4) and

$$[l_1,m_1,n_1]\,\mathbf{D}\,[l_2,m_2,n_2]^T = l_1l_2\left(\alpha^2-1\right)-\alpha^2m_1m_2-\alpha^2\left(\alpha^2-1\right)n_1n_2$$

the formula follows using an identity calculation. □

Theorem 11 (Axis reflection) *The reflection r_a in the point a has the form*

$$r_a([x:y:z]) = [x:-y:z].$$

Proof. We use the usual formula for reflection in a vector:

$$r_v(u) = 2\frac{(u\cdot v)\,v}{v\cdot v} - u = 2\frac{\left(uCv^T\right)v}{vCv^T} - u.$$

With the matrix C above, and working with regular vectors, we get

$$\begin{aligned} r_{[0,1,0]}([x,y,z]) &= 2\frac{[0,1,0]\,C\,[x,y,z]^T}{[0,1,0]\,C\,[0,1,0]^T}[0,1,0]-[x,y,z] \\ &= [-x,y,-z] = [x,-y,z]. \end{aligned}$$ □

3.4 Duality with respect to a conic and parametrizations

Let's recall some basic facts from the general theory of points and tangents to a projective conic. Suppose that a general conic $\mathcal{C}$ is given by the projective symmetric 3×3 matrix $\mathbf{A}$, with adjugate $\mathbf{B}$, so that a general point $p = [x:y:z]$ lies on $\mathcal{C}$ precisely when $p\mathbf{A}p^T = 0$. The tangent line P to a point p lying on $\mathcal{C}$ is $P = p^{\perp} \equiv \mathbf{A}p^T$. Dually, the point at which a tangent line L meets the conic is $l = L^{\perp} \equiv L^T\mathbf{B}$. While a point p on the conic satisfies the equation $p\mathbf{A}p^T = 0$, a line L on the conic (that is, a tangent line to the conic at some point) satisfies the dual equation $L^T\mathbf{B}L = 0$ (where we regard lines as projective column vectors).

More generally, we can regard the projective matrix $\mathbf{A}$ as determining a projective bilinear form, which is equivalent to a duality between points and lines. For a general point p, not necessarily lying on $\mathcal{C}$, its **dual** with respect to $\mathcal{C}$ is the line $p^{\perp} = \mathbf{A}p^T$, while for a general point L, its **dual** with respect to $\mathcal{C}$ is the point $L^{\perp} = L^T\mathbf{B}$. These are inverse procedures.

These notions of course go back to Apollonius, and it could be argued that this duality between points and lines is the essential feature or characteristic of a conic. But this modern formulation in the language of linear algebra and matrices makes many of its aspects much easier to understand, see [3], [12].

In this work, the main example of duality is with respect to the null circle $\mathcal{C}$, for which we will stick with the notation that if x_j is a point, then $X_j = \mathbf{C}x_j^T$ refers to the dual line and conversely. However the secondary duality with respect to the parabola $\mathcal{P}_0$ will also be involved, as we now see.

The equation (12) for the parabola $\mathcal{P}_0$ in standard coordinates, namely $p(x,y,z) = xz - y^2 = 0$, can be expressed in homogeneous matrix form as $p\mathbf{A}p^T = 0$ or

$$\begin{bmatrix} x & y & z \end{bmatrix}\mathbf{A}\begin{bmatrix} x & y & z \end{bmatrix}^T = 0$$

where

$$\mathbf{A} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & -2 & 0 \\ 1 & 0 & 0 \end{bmatrix} \quad \text{and} \quad \operatorname{adj}(\mathbf{A}) \equiv \mathbf{B} = \begin{bmatrix} 0 & 0 & 2 \\ 0 & -1 & 0 \\ 2 & 0 & 0 \end{bmatrix}.$$

Theorem 12 (Parabola parametrization) *The parabola $\mathcal{P}_0$ is parametrized, using an affine parameter t, by $p_0 = \left[t^2 : t : 1\right] \equiv p(t)$ or by using a projective parameter $[t:r]$ as $p_0 = \left[t^2 : tr : r^2\right] \equiv p(t:r)$. The tangent line P^0 to the parabola at $p_0 = \left[t^2 : t : 1\right]$ is $P^0 = \left\langle 1 : -2t : t^2\right\rangle \equiv P(t)$ or projectively the tangent to $p_0 = \left[t^2 : tr : r^2\right]$ is $P^0 = \left\langle r^2 : -2rt : t^2\right\rangle \equiv P(t:r)$. A line $L = \langle l : m : n\rangle$ is tangent to the parabola precisely when $m^2 = 4nl$.*

Proof. The simple form of the equation $xz = y^2$ makes the parametrization immediate. The formula for the tangent line is a direct application of the discussion above, so that

$$\begin{aligned} P^0 \equiv \mathbf{A}p_0^T &= \begin{bmatrix} 0 & 0 & -1 \\ 0 & 2 & 0 \\ -1 & 0 & 0 \end{bmatrix}\begin{bmatrix} t^2 & t & 1 \end{bmatrix}^T = \begin{bmatrix} 1 \\ -2t \\ t^2 \end{bmatrix} \\ &= \left\langle 1 : -2t : t^2\right\rangle \end{aligned}$$

or using projective parameters

$$P^0 \equiv \mathbf{A}p_0^T = \begin{bmatrix} 0 & 0 & -1 \\ 0 & 2 & 0 \\ -1 & 0 & 0 \end{bmatrix} \begin{bmatrix} t^2 & tr & r^2 \end{bmatrix}^T = \begin{bmatrix} r^2 \\ -2rt \\ t^2 \end{bmatrix}$$
$$= \left\langle r^2 : -2rt : t^2 \right\rangle.$$

The relation $m^2 = 4nl$ is exactly satisfied by those lines of this form. □

Theorem 13 (Tangent meets) *If $p_0 \equiv p(t)$ and $q_0 \equiv p(u)$ are two distinct points on $\mathcal{P}_0$, then their tangents P^0 and Q^0 meet at the* ***polar point*** *$z \equiv P^0Q^0 = [2tu : t+u : 2]$ while $Z \equiv p_0q_0 = \langle 1 : -(t+v) : tv \rangle$.*

Proof. We compute that

$$z \equiv P^0Q^0 = \left\langle 1 : -2t : t^2 \right\rangle \times \left\langle 1 : -2u : u^2 \right\rangle = [2tu : t+u : 2]$$

and

$$p_0q_0 = \left[t^2 : t : 1\right] \times \left[v^2 : v : 1\right] = \langle 1 : -(t+v) : tv \rangle.$$ □

The projective parametrization of $\mathcal{P}_0$ has the advantage that it includes the important point at infinity $p(1:0) = [1:0:0] = v_2$. We can recover the affine parametrization by setting $r = 1$, and we can go from the affine to the projective parametrization by replacing t with t/r and clearing denominators. In practice we will generally use the affine parametrization, since it is requires only one variable, not two. The existence of this simple parametrization will be extremely useful for us: giving us the same amount of control over the hyperbolic parabola as we have over the much simpler Euclidean parabola (which of course can be positioned to have exactly the same equation!)

Theorem 14 *The dual of the point $p_0 = \left[t^2 : t : 1\right]$ on $\mathcal{P}_0$ is $P_0 = \left\langle t^2\alpha^2 : t\left(1-\alpha^2\right) : -1 \right\rangle$. The dual of the tangent line $P^0 = \left\langle 1 : -2t : t^2 \right\rangle$ is $p^0 = \left[\alpha^2 - 1 : 2t\alpha^2 : -t^2\alpha^2\left(\alpha^2-1\right)\right]$.*

Proof. We compute that

$$P_0 = \mathbf{C}p_0^T = \begin{bmatrix} \alpha^2 & 0 & 0 \\ 0 & 1-\alpha^2 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} t^2 & t & 1 \end{bmatrix}^T$$
$$= \left\langle t^2\alpha^2 : t\left(1-\alpha^2\right) : -1 \right\rangle$$

and

$$p^0 = \left(P^0\right)^T \mathbf{D} = \begin{bmatrix} 1 & -2t & t^2 \end{bmatrix} \begin{bmatrix} \alpha^2-1 & 0 & 0 \\ 0 & -\alpha^2 & 0 \\ 0 & 0 & \alpha^2\left(1-\alpha^2\right) \end{bmatrix}$$
$$= \left[\alpha^2 - 1 : 2t\alpha^2 : -t^2\alpha^2\left(\alpha^2-1\right)\right].$$ □

We will say that p^0 is the **twin point** to p_0. Later we will see that the locus of p^0 is also a parabola, whose foci f^1 and f^2 are the sydpoints of $\overline{f_1 f_2}$.

Theorem 15 (Focus directrix polarity) *The focus f_1 is the pole of the directrix F_2 with respect to the parabola $\mathcal{P}_0$, and similarly the focus f_2 is the pole of the directrix F_1.*

Proof. We check that

$$F_2^T\mathbf{B} = \begin{bmatrix} \alpha(\alpha-1) & 0 & \alpha+1 \end{bmatrix}\mathbf{B} = [\alpha+1 : 0 : \alpha(\alpha-1)] = f_1$$

or

$$\mathbf{A}f_1^T = \mathbf{A}\begin{bmatrix} \alpha+1 & 0 & \alpha(\alpha-1) \end{bmatrix}^T = \langle \alpha(\alpha-1) : 0 : \alpha+1 \rangle = F_2.$$

Similarly,

$$F_1^T\mathbf{B} = \begin{bmatrix} \alpha(\alpha+1) & 0 & 1-\alpha \end{bmatrix}\mathbf{B} = [-(\alpha-1) : 0 : \alpha(\alpha+1)] = f_2$$

or

$$\mathbf{A}f_2^T = \mathbf{A}\begin{bmatrix} 1-\alpha & 0 & \alpha(\alpha+1) \end{bmatrix}^T = \langle \alpha(\alpha+1) : 0 : 1-\alpha \rangle = F_1.$$

□

In order for the parabola $y^2 = xz$ to have a null point $p(t)$, the parameter t must satisfy $\left[t^2 : t : 1\right]\mathbf{C}\left[t^2 : t : 1\right]^T = 0$, which yields $\left(t^2-1\right)\left(t^2\alpha^2+1\right) = 0$. Over the rational field, the values $t = \pm 1$ agree with the null points $\alpha_0 = [1:1:1]$ and $\overline{\alpha_0} = [1:-1:1]$ with which we begun our work.

However, there are also another two solutions which are invisible over the rational field, but exist in an extension field obtained by adjoining a square root i of -1. These points are $\zeta_1 \equiv \left[1 : i\alpha : -\alpha^2\right]$ and $\zeta_2 \equiv \left[1 : -i\alpha : -\alpha^2\right]$. In this paper we will not mention these points too much.

3.5 Formulas for directrices, vertex lines, base points and base lines

We can now augment our formulas using standard coordinates. The directrices are

$$F_1 \equiv f_1^\perp = \mathbf{C}\,[\alpha+1 : 0 : \alpha(\alpha-1)]^T = \langle \alpha(\alpha+1) : 0 : 1-\alpha \rangle$$
$$F_2 \equiv f_2^\perp = \mathbf{C}\,[1-\alpha : 0 : \alpha(\alpha+1)]^T = \langle \alpha(\alpha-1) : 0 : 1+\alpha \rangle.$$

The base points are the meets of the directrices and the axis line. They are

$$b_1 \equiv F_1A = \left\langle \alpha^2(\alpha+1) : 0 : \alpha(1-\alpha) \right\rangle \times \langle 0:1:0 \rangle$$
$$= [\alpha-1 : 0 : \alpha(\alpha+1)]$$
$$b_2 \equiv F_2A = \left\langle \alpha^2(\alpha-1) : 0 : \alpha(1+\alpha) \right\rangle \times \langle 0:1:0 \rangle$$
$$= [\alpha+1 : 0 : \alpha(1-\alpha)].$$

The duals are the **base lines** B_1, B_2, which are the altitudes to the axis A through the foci f_1, f_2 of the parabola:

$$B_1 \equiv b_1^\perp = \mathbf{C}\,[(\alpha-1) : 0 : \alpha(\alpha+1)]$$
$$= \langle -\alpha(\alpha-1) : 0 : \alpha+1 \rangle$$
$$B_2 \equiv b_2^\perp = \mathbf{C}\,[(\alpha+1) : 0 : \alpha(1-\alpha)]$$
$$= \langle \alpha(\alpha+1) : 0 : \alpha-1 \rangle.$$

The **vertex lines** V_1, V_2 are the altitudes to the axis A through the vertices v_1, v_2 of the parabola**:**

$$V_1 \equiv v_1^{\perp} = \mathbf{C}\,[0:0:1] = [0:0:1] \qquad \text{and}$$
$$V_2 \equiv v_2^{\perp} = \mathbf{C}\,[1:0:0] = [1:0:0].$$

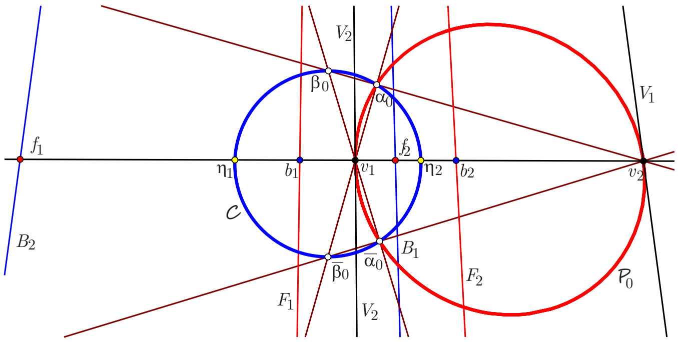

Figure 9: *Some basic points associated to a parabola $\mathcal{P}_0$*

3.6 The j, h and d points and lines

We define the **axis null points** to be the meets of the axis A and the null conic C. These points exist under our assumptions, and are

$$\eta_1 \equiv AC = [1:0:\alpha] \quad \text{and} \quad \eta_2 = AC = [-1:0:\alpha].$$

We now introduce some other secondary points and lines associated to a generic point p_0 on the parabola $\mathcal{P}_0$. The reflection of $p_0 = \left[t^2:t:1\right]$ in the axis is the **opposite point**

$$\overline{p_0} = r_a(p_0) = \left[t^2:-t:1\right].$$

Clearly $\overline{p_0}$ also lies on the parabola.
The meet of the dual line P_0 with the axis A is the j**-point**

$$j_0 \equiv P_0A = \left\langle t^2\alpha^2 : t\left(1-\alpha^2\right) : -1\right\rangle \times \langle 0:1:0\rangle = \left[1:0:t^2\alpha^2\right]$$

with dual the J**-line**

$$J_0 = ap_0 = [0:1:0] \times \left[t^2:t:1\right] = \left\langle 1:0:-t^2\right\rangle.$$

By duality J_0 is the altitude from p_0 to the axis, and so also $J_0 = p_0\overline{p_0}$. The meet of the J-line with the axis is the foot of this altitude; it is the h**-point**

$$h_0 \equiv AJ_0 = \langle 0:1:0\rangle \times \left\langle 1:0:-t^2\right\rangle = \left[t^2:0:1\right]$$

and its dual is the H-line

$$H_0 \equiv h_0^{\perp} = a j_0 = [0:1:0] \times \left[1:0:t^2\alpha^2\right] = \left\langle t^2\alpha^2:0:-1\right\rangle.$$

The meet of the tangent line P^0 with the axis is the **twin** j**-point**

$$j^0 \equiv P^0A = \left\langle 1:-2t:t^2\right\rangle \times \langle 0:1:0\rangle = \left[-t^2:0:1\right]$$

with dual the **twin** J**-line**

$$J^0 = ap^0 = [0:1:0] \times \left[\alpha^2-1:2t\alpha^2:-t^2\alpha^2\left(\alpha^2-1\right)\right] = \left\langle t^2\alpha^2:0:1\right\rangle.$$

The meet of the twin J-line with the axis is the **twin** h**-point**

$$h^0 \equiv AJ^0 = \langle 0:1:0\rangle \times \left\langle t^2\alpha^2:0:1\right\rangle = \left[-1:0:t^2\alpha^2\right]$$

and its dual is the **twin** H**-line**

$$H^0 \equiv \left(h^0\right)^{\perp} = a j^0 = [0:1:0] \times \left[-t^2:0:1\right] = \left\langle 1:0:t^2\right\rangle.$$

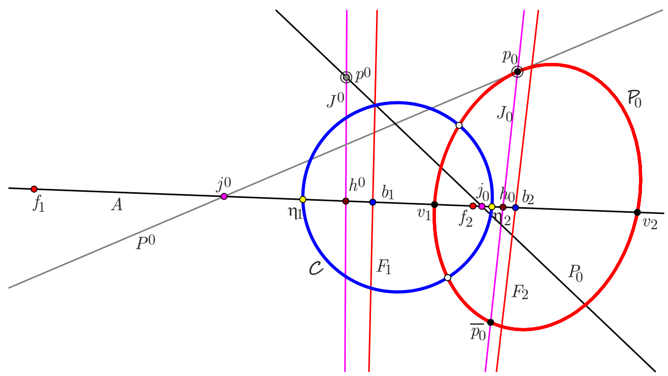

Figure 10: *The j and h points and lines*

Theorem 16 (Null tangent) *The tangent P^0 to the parabola $\mathcal{P}_0$ at p_0 is a null line precisely when p_0 lies on a directrix, and in this case the twin point p^0 is a null point lying on the other directrix, j_0 coincides with a focus, and j^0 with the other focus.*

Proof. If the tangent $P^0 = \left\langle 1:-2t:t^2\right\rangle$ at $p_0 = \left[t^2:t:1\right]$ is a null line, then by the Null points/lines theorem

$$\left(1-\alpha^2\right) + 4\alpha^2t^2 + \alpha^2\left(\alpha^2-1\right)t^4 = 0.$$

This factors as

$$\left(\alpha(\alpha+1)t^2 - (\alpha-1)\right)\left(\alpha(\alpha-1)t^2 + (\alpha+1)\right) = 0$$

so that

$$t^2 = \frac{\alpha-1}{\alpha(\alpha+1)} \qquad \text{or} \qquad t^2 = -\frac{\alpha+1}{\alpha(\alpha-1)}. \tag{13}$$

Now $p_0 = \left[t^2:t:1\right]$ is on the directrix F_1 or F_2, precisely when

$$\left[t^2:t:1\right]\left[\alpha^2(\alpha+1):0:\alpha(1-\alpha)\right]^T = 0 \qquad \text{or}$$
$$\left[t^2:t:1\right]\left[\alpha^2(\alpha-1):0:\alpha(1+\alpha)\right]^T = 0$$

and similarly, the point $p^0 = \left[\alpha^2-1 : 2t\alpha^2 : -t^2\alpha^2\left(\alpha^2-1\right)\right]$ is on the directrix F_1 or F_2, precisely when

$$\left[\alpha^2-1 : 2t\alpha^2 : -t^2\alpha^2\left(\alpha^2-1\right)\right]\left[\alpha^2(\alpha+1) : 0 : \alpha(1-\alpha)\right]^T = 0$$

or

$$\left[\alpha^2-1 : 2t\alpha^2 : -t^2\alpha^2\left(\alpha^2-1\right)\right]\left[\alpha^2(\alpha-1) : 0 : \alpha(1+\alpha)\right]^T = 0.$$

These conditions are exactly the same as (13). Using (13) we get either $j_0 = \left[1:0:t^2\alpha^2\right] = [\alpha+1:0:\alpha(\alpha-1)] = f_1$ and $j^0 = \left[-t^2:0:1\right] = [1-\alpha:0:\alpha(\alpha+1)] = f_2$ or $j_0 = [1-\alpha:0:\alpha(\alpha+1)] = f_2$ and $j^0 = [\alpha+1:0:\alpha(\alpha-1)] = f_1$. □

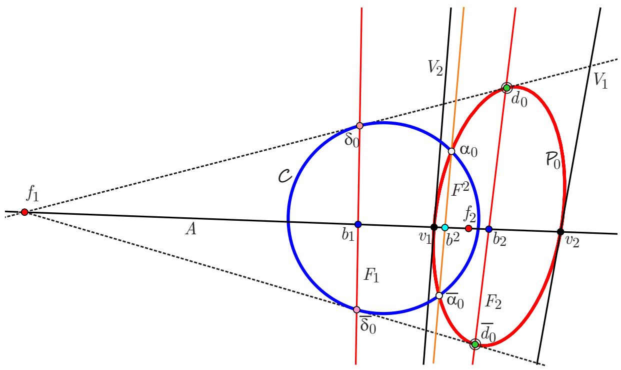

Figure 11: *Null tangents and d_0, $\overline{d_0}$ points*

We introduce the points d_0 and $\overline{d_0}$ to be the meets of the directrix F_2 with the parabola $\mathcal{P}_0$, should they exist, and the corresponding twin null points δ_0 and $\overline{\delta_0}$ lying on the directrix F_1. These are important canonical points associated with the parabola. Since their existence requires solutions to (13), and so a number τ satisfying $\tau^2 = \alpha\left(\alpha^2-1\right)$, we may write

$$d_0 = F_2\mathcal{P}_0 = [\alpha-1 : \tau : \alpha(\alpha+1)]$$
$$\overline{d_0} = F_2\mathcal{P}_0 = [\alpha-1 : -\tau : \alpha(\alpha+1)]$$

and

$$d^0 \equiv \delta_0 = \left[(\alpha-1)^2(\alpha+1) : -2\alpha i\tau : \alpha(\alpha+1)^2(\alpha-1)\right]$$
$$\overline{d_0} = \overline{\delta_0} = \left[(\alpha-1)^2(\alpha+1) : 2\alpha i\tau : \alpha(\alpha+1)^2(\alpha-1)\right]$$

where $(i\tau)^2 = -\alpha\left(\alpha^2-1\right)$.

In Figure 11, notice that the lines $f_1\delta_0$ and $f_1\overline{\delta_0}$ are joint tangents to both $\mathcal{C}$ and the parabola $\mathcal{P}_0$, touching $\mathcal{P}_0$ at the points d_0 and $\overline{d_0}$.

3.7 The sydpoints of a parabola

It is a remarkable fact that the theory of sydpoints that we developed in [20] plays a crucial role in the theory of the parabola. Define the lines

$$F^2 \equiv \alpha_0\overline{\alpha_0} = [1:1:1]\times[1:-1:1] = \langle 1:0:-1\rangle$$
$$B^1 \equiv \beta_0\overline{\beta_0} = [-1:1:1]\times[-1:-1:1] = \langle 1:0:1\rangle$$

with corresponding axis meets

$$b^2 \equiv F^2A = \langle 1:0:-1\rangle\times\langle 0:1:0\rangle = [1:0:1]$$
$$f^1 \equiv B^1A = \langle 1:0:1\rangle\times\langle 0:1:0\rangle = [-1:0:1].$$

The duals of these points and lines are

$$f^2 \equiv \left(F^2\right)^{\perp} = \begin{bmatrix}1 & 0 & -1\end{bmatrix}\mathbf{D} = \left[1:0:\alpha^2\right]$$
$$b^1 \equiv \left(B^1\right)^{\perp} = \begin{bmatrix}1 & 0 & 1\end{bmatrix}\mathbf{D} = \left[1:0:-\alpha^2\right]$$
$$B^2 \equiv \left(b^2\right)^{\perp} = \mathbf{C}\begin{bmatrix}1 & 0 & 1\end{bmatrix}^T = \left\langle -\alpha^2:0:1\right\rangle$$
$$F^1 \equiv \left(f^1\right)^{\perp} = \mathbf{C}\begin{bmatrix}-1 & 0 & 1\end{bmatrix}^T = \left\langle \alpha^2:0:1\right\rangle.$$

The points f^1 and f^2 are the **twin foci**, or **t-foci** for short**,** of the parabola $\mathcal{P}_0$. They will play a major role in the theory. The dual lines of f^1 and f^2, namely F^1 and F^2 respectively, are the **t-directrices of** $\mathcal{P}_0$. The meets of the t-directrices and the axis A are $F^1A \equiv b^1$ and $F^2A \equiv b^2$ respectively; these are the **t-base points** of $\mathcal{P}_0$. The dual lines of b^1 and b^2, namely B^1 and B^2 respectively, are the **t-base lines** of $\mathcal{P}$. These are all shown in Figure 12.

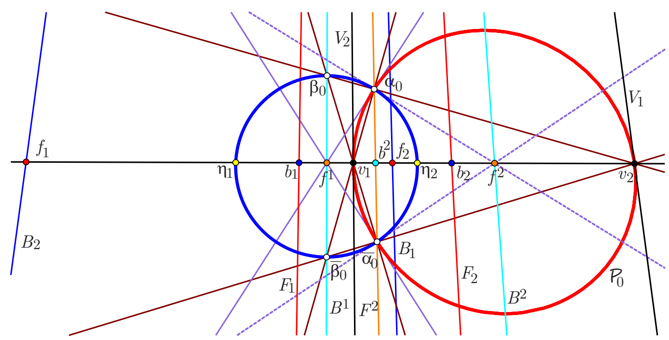

Figure 12: *Sydpoints and the twin foci f^1 and f^2 of $\mathcal{P}_0$*

Theorem 17 (Parabola sydpoints) *The points f^1 and f^2 are the sydpoints of the side $\overline{f_1f_2}$.*

Proof. We calculate that

$$q\left(f_1, f^1\right) = q\left([\alpha+1:0:\alpha(\alpha-1)], [1:0:-1]\right)$$
$$= 1+\frac{\left(\alpha(\alpha-1)+\alpha^2(\alpha+1)\right)^2}{4\alpha^3-4\alpha^5} = -\frac{\left(\alpha^2+1\right)^2}{4\alpha\left(\alpha^2-1\right)}$$
$$q\left(f_2, f^1\right) = q\left([1-\alpha:0:\alpha(\alpha+1)], [1:0:-1]\right)$$
$$= 1-\frac{\left(\alpha(\alpha+1)-\alpha^2(\alpha-1)\right)^2}{4\alpha^3-4\alpha^5} = \frac{\left(\alpha^2+1\right)^2}{4\alpha\left(\alpha^2-1\right)}$$

$$q\left(f_1,f^2\right)=q\left(\left[\alpha+1:0:\alpha(\alpha-1)\right],\left[1:0:\alpha^2\right]\right)$$
$$=1-\frac{\left(\alpha^2(\alpha+1)-\alpha^3(\alpha-1)\right)^2}{4\alpha^5-4\alpha^7}=\frac{1}{4}\frac{\left(\alpha^2+1\right)^2}{\alpha\left(\alpha^2-1\right)}$$
$$q\left(f_2,f^2\right)=q\left(\left[1-\alpha:0:\alpha(\alpha+1)\right],\left[1:0:\alpha^2\right]\right)$$
$$=1+\frac{\left(\alpha^2(\alpha-1)+\alpha^3(\alpha+1)\right)^2}{4\alpha^5-4\alpha^7}=-\frac{1}{4}\frac{\left(\alpha^2+1\right)^2}{\alpha\left(\alpha^2-1\right)}.$$

Clearly $q\left(f_1,f^1\right)=-q\left(f_2,f^1\right)$ and $q\left(f_1,f^2\right)=-q\left(f_2,f^2\right)$ so f^1 and f^2 are the sydpoints of the side $\overline{f_1f_2}$. □

Theorem 18 (Parabola null tangents) *The tangents to the null circle at α_0 and $\overline{\alpha_0}$ meet at f^2. The tangents to $\mathcal{P}_0$ at α_0 and $\overline{\alpha_0}$ meet at f^1.*

Proof. The tangents to the null circle at α_0 and $\overline{\alpha_0}$ are the dual lines

$$\alpha_0^{\perp}=C[1:1:1]^T=\left\langle\alpha^2:1-\alpha^2:-1\right\rangle \qquad \text{and}$$
$$\left(\overline{\alpha_0}\right)^{\perp}=C[1:-1:1]^T=\left\langle\alpha^2:\alpha^2-1:-1\right\rangle$$

and these meet at

$$\alpha_0^{\perp}\left(\overline{\alpha_0}\right)^{\perp}=\left\langle\alpha^2:1-\alpha^2:-1\right\rangle\times\left\langle\alpha^2:\alpha^2-1:-1\right\rangle$$
$$=\left[1:0:\alpha^2\right]=f^2.$$

The tangents to the parabola $\mathcal{P}_0$ at α_0 and $\overline{\alpha_0}$ are the lines

$$\mathbf{A}\begin{bmatrix}1&1&1\end{bmatrix}^T=\langle 1:-2:1\rangle \quad \text{and} \quad \mathbf{A}\begin{bmatrix}1&-1&1\end{bmatrix}^T=\langle 1:2:1\rangle$$

and these meet at
$\langle 1:-2:1\rangle\times\langle 1:2:1\rangle=[-1:0:1]=f^1$. □

3.8 A rational parabola

In this section we show the existence of a two-parameter family of rational hyperbolic parabolas, and give the associated transformations to parabolic standard coordinates.
The conic $\mathcal{P}_0$ with equation

$$\left(t_1^2t_2^2-1\right)x^2+2\left(t_1^2t_2^2+1\right)x+\left(t_1^2-t_2^2\right)y^2+\left(t_1^2t_2^2-1\right)=0$$

meets the null circle at the null points $\alpha_0=\left[1-t_1^2:2t_1:t_1^2+1\right]$ and $\overline{\alpha_0}=\left[1-t_1^2:-2t_1:t_1^2+1\right]$. This is a parabola with foci
$f_1=\left[t_1+t_2-t_1t_2^2+t_1^2t_2:0:t_1+t_2+t_1t_2^2-t_1^2t_2\right]$ and $f_2=\left[t_1-t_2-t_1t_2^2-t_1^2t_2:0:t_1-t_2+t_1t_2^2+t_1^2t_2\right]$, axis $A=\langle 0:1:0\rangle$, and t-foci $f^1=\left[t_1^2+1:0:-\left(t_1^2-1\right)\right]$ and $f^2=\left[t_2^2-1:0:-\left(t_2^2+1\right)\right]$. The null points $\beta_0,\overline{\beta_0}$ are $\beta_0=\left[1-t_2^2:2t_2:t_2^2+1\right]$ and $\overline{\beta_0}=\left[1-t_2^2:-2t_2:t_2^2+1\right]$, and the vertices are $v_1=\left[t_1t_2-1:0:-(t_1t_2+1)\right]$ and $v_2=\left[t_1t_2+1:0:-(t_1t_2-1)\right]$. Note that

$$q\left(f^1,f^2\right)-1$$
$$=q\left(\left[t_1^2+1:0:-\left(t_1^2-1\right)\right],\left[t_2^2-1:0:-\left(t_2^2+1\right)\right]\right)-1$$
$$=\frac{1}{4}\left(t_1-t_2\right)^2\frac{\left(t_1+t_2\right)^2}{t_1^2t_2^2}$$

is a square.
We are now interested in sending these points $\alpha_0,\overline{\alpha_0},\beta_0,\overline{\beta_0}$ to the points $[1:1:1],[1:-1:1],[-1:1:1],[-1:-1:1]$ respectively, using a projective transformation. Firstly, we send $[1:1:1],[1:0:0],[0:1:0],[0:0:1]$ to $\alpha_0,\overline{\alpha_0},\beta_0,\overline{\beta_0}$ respectively by the linear transformation $T_1(v)=vN$ where N is

$$N=\begin{bmatrix}-t_2\left(t_1^2-1\right)&-2t_1t_2&t_2\left(t_1^2+1\right)\\-t_1\left(t_2^2-1\right)&2t_1t_2&t_1\left(t_2^2+1\right)\\t_1\left(t_2^2-1\right)&2t_1t_2&-t_1\left(t_2^2+1\right)\end{bmatrix}.$$

Its inverse sends $\alpha_0,\overline{\alpha_0},\beta_0,\overline{\beta_0}$ back to $[1:1:1]$, $[1:0:0]$, $[0:1:0]$, $[0:0:1]$ by $T(v)=vR$ where R is the adjugate of N :

$$R=\begin{bmatrix}-2t_1\left(t_2^2+1\right)&(t_1-t_2)(t_1t_2-1)&-(t_1t_2+1)(t_1+t_2)\\0&t_1^2-t_2^2&t_1^2-t_2^2\\-2t_1\left(t_2^2-1\right)&(t_1-t_2)(t_1t_2+1)&-(t_1t_2-1)(t_1+t_2)\end{bmatrix}.$$

Secondly, the linear transformation $T_2(v)=vM$, where M is

$$M=\begin{bmatrix}1&-1&1\\-1&1&1\\1&1&-1\end{bmatrix},$$

sends $[1:1:1]$, $[1:0:0]$, $[0:1:0]$, $[0:0:1]$ to $[1:1:1]$, $[-1:1:1]$, $[-1:1:1]$, $[-1:-1:1]$ respectively. Thus, the required transformation is $T(v)=v(RM)$ where RM is

$$\begin{bmatrix}-(t_1t_2+1)(t_1+t_2)&0&(t_1-t_2)(t_1t_2-1)\\0&(t_1-t_2)(t_1+t_2)&0\\-(t_1t_2-1)(t_1+t_2)&0&(t_1-t_2)(t_1t_2+1)\end{bmatrix}.$$

After applying this linear transformation, the matrix J is replaced by

$$\mathbf{C}=(RM)^{-1}J\left((RM)^{-1}\right)^T$$
$$=\begin{bmatrix}t_1t_2(t_1-t_2)^2&0&0\\0&4t_1^2t_2^2&0\\0&0&-t_1t_2(t_1+t_2)^2\end{bmatrix} \qquad \text{and}$$
$$\mathbf{D}=(RM)^TJ(RM)$$
$$=\begin{bmatrix}4t_1t_2(t_1+t_2)^2&0&0\\0&\left(t_1^2-t_2^2\right)^2&0\\0&0&-4t_1t_2(t_1-t_2)^2\end{bmatrix}.$$

and we get $\alpha = \frac{t_1-t_2}{t_1+t_2}$. In this new coordinate system, the parabola is $y^2 = xz$ with foci $f_1 = [t_1(t_1+t_2) : 0 : -t_2(t_1-t_2)]$ and $f_2 = [t_2(t_1+t_2) : 0 : t_1(t_1-t_2)]$.

Example 1 *If $t_1 = 1/2$ and $t_2 = 3$ then the parabola $\mathcal{P}_0$ has equation $26x+5x^2-35y^2+5=0$ which meets the null circle at the null points $\alpha_0 = [3:4:5]$ and $\overline{\alpha_0} = [3:-4:5]$; has axis $A = \langle 0:1:0\rangle$, foci $f_1 = [-1:0:29]$ and $f_2 = [-31:0:11]$, vertices $v_1 = [1:0:-5]$ and $v_2 = [5:0:-1]$, t-foci $f^1 = [5:0:3]$ and $f^2 = [4:0:-5]$, and $\beta_0 = [-4:3:5]$ and $\overline{\beta_0} = [4:3:-5]$.*

3.9 Focal and base lines

We now define some other fundamental points and lines associated with a point $p_0 \equiv [t^2:t:1]$ on the parabola $\mathcal{P}_0$. It will be convenient to introduce the quantities

$$\Delta_1(t) \equiv \alpha+1+t^2\alpha-t^2\alpha^2$$
$$\Delta_2(t) \equiv \alpha-1+t^2\alpha+t^2\alpha^2$$
$$\Delta_3(t) \equiv \alpha+1-t^2\alpha+t^2\alpha^2$$
$$\Delta_4(t) \equiv \alpha-1-t^2\alpha^2-t^2\alpha$$

which depends on t, and so on p_0, and which will appear in many formulas to follow. Notice that

$$\Delta_1^2-\Delta_2^2 = -4\alpha\left(t^2\alpha^2-1\right)\left(t^2+1\right), \quad \Delta_1^2-\Delta_3^2 = -4\alpha t^2\left(\alpha^2-1\right)$$
$$\Delta_1^2-\Delta_4^2 = -4\alpha\left(t^4\alpha^2-1\right), \quad \Delta_2^2-\Delta_3^2 = 4\alpha\left(t^4\alpha^2-1\right)$$
$$\Delta_2^2-\Delta_4^2 = 4t^2\alpha\left(\alpha^2-1\right), \quad \Delta_3^2-\Delta_4^2 = -4\alpha\left(t^2-1\right)\left(t^2\alpha^2+1\right).$$

The **focal lines** R_1, R_2 and the dual **focal line points** r_1, r_2 are defined by, and calculated as:

$$R_1 \equiv f_1p_0 = [\alpha+1:0:\alpha(\alpha-1)] \times [t^2:t:1]$$
$$= \langle t\alpha(\alpha-1) : \Delta_1 : -t(\alpha+1)\rangle$$
$$R_2 \equiv f_2p_0 = [1-\alpha:0:\alpha(\alpha+1)] \times [t^2:t:1]$$
$$= \langle t\alpha(\alpha+1) : -\Delta_2 : t(\alpha-1)\rangle$$
$$r_1 \equiv R_1^\perp = F_1P_0$$
$$= \left[t(\alpha-1)^2(\alpha+1) : -\alpha\Delta_1 : t\alpha(\alpha-1)(\alpha+1)^2\right]$$
$$r_2 \equiv R_2^\perp = F_2P_0$$
$$= \left[t(\alpha-1)(\alpha+1)^2 : \alpha\Delta_2 : -t\alpha(\alpha-1)^2(\alpha+1)\right].$$

Since R_1, R_2 and P^0 are concurrent at p_0, dually we see that r_1, r_2 and p^0 are collinear on P_0.

The **altitude base points** t_1 and t_2 and the dual **altitude base lines** T_1, T_2 are defined by, and calculated as:

$$t_1 \equiv F_1R_1 = \left[(\alpha-1)\Delta_1 : 4t\alpha^2 : \alpha(\alpha+1)\Delta_1\right]$$
$$t_2 \equiv F_2R_2 = \left[(\alpha+1)\Delta_2 : 4t\alpha^2 : -\alpha(\alpha-1)\Delta_2\right]$$
$$T_1 \equiv t_1^\perp = f_1r_1 = \left\langle \alpha(\alpha-1)\Delta_1 : -4t\alpha\left(\alpha^2-1\right) : -(\alpha+1)\Delta_1\right\rangle$$
$$T_2 \equiv t_2^\perp = f_2r_2 = \left\langle \alpha(\alpha+1)\Delta_2 : -4t\alpha\left(\alpha^2-1\right) : (\alpha-1)\Delta_2\right\rangle.$$

The focal lines R_1 and R_2 also meet the directrices at the **second altitude base points** u_1, u_2, with dual lines U_1, U_2:

$$u_1 \equiv R_2F_1 = \left[(\alpha-1)\Delta_2 : 2t\alpha\left(\alpha^2-1\right) : \alpha(\alpha+1)\Delta_2\right]$$
$$u_2 \equiv R_1F_2 = \left[-(\alpha+1)\Delta_1 : 2t\alpha\left(\alpha^2-1\right) : \alpha(\alpha-1)\Delta_1\right]$$
$$U_1 \equiv u_1^\perp = \left\langle \alpha(\alpha+1)\Delta_1 : 2t\left(\alpha^2-1\right)^2 : (\alpha-1)\Delta_1\right\rangle$$
$$U_2 \equiv u_2^\perp = \left\langle -\alpha(\alpha-1)\Delta_2 : 2t\left(\alpha^2-1\right)^2 : (\alpha+1)\Delta_2\right\rangle.$$

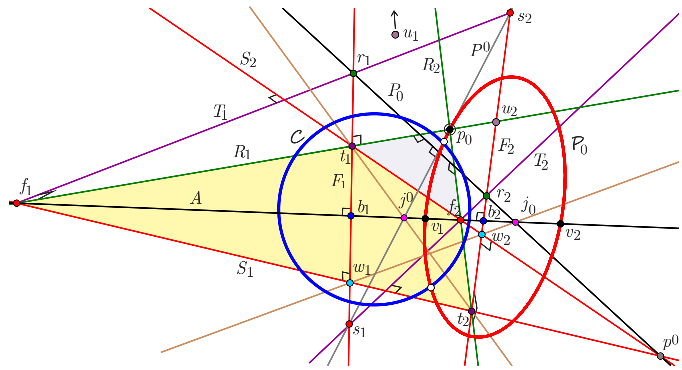

Figure 13: *The r, s, t and w points of p_0 on $\mathcal{P}_0$*

The **t-base lines** S_1, S_2 and their duals the **t-base points** s_1, s_2 are defined by, and calculated as:

$$S_1 \equiv f_1t_2 = \left\langle -2t\alpha^2(\alpha-1) : \left(\alpha^2-1\right)\Delta_2 : 2t\alpha(\alpha+1)\right\rangle$$
$$S_2 \equiv f_2t_1 = \left\langle 2t\alpha^2(\alpha+1) : -\left(\alpha^2-1\right)\Delta_1 : 2t\alpha(\alpha-1)\right\rangle$$
$$s_1 \equiv S_1^\perp = F_1T_2 = [2t(\alpha-1) : \Delta_2 : 2t\alpha(\alpha+1)]$$
$$s_2 \equiv S_2^\perp = F_2T_1 = [2t(\alpha+1) : \Delta_1 : -2t\alpha(\alpha-1)].$$

Theorem 19 (T-base) *Both s_1 and s_2 lie on the tangent P^0. Dually the lines S_1 and S_2 meet at p^0.*

Proof. We verify that s_1 and s_2 lie on the tangent $P^0 = \langle 1:-2t:t^2\rangle$ by computing

$$[2t(\alpha-1) : \Delta_2 : 2t\alpha(\alpha+1)]\left[1:-2t:t^2\right]^T = 0$$
$$[2t(\alpha+1) : \Delta_1 : -2t\alpha(\alpha-1)]\left[1:-2t:t^2\right]^T = 0.$$

The statement that S_1 and S_2 meet at p^0 follows from duality. □

The **w-points** w_1 and w_2, and their duals W_1 and W_2, are defined and computed as:

$$w_1 \equiv F_1S_1 = \left[\left(\alpha^2-1\right)(\alpha-1)\Delta_2 : -8t\alpha^3 : \alpha\left(\alpha^2-1\right)(\alpha+1)\Delta_2\right]$$
$$w_2 \equiv F_2S_2 = \left[\left(\alpha^2-1\right)(\alpha+1)\Delta_1 : 8t\alpha^3 : -\alpha\left(\alpha^2-1\right)(\alpha-1)\Delta_1\right]$$
$$W_1 \equiv f_1s_1 = \left(\alpha(\alpha-1)\Delta_2 : 8t\alpha^2 : -(\alpha+1)\Delta_2\right)$$
$$W_2 \equiv f_2s_2 = \left(\alpha(\alpha+1)\Delta_1 : -8t\alpha^2 : (\alpha-1)\Delta_1\right).$$

Theorem 20 (J points collinearities) *We have collinearities* $\left[\left[t_1 t_2 j^0\right]\right]$, $[[j_0 u_1 u_2]]$ *and* $[[w_1 w_2 j_0]]$.

Proof. Using the various formulas above, we compute

$$\det\begin{pmatrix}(\alpha-1)\Delta_1 & 4t\alpha^2 & \alpha(\alpha+1)\Delta_1\\ (\alpha+1)\Delta_2 & 4t\alpha^2 & -\alpha(\alpha-1)\Delta_2\\ -t^2 & 0 & 1\end{pmatrix}=0,$$

$$\det\begin{pmatrix}1 & 0 & t^2\alpha^2\\ -(\alpha+1)\Delta_1 & 2t\alpha\left(\alpha^2-1\right) & \alpha(\alpha-1)\Delta_1\\ (\alpha-1)\Delta_2 & 2t\alpha\left(\alpha^2-1\right) & \alpha(\alpha+1)\Delta_2\end{pmatrix}=0$$

and

$$\det\begin{pmatrix}\left(\alpha^2-1\right)(\alpha-1)\Delta_2 & -8t\alpha^3 & \alpha\left(\alpha^2-1\right)(\alpha+1)\Delta_2\\ \left(\alpha^2-1\right)(\alpha+1)\Delta_1 & 8t\alpha^3 & -\alpha\left(\alpha^2-1\right)(\alpha-1)\Delta_1\\ 1 & 0 & t^2\alpha^2\end{pmatrix}$$
$$=0.$$ □

Theorem 21 (Null focal line) *The focal line R_1 of a point p_0 on the parabola $\mathcal{P}_0$ is a null line precisely when $\Delta_3=0$. Similarly, the focal line R_2 is a null line precisely when $\Delta_4=0$.*

Proof. By the Null points/lines theorem, the focal line $R_1=\langle t\alpha(\alpha-1):\Delta_1:-t(\alpha+1)\rangle$ of $p_0=\left[t^2:t:1\right]$ is a null line precisely when

$$\langle t\alpha(\alpha-1):\Delta_1:-t(\alpha+1)\rangle\,\mathbf{D}\,\langle t\alpha(\alpha-1):\Delta_1:-t(\alpha+1)\rangle^T=0$$

or

$$-\alpha^2\left(\alpha+t^2\alpha^2-t^2\alpha+1\right)^2=0.$$

Since $\alpha\neq 0$, this is equivalent to $\Delta_3=0$. Similarly $R_2=\langle t\alpha(\alpha+1):-\Delta_2:t(\alpha-1)\rangle$ is a null line precisely when

$$-\alpha^2\left(-\alpha+t^2\alpha^2+t^2\alpha+1\right)^2=0$$

or $\Delta_4=0$. □

4 Parallels between the Euclidean and hyperbolic parabolas

4.1 Some congruent triangles

Recall that the focal line $R_1\equiv p_0 f_1$ meets the directrix F_1 in the point t_1. We will assume that the focal lines R_1 and R_2 are non-null line so that we have $\Delta_3\neq 0$ and $\Delta_4\neq 0$.

Theorem 22 (Congruent triangles) *Suppose that the tangent P^0 to $\mathcal{P}_0$ at p_0 meets $S_2=t_1 f_2$ at the point m^1. Then the triangles $\overline{p_0 t_1 m^1}$ and $\overline{p_0 f_2 m^1}$ are congruent. In particular i) $q(p_0,t_1)=q(p_0,f_2)$; ii) $q\left(t_1,m^1\right)=q\left(m^1,f_2\right)$; iii) $S_2\perp P^0$; iv) the tangent P^0 is a bisector of the vertex $\overline{R_1R_2}$; v) $S(S_2,R_1)=S(S_2,R_2)$; and vi) the tangent P^0 is a midline of the side $\overline{t_1 f_2}$. The same statements are true by f_1-f_2 symmetry if we interchange the indices 1 and 2.*

Proof. i) The first statement $q(p_0,t_1)=q(p_0,f_2)$ comes from the definition of the parabola $\mathcal{P}_0$, and we can also calculate quadrances to obtain

$$\begin{aligned}q(p_0,t_1)&=q\left(\left[t^2:t:1\right],\left[(\alpha-1)\Delta_1:4t\alpha^2:\alpha(\alpha+1)\Delta_1\right]\right)\\&=\frac{\Delta_4^2}{\Delta_4^2-\Delta_3^2}=q\left(\left[t^2:t:1\right],\left[1-\alpha:0:\alpha(\alpha+1)\right]\right)\\&=q(p_0,f_2).\end{aligned}$$

ii) Calculate

$$\begin{aligned}m^1&=P^0S_2\\&=\left\langle 1:-2t:t^2\right\rangle\times\left\langle 2t\alpha^2(\alpha+1):-\left(\alpha^2-1\right)\Delta_1:2t\alpha(\alpha-1)\right\rangle\\&=\left[t^2(\alpha-1)^2\Delta_4:-2t\alpha\Delta_4:-(\alpha+1)^2\Delta_4\right]\\&=\left[-t^2(\alpha-1)^2:2t\alpha:(\alpha+1)^2\right].\end{aligned}$$

Here we have used the fact that the focal line R_2 is non-null so that Δ_4 is nonzero. Thus

$$\begin{aligned}&q\left(t_1,m^1\right)=\\&q\left(\left[(\alpha-1)\Delta_1:4t\alpha^2:\alpha(\alpha+1)\Delta_1\right],\left[-t^2(\alpha-1)^2:2t\alpha:(\alpha+1)^2\right]\right)\\&=-\frac{1}{4}\frac{\left(\alpha^2-1\right)\Delta_4}{\alpha\Delta_3}\\&=q\left(\left[-t^2(\alpha-1)^2:2t\alpha:(\alpha+1)^2\right],\left[1-\alpha:0:\alpha(\alpha+1)\right]\right)\\&=q\left(m^1,f_2\right).\end{aligned}$$

iii) Since the tangent line P^0 passes through s_2, which is the dual of the line $S_2=t_1 f_2$, the tangent P^0 is perpendicular to the line S_2; and we can also check that

$$\left\langle 1:-2t:t^2\right\rangle\mathbf{D}\left\langle 2t\alpha^2(\alpha+1):\left(\alpha^2-1\right)\Delta_1:2t\alpha(\alpha-1)\right\rangle^T=0.$$

iv) The tangent P^0 is a bisector of the vertex $\overline{R_1R_2}$ since

$$\begin{aligned}S\left(R_1,P^0\right)&=S\left(\langle t\alpha(\alpha-1):\Delta_1:-t(\alpha+1)\rangle,\left\langle 1:-2t:t^2\right\rangle\right)\\&=\frac{\left(\alpha^2-1\right)\left(\Delta_3^2-\Delta_4^2\right)}{4\alpha\Delta_4\Delta_3}\\&=S\left(\langle t\alpha(\alpha+1):-\Delta_2:t(\alpha-1)\rangle,\left\langle 1:-2t:t^2\right\rangle\right)\\&=S\left(R_2,P^0\right).\end{aligned}$$

v) Now calculate the spreads

$$\begin{aligned}S(S_2,R_1)&=S\left(\left\langle 2t\alpha^2(\alpha+1):-\left(\alpha^2-1\right)\Delta_1:2t\alpha(\alpha-1)\right\rangle,\right.\\&\qquad\left.\langle t\alpha(\alpha-1):\Delta_1:-t(\alpha+1)\rangle\right)\\&=\frac{4t^2\alpha\left(\alpha^2+1\right)^2}{16t^2\alpha^3-\Delta_1^2\left(\alpha^2-1\right)}\\&=S\left(\left\langle 2t\alpha^2(\alpha+1):-\left(\alpha^2-1\right)\Delta_1:2t\alpha(\alpha-1)\right\rangle,\right.\\&\qquad\left.\langle t\alpha(\alpha+1):-\Delta_2:t(\alpha-1)\rangle\right)\\&=S(S_2,R_2).\end{aligned}$$

vi) It is obvious that the tangent P^0 is a midline of the side $\overline{t_1 f_2}$, since P^0 is perpendicular to the line $S_2 = t_1 f_2$ through the point m^1 which is, from ii), a midpoint of $\overline{t_1 f_2}$. The symmetry between f_1 and f_2 in the definition of the parabola $\mathcal{P}_0$ ensures that all these statements hold also if we interchange the indices $1, 2$. $\square$

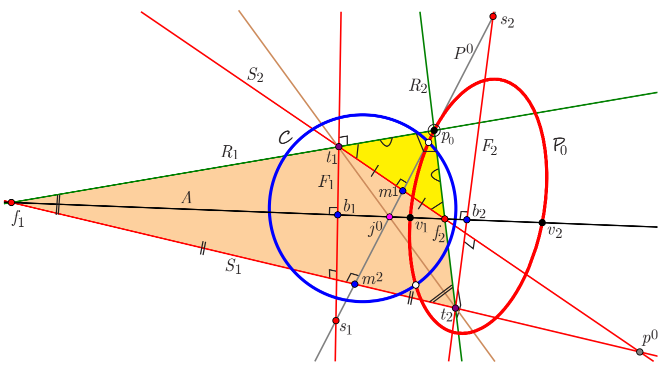

Figure 14: *Two pairs of congruent triangles*

In Figure14 we see also the point $m^2 = P_0 S_1$ and the congruent triangles $\overline{p_0 t_2 m^2}$ and $\overline{p_0 f_1 m^2}$. We call m^1 and m^2 the **t-perpendicular points** of p_0. Note that the theorem allows us a simple construction of the tangent P^0 at p_0: drop the perpendicular to the line $t_1 f_2$.

Corollary 1 *We have i) the triangles $\overline{m^1 t_1 j^0}$ and $\overline{m^1 f_2 j^0}$ are congruent, and ii) the triangles $\overline{p_0 t_1 j^0}$ and $\overline{p_0 f_2 j^0}$ are congruent. The same statements are true by $f_1 - f_2$ symmetry if we interchange the indices 1 and 2.*

Proof. The triangles $\overline{m^1 f_2 j^0}$ and $\overline{m^1 t_1 j^0}$ are right triangles since P^0 is perpendicular to S_2; we also have $q(t_1, m^1) = q(m^1, f_2)$ and $\overline{m^1 j^0}$ is a common side.
i) We calculate the quadrances

$$\begin{aligned} q\left(t_1, j^0\right) &= q\left(\left[(\alpha-1)\Delta_1 : 4t\alpha^2 : \alpha(\alpha+1)\Delta_1\right], \left[-t^2 : 0 : 1\right]\right) \\ &= \frac{\Delta_4^2}{\Delta_4^2 - \Delta_1^2} \\ &= q\left(\left[1-\alpha : 0 : \alpha(\alpha+1)\right], \left[-t^2 : 0 : 1\right]\right) = q\left(j^0, f_2\right). \end{aligned}$$

and spreads

$$\begin{aligned} S\left(t_1 m^1, t_1 j^0\right) &= \frac{q\left(m^1, j^0\right)}{q\left(t_1, j^0\right)} = \frac{16t^2\alpha^3}{16t^2\alpha^3 - \Delta_1^2\left(\alpha^2-1\right)} \\ &= \frac{q\left(m^1, j^0\right)}{q\left(j^0, f_2\right)} = S\left(f_2 j^0, f_2 m^1\right). \end{aligned}$$

$$\begin{aligned} S\left(j^0 m^1, j^0 t_1\right) &= \frac{q\left(m^1, t_1\right)}{q\left(t_1, j^0\right)} = \frac{\left(\alpha^2-1\right)\Delta_1^2}{16t^2\alpha^3 - \Delta_1^2\left(\alpha^2-1\right)} \\ &= \frac{q\left(m^1, f_2\right)}{q\left(j^0, f_2\right)} = S\left(j^0 m^1, j^0 f_2\right). \end{aligned}$$

Therefore, the triangles $\overline{m^1 t_1 j^0}$ and $\overline{m^1 f_2 j^0}$ are congruent.
ii) The triangles $\overline{p_0 f_2 j^0}$ and $\overline{p_0 t_1 j^0}$ have one common side $\overline{p_0 j^0}$. Using the Spread law and the congruences above,

$$\begin{aligned} S\left(t_1 p_0, t_1 j^0\right) &= \frac{S\left(p_0 t_1, p_0 j^0\right) q\left(p_0, j^0\right)}{q\left(t_1, j^0\right)} \\ &= \frac{S\left(p_0 f_2, p_0 j^0\right) q\left(p_0, j^0\right)}{q\left(f_2, j^0\right)} = \frac{\Delta_1^2 - \Delta_3^2}{\Delta_4^2} \\ &= S\left(f_2 p_0, f_2 j^0\right). \end{aligned}$$

Therefore, the triangles $\overline{p_0 f_2 j^0}$ and $\overline{p_0 t_1 j^0}$ are congruent. $\square$

Theorem 23 (Tangent base symmetry) *Let $j^0 \equiv AP^0$ be the meet of the axis A and the tangent P^0, and h_0 the base of the altitude from p_0 to A. Then i) $q\left(b_1, j^0\right) = q\left(f_2, h_0\right)$ and ii) $q\left(v_1, j^0\right) = q\left(v_1, h_0\right)$. The same statements are true if we interchange the indices 1 and 2 by $f_1 - f_2$ symmetry.*

Proof. i) We calculate the quadrances

$$\begin{aligned} q\left(b_1, j^0\right) &= q\left(\left[\alpha(\alpha-1) : 0 : \alpha^2(\alpha+1)\right], \left[-t^2 : 0 : 1\right]\right) \\ &= \frac{\Delta_2^2}{\Delta_2^2 - \Delta_3^2} = q\left(\left[1-\alpha : 0 : \alpha(\alpha+1)\right], \left[t^2 : 0 : 1\right]\right) \\ &= q\left(f_2, h_0\right). \end{aligned}$$

ii) Similarly, we calculate the quadrances

$$\begin{aligned} q\left(v_1, j^0\right) &= q\left(\left[0:0:1\right], \left[-t^2 : 0 : 1\right]\right) = \frac{t^4\alpha^2}{t^4\alpha^2 - 1} \\ &= q\left(\left[0:0:1\right], \left[t^2 : 0 : 1\right]\right) = q\left(v_1, h_0\right). \end{aligned}$$ $\square$

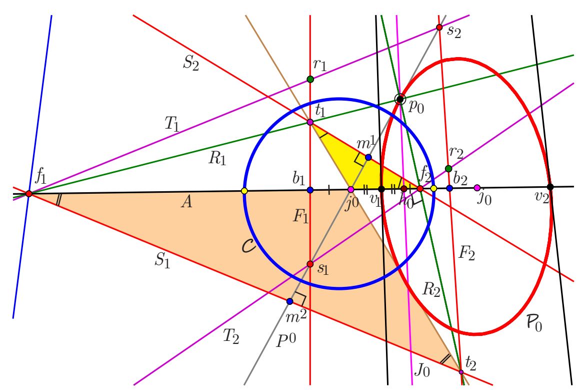

Figure 15: *The j^0 and h_0 points*

Theorem 24 (Two chord midpoints) *Let $p_0 \equiv p(t)$, $q_0 \equiv p(u)$ be two points on a hyperbolic parabola $\mathcal{P}_0$, with $\overline{p_0}$ the opposite point of p_0 with respect to the axis A. Suppose that the chords $\overline{p_0 q_0}$ and $\overline{q_0 \overline{p_0}}$ meet A at x and y respectively. Then the vertices v_1, v_2 of $\mathcal{P}_0$ are the midpoints of the side $\overline{xy}$.*

Proof. Suppose $p_0 = \left[t^2 : t : 1\right]$ and $q_0 = \left[u^2 : u : 1\right]$. The line $p_0q_0 = \langle 1 : -(t+u) : tu\rangle$ meets the axis $A = [0:1:0]$ at $x = [-tu : 0 : 1]$. The chord $\overline{p_0}q_0 = \langle 1 : t-u : -tu\rangle$ intersects $A = \langle 0:1:0\rangle$ at $y = [tu : 0 : 1]$. Thus

$$\begin{aligned} q(v_1, x) &= q([0:0:1], [-tu:0:1]) = \frac{\alpha^2 t^2 u^2}{(t^2u^2\alpha^2 - 1)} \\ &= q([0:0:1], [tu:0:1]) = q(v_1, y) \end{aligned}$$

which shows v_1 is a midpoint of the side $\overline{xy}$. The other midpoint will be perpendicular to v_1, which must be v_2 without calculation. □

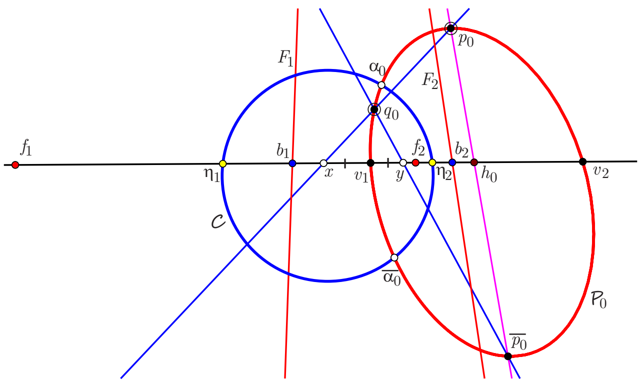

Figure 16: *Two chord midpoints*

4.2 The optical property

Recall the famous *optical property* of a parabola $\mathcal{P}$ in Euclidean geometry: if P is a point lying on $\mathcal{P}$, and light emanates from the focus F heading towards the point P, then the light will be reflected to be parallel to the axis. An analogous result in the hyperbolic case is the statement iv) of the Congruent triangles theorem: that the tangent line P_0 to a point p_0 is a biline (bisector) of the vertex $\overline{R_1R_2}$. So reflecting the focal line $R_1 \equiv f_1p_0$ in the tangent P^0 results in the other focal line R_2, which is perpendicular to the directrix F_2.

Recall from [16] that in Universal Hyperbolic Geometry there is an important notion of parallelism, which is quite different from the usage in classical hyperbolic geometry. We say rather generally that the **parallel line** P **through a point** a **to a line** L is the line through a perpendicular to the altitude from a to L.

Now recall that given a point p_0 on the hyperbolic parabola $\mathcal{P}_0$, the perpendicular to the axis A through p_0 is $J_0 \equiv (j_0)^{\perp} = ap_0 = \left\langle 1:0:-t^2\right\rangle$ with dual point $j_0 = P_0A = \left[1:0:t^2\alpha^2\right]$. Therefore, the parallel line to the axis A through the point p_0 is

$$L_0 = j_0p_0 = \left\langle -t^3\alpha^2 : t^4\alpha^2 - 1 : t\right\rangle.$$

Here then is another analog of the optical property, dealing with the relationship between two spreads formed by the tangent line P_0. Recall that the quadrance of the parabola was defined as $q_{\mathcal{P}_0} \equiv q(f_1, f_2)$.

Theorem 25 (Parallel line spread relation) *Let p_0 be a point on the hyperbolic parabola $\mathcal{P}_0$. If $\widehat{T}$ is the spread between the tangent line P^0 at p_0 and the parallel line L_0 to the axis through p_0, and $\widehat{S}$ is the common spread between the tangent P^0 and the lines R_1 and R_2, then*

$$\frac{(\widehat{S} - \widehat{T})\widehat{S}}{1 - \widehat{T}} = 1 - q_{\mathcal{P}_0}.$$

Proof. Using the Spread formula, we compute that

$$\widehat{S} = S\left(R_1, P^0\right) = \frac{\left(\alpha^2 - 1\right)\left(\Delta_3^2 - \Delta_4^2\right)}{4\alpha\Delta_4\Delta_3}$$

and

$$\widehat{T} = S\left(L_0, P^0\right) = \frac{-\left(\alpha^2 - 1\right)\left(t^4\alpha^2 + 1\right)^2}{\left(t^4\alpha^2 - 1\right)\Delta_3\Delta_4}.$$

So now

$$\frac{(\widehat{S} - \widehat{T})\widehat{S}}{1 - \widehat{T}} = -\frac{1}{4}\frac{\left(\alpha^2 - 1\right)^2}{\alpha^2} = 1 - q_{\mathcal{P}_0}.$$

□

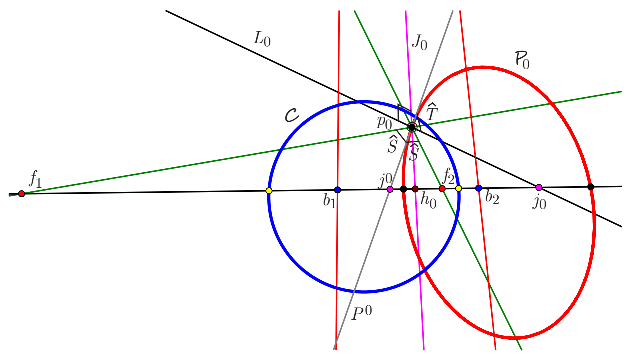

Figure 17: *The parallel line spread relation*

Note that $1 - q_{\mathcal{P}_0} = q(b_1, f_2)$ since b_1 and f_1 are perpendicular points. So in the limiting Euclidean case when b_1 is very close to f_2, it follows that $\widehat{S}$ is very close to $\widehat{T}$.

4.3 The s points and $\mathcal{S}$ circles

Recall that $s_1 \equiv F_1P^0$ and $s_2 \equiv F_2P^0$.

Theorem 26 (The $\mathcal{S}_1$ and $\mathcal{S}_2$ circles) *The circle $\mathcal{S}_1$ with center s_1 passing through f_2 also passes through t_1, and is tangent at these points to R_2 and R_1 respectively. In particular i) $q(s_1, t_1) = q(s_1, f_2)$; ii) $R_1 \perp F_1$; iii) $R_2 \perp T_2$ and iv) $S(s_1t_1, t_1f_2) = S(s_1f_2, f_2t_1)$. The same statements are true if we interchange the indices 1 and 2, giving also a circle $\mathcal{S}_2$ with center s_2.*

Proof. i) Calculate

$$\begin{aligned} q(s_1,t_1) &= q\big([2t(\alpha-1):\Delta_2:2t\alpha(\alpha+1)],\\ &\qquad [(\alpha-1)\Delta_1:4t\alpha^2:\alpha(\alpha+1)\Delta_1]\big)\\ &= \frac{(\alpha^2-1)\Delta_4^2}{16t^2\alpha^3+\Delta_2^2(\alpha^2-1)} = q(s_1,f_2). \end{aligned}$$

ii) The line $R_1 = f_1p_0$ is clearly perpendicular to the directrix F_1 since it passes through the focus $f_1 = F_1^{\perp}$.
iii) Since $t_2 = R_2F_2$, $S_1 = f_1t_2$, the lines R_2, F_2 and S_1 are concurrent at t_2, so the line $T_2 = t_2^{\perp}$ passes through r_2, f_2 and s_1. Therefore T_2 is perpendicular to the line R_2.
iv) Calculate

$$\begin{aligned} S(t_1s_1,t_1f_2) &= S(\langle\alpha(\alpha+1):0:1-\alpha\rangle,\\ &\qquad \langle 2t\alpha^2(\alpha+1):-(\alpha^2-1)\Delta_1:2t\alpha(\alpha-1)\rangle)\\ &= \frac{(\alpha^2-1)\Delta_3^2}{16t^2\alpha^3-\Delta_1^2(\alpha^2-1)}\\ &= S(f_2s_1,f_2t_1). \end{aligned}$$

□

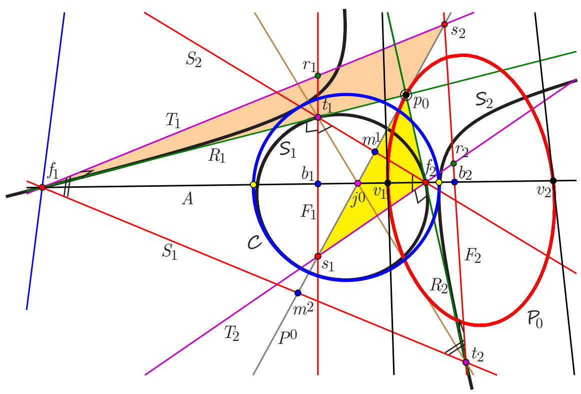

Figure 18: *The $\mathcal{S}_1$ and $\mathcal{S}_2$ circles*

In particular, property iii) provides us with an important alternate construction of the tangent P^0 to the parabola $\mathcal{P}_0$ at p_0 : namely we construct the altitude T_2 to p_0f_2 through f_2, and obtain $s_1 = F_1T_2$, giving $P^0 = p_0s_1$ (or similarly construct p_0s_2). In Figure 18 we see the circles $\mathcal{S}_1$ and $\mathcal{S}_2$. Note that $\mathcal{S}_2$ looks like a hyperbola tangent to the null circle, in fact it is tangent at exactly the points where S_2 meets the null circle $\mathcal{C}$ – see the discussion in [18].

4.4 Focal chords and conjugates

A chord $\overline{p_0q_0}$ is a **focal chord** precisely when p_0q_0 passes through a focus. Such chords play an important role both in the Euclidean and the hyperbolic theory.

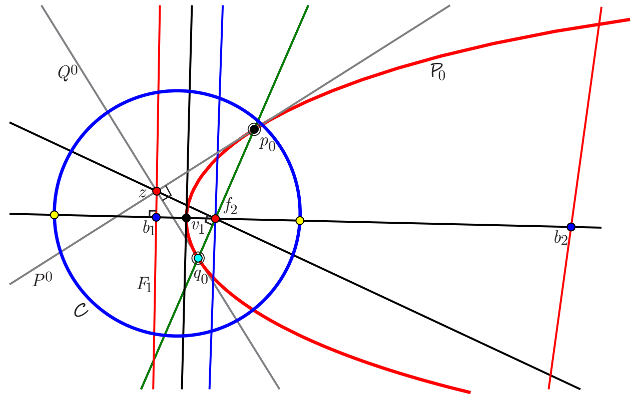

Figure 19: *A focal chord $\overline{p_0q_0}$ with polar point z on directrix*

Theorem 27 (Focal tangents perpendicularity) *If $p_0 \equiv p(t)$ and $q_0 \equiv p(u)$ are two points on $\mathcal{P}_0$ then $\overline{p_0q_0}$ is a focal chord precisely when the respective tangents P^0 and Q^0 are perpendicular; and precisely when the polar point $z \equiv P^0Q^0$ lies on a directrix.*

Proof. Suppose $p_0 = [t^2:t:1]$ and $q_0 = [u^2:u:1]$ lie on $\mathcal{P}_0$. Then $p_0q_0 = \langle 1:-(t+u):tu\rangle$ is a focal line precisely when it passes through either f_1 of f_2, in other words precisely when

$$\begin{aligned} &(1:-(t+u):tu)[\alpha+1:0:\alpha(\alpha-1)]^T\\ &\quad = \alpha+1+tu\alpha(\alpha-1) = 0 \qquad \text{or}\\ &(1:-(t+u):tu)[1-\alpha:0:\alpha(\alpha+1)]^T\\ &\quad = -\alpha+1+tu\alpha(\alpha+1) = 0. \end{aligned}$$

On the other hand the tangents $P^0 = \langle 1:-2t:t^2\rangle$ and $Q^0 = \langle 1:-2u:u^2\rangle$ are perpendicular precisely when

$$\begin{aligned} 0 &= \langle 1:-2t:t^2\rangle \mathbf{D} \langle 1:-2u:u^2\rangle^T\\ &= \alpha^2-4tu\alpha^2-t^2u^2\alpha^2(\alpha^2-1)-1\\ &= (\alpha+1+tu\alpha(\alpha-1))(\alpha-1-tu\alpha(\alpha+1)). \end{aligned}$$

Thus the two conditions are equivalent.
As in the Tangent meets theorem, the tangents P^0 and Q^0 meet at $z = [2tu:t+u:2]$. This point lies on $F_1 = \langle\alpha(\alpha+1):0:1-\alpha\rangle$ or $F_2 = \langle\alpha(\alpha-1):0:\alpha+1\rangle$ precisely when

$$\begin{aligned} &[2tu:t+u:2](\alpha(\alpha+1):0:1-\alpha)^T\\ &\quad = 2(-\alpha+tu\alpha+tu\alpha^2+1) = 0 \qquad \text{or}\\ &[2tu:t+u:2](\alpha(\alpha-1):0:\alpha+1)^T\\ &\quad = 2(\alpha-tu\alpha+tu\alpha^2+1) = 0. \end{aligned}$$

Since we work over a field not of characteristic two, the conditions are equivalent to the previous ones. □

Given a point p_0 on the parabola $\mathcal{P}_0$, we define the **conjugate points** n_1, n_2 as the second meets of the focal lines R_1 and R_2 with the parabola $\mathcal{P}_0$ respectively. Since one meet is known, solving the quadratic equations is straightforward and yields

$$n_1 = \left[(\alpha+1)^2 : t\alpha\left(1-\alpha^2\right) : t^2\alpha^2(\alpha-1)^2\right]$$
$$n_2 = \left[(\alpha-1)^2 : t\alpha\left(\alpha^2-1\right) : t^2\alpha^2(\alpha+1)^2\right]. \qquad (14)$$

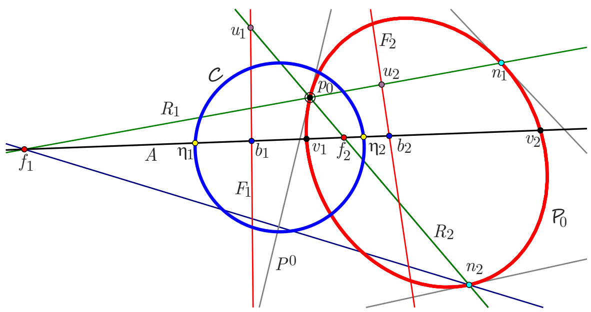

Figure 20: *Focal conjugates n_1 and n_2*

The dual lines are the **conjugate lines;**

$$N_1 \equiv n_1^{\perp} = \left\langle \alpha(\alpha+1)^2 : t\left(\alpha^2-1\right)^2 : -t^2\alpha(\alpha-1)^2\right\rangle$$
$$N_2 \equiv n_2^{\perp} = \left\langle -\alpha(\alpha-1)^2 : t\left(\alpha^2-1\right)^2 : t^2\alpha(\alpha+1)^2\right\rangle.$$

Theorem 28 (Conjugate points parameter) *Let $p_0 \equiv p(t)$ be a point on the parabola $\mathcal{P}_0$, then the point $p(u)$ is the conjugate point n_1 of p_0 with respect to the focus f_1 precisely when $u = -\frac{\alpha+1}{\alpha t(\alpha-1)}$, while $p(u)$ is the conjugate point n_2 of p_0 with respect to the focus f_2 precisely when $u = \frac{\alpha-1}{\alpha t(\alpha+1)}$.*

Proof. Let $p_0 = \left[t^2 : t : 1\right]$ and $p(u) = \left[u^2 : u : 1\right]$ lie on $\mathcal{P}_0$. Then, the line $p_0q_0 = \langle 1 : -(t+u) : tu\rangle$ is a focal line with respect to the focus f_1 when it passes through the focus f_1 and then we have

$$[1 : -(t+u) : tu]\,[\alpha+1 : 0 : \alpha(\alpha-1)]^T = 0 \qquad \text{so that}$$
$$\alpha - tu\alpha + tu\alpha^2 + 1 = 0.$$

This gives the condition $u = -\frac{\alpha+1}{\alpha t(\alpha-1)}$. Similarly, the other direction is straightforward.
When the line $p_0q_0 = \langle 1 : -(t+u) : tu\rangle$ is a focal line with respect to the focus f_2, then the focal line passes through the focus f_2 and we have

$$[1 : -(t+u) : tu]\,[1-\alpha : 0 : \alpha(\alpha+1)]^T = 0 \qquad \text{so that}$$
$$-\alpha + tu\alpha + tu\alpha^2 + 1 = 0.$$

This gives the condition $u = \frac{\alpha-1}{\alpha t(\alpha+1)}$. Similarly, the other direction is straightforward. □

4.5 Quadrance cross ratios

Theorem 29 (Quadrance cross ratio) *Suppose that a, b, c, d are a harmonic range of points on a line L in UHG. Then*

$$\frac{q(a,c)}{q(a,d)} = \frac{q(b,c)}{q(b,d)}.$$

Proof. We know from projective geometry that a harmonic range of points a, b, c, d in the projective space can be realized as $[v], [u], [\alpha v + \beta u], [\alpha v - \beta u]$ for two vectors v and u and two scalars α and β. Then using the short hand notation $v^2 \equiv v \cdot v$ and $uv = u \cdot v$, we calculate that

$$\begin{aligned} q([v],[\alpha v+\beta u]) &= 1 - \frac{(v\cdot(\alpha v+\beta u))^2}{(v\cdot v)\left((\alpha v+\beta u)\cdot(\alpha v+\beta u)\right)} \\ &= \frac{v^2\left(\alpha^2v^2+2\alpha\beta(uv)+\beta^2u^2\right)-\left(\alpha v^2+\beta uv\right)^2}{v^2\left(\alpha^2v^2+2\alpha\beta(uv)+\beta^2u^2\right)} \\ &= \frac{\beta^2\left(u^2v^2-(uv)^2\right)}{v^2\left(\alpha^2v^2+2\alpha\beta(uv)+\beta^2u^2\right)} \end{aligned}$$

and similarly

$$\begin{aligned} q([u],[\alpha v+\beta u]) &= 1 - \frac{(u\cdot(\alpha v+\beta u))^2}{(u\cdot u)\left((\alpha v+\beta u)\cdot(\alpha v+\beta u)\right)} \\ &= \frac{u^2\left(\alpha^2v^2+2\alpha\beta(uv)+\beta^2u^2\right)-\left(\alpha(uv)+\beta u^2\right)^2}{u^2\left(\alpha^2v^2+2\alpha\beta(uv)+\beta^2u^2\right)} \\ &= \frac{\alpha^2\left(u^2v^2-(uv)^2\right)}{u^2\left(\alpha^2v^2+2\alpha\beta(uv)+\beta^2u^2\right)}. \end{aligned}$$

It follows that

$$\frac{q(a,c)}{q(b,c)} = \frac{q([v],[\alpha v+\beta u])}{q([u],[\alpha v+\beta u])} = \frac{\beta^2u^2}{\alpha^2v^2}.$$

But this quantity is then unchanged if we replaced α with $-\alpha$, or β with $-\beta$. □

Theorem 30 (Conjugate cross ratios) *Let p_0 be a point on the parabola $\mathcal{P}_0$, with n_1 and n_2 the focal conjugates and u_1 and u_2 the meets of R_2 and R_1 with the directrices F_1 and F_2 respectively. Then*

$$\frac{q(p_0,f_1)}{q(f_1,n_1)} = \frac{q(p_0,u_2)}{q(u_2,n_1)} \quad \text{and} \quad \frac{q(p_0,f_2)}{q(f_2,n_2)} = \frac{q(p_0,u_1)}{q(u_1,n_2)}.$$

Proof. From the Focus directrix polarity theorem, we know that f_2 and F_1 are a pole-polar pair with respect to the parabola $\mathcal{P}_0$. Hence $f_1, u_2; p_0, n_1$ is a harmonic range. From the previous theorem, that implies that

$$\frac{q(p_0,f_1)}{q(f_1,n_1)} = \frac{q(p_0,u_2)}{q(u_2,n_1)}.$$

The other relation follows similarly since $f_2, u_1; p_0, n_2$ is also a harmonic range of points. □

4.6 Spreads related to chords of a parabola

Theorem 31 (Polar point spreads) *If the tangents P^0 and Q^0 at the points $p_0 \equiv p(t)$ and $q_0 \equiv p(u)$ lying on the parabola $\mathcal{P}_0$ meet at the polar point z, then $S(f_1p_0, f_1z) = S(f_1q_0, f_1z)$ and $S(f_2p_0, f_2z) = S(f_2q_0, f_2z)$.*

Proof. Suppose that $p_0 \equiv [t^2 : t : 1]$ and $q_0 \equiv [u^2 : u : 1]$ are on the parabola $\mathcal{P}_0$. Then $z = [2tu : t+u : 2]$ and we calculate

$$S(f_1p_0, f_1z) = \frac{\alpha(t-u)^2(\alpha^2-1)}{(\alpha+\alpha^2t^2-\alpha t^2+1)(\alpha+\alpha^2u^2-\alpha u^2+1)} = \frac{\alpha(t-u)^2(\alpha^2-1)}{\Delta_3(t)\Delta_3(u)} = S(f_1q_0, f_1z)$$

and

$$S(f_2p_0, f_2z) = \frac{\alpha(\alpha^2-1)(t-u)^2}{(\alpha-u^2\alpha^2-u^2\alpha-1)(-\alpha+t^2\alpha^2+t^2\alpha+1)} = \frac{-\alpha(\alpha^2-1)(t-u)^2}{\Delta_4(t)\Delta_4(u)} = S(f_2q_0, f_2z).$$

□

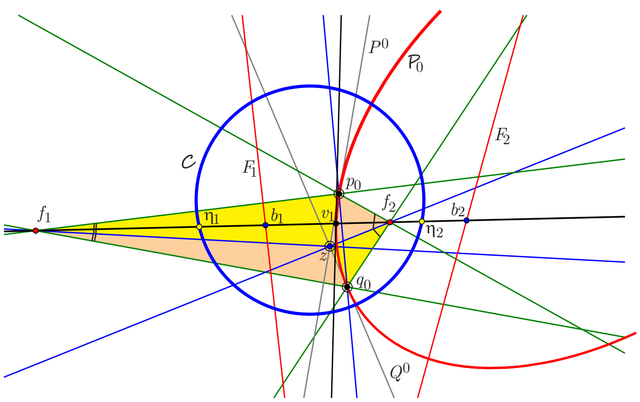

Figure 21: *The polar point z of the chord $\overline{p_0q_0}$*

Theorem 32 (Chord directrix meets) *Let $p_0 \equiv p(t)$ and $q_0 \equiv p(u)$ be two points on a parabola $\mathcal{P}_0$. Let z be the polar point of the chord $\overline{p_0q_0}$, and $x_1 \equiv F_1(p_0q_0)$ and $x_2 \equiv F_2(p_0q_0)$. Then i) $f_1z \perp f_1x_2$, ii) $f_2z \perp f_2x_1$ and iii) $S(x_1z, zf_2) = S(x_2z, zf_1)$.*

Proof. We suppose as usual that $p_0 = [t^2 : t : 1]$ and $q_0 = [u^2 : u : 1]$. Then
i) We compute that

$$x_2 \equiv F_2(p_0q_0) = \langle \alpha^2(\alpha-1) : 0 : \alpha(1+\alpha) \rangle \times \langle 1 : -(t+u) : tu \rangle = [(\alpha+1)(t+u) : \alpha+tu\alpha-tu\alpha^2+1 : -\alpha(\alpha-1)(t+u)].$$

Also

$$f_1z = \langle -\alpha(\alpha-1)(t+v) : 2(-\alpha-tv\alpha+tv\alpha^2-1) : (\alpha+1)(t+v) \rangle$$
$$f_1x_2 = \langle \alpha(\alpha-1)(-\alpha-tv\alpha+tv\alpha^2-1) : 2\alpha(\alpha^2-1)(t+v) : (\alpha+1)(\alpha+tv\alpha-tv\alpha^2+1) \rangle$$

and so we may verify that

$$0 = \langle -\alpha(\alpha-1)(t+v) : 2(-\alpha-tv\alpha+tv\alpha^2-1) : (\alpha+1)(t+v) \rangle D \times \langle \alpha(\alpha-1)(-\alpha-tv\alpha+tv\alpha^2-1) : 2\alpha(\alpha^2-1)(t+v) : (\alpha+1)(\alpha+tv\alpha-tv\alpha^2+1) \rangle^T.$$

Thus

$$S(f_1z, f_1x_2) = 1.$$

ii) Similarly

$$x_1 \equiv F_1(p_0q_0) = \langle \alpha(\alpha+1) : 0 : 1-\alpha \rangle \times \langle 1 : -(t+u) : tu \rangle = [(\alpha-1)(t+u) : \alpha+tu\alpha+tu\alpha^2-1 : \alpha(\alpha+1)(t+u)]$$

and the lines

$$f_2z = \langle -\alpha(\alpha+1)(t+u) : 2(\alpha+tu\alpha+tu\alpha^2-1) : -(\alpha-1)(t+u) \rangle$$
$$f_2x_1 = \langle \alpha(\alpha+1)(\alpha+tu\alpha+tu\alpha^2-1) : -2\alpha(\alpha^2-1)(t+u) : (\alpha-1)(\alpha+tu\alpha+tu\alpha^2-1) \rangle$$

are perpendicular, so that

$$S(f_2z, f_2x_1) = 1.$$

iii) Another calculation shows that

$$S(x_1z, zf_2) = \frac{1}{4}\frac{(\alpha^2-1)\left((2tu)^2-(t+u)^2\right)\alpha^2+(t+u)^2-4}{(t^2u^2)\alpha^4+\left((t+u)^2-(t^2u^2+1)\right)\alpha^2+1} = S(x_2z, zf_1).$$

□

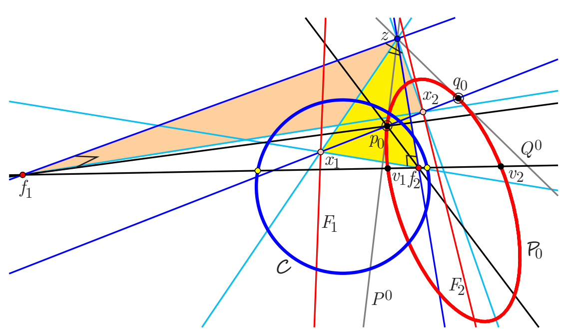

Figure 22: *Chord directrix meets x_1 and x_2*

In Figure 22 we see the two triangles $\overline{f_1zx_2}$ and $\overline{f_2zx_1}$, which are both right triangles sharing a common spread.

Theorem 33 (Tangent directrix meets) *If the two tangents P^0 and Q^0 to a parabola $\mathcal{P}_0$ at $p_0 \equiv p(t)$ and $q_0 \equiv p(u)$ respectively meet the directrix F_1 at s_1 and s_1' respectively, and meet F_2 at s_2 and s_2' respectively, then $S(f_1p_0, f_1q_0) = S(f_1s_2, f_1s_2')$ and $S(f_2p_0, f_2q_0) = S(f_2s_1, f_2s_1')$.*

Proof. Suppose that $p_0 \equiv [t^2 : t : 1]$ and $q_0 \equiv [u^2 : u : 1]$ are on the parabola $\mathcal{P}_0$. Then

$$S(f_1p_0, f_1q_0) = \frac{4\alpha(\alpha^2-1)(t-u)^2(\alpha+\alpha^2tu-\alpha tu+1)^2}{\Delta_3^2(t)\Delta_3^2(u)} = S(f_1s_2, f_1s_2').$$

Also, we have that

$$S(f_2p_0, f_2q_0) = \frac{-4\alpha(\alpha^2-1)(t-u)^2(-\alpha+tu\alpha+tu\alpha^2+1)^2}{\Delta_4^2(t)\Delta_4^2(u)} = S(f_2s_1, f_2s_1').$$

□

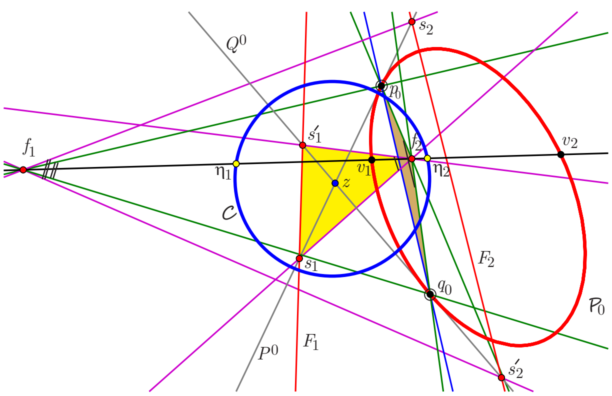

Figure 23: *Tangent directrix meets s_1 and s_2*

Recall that in universal hyperbolic geometry, a triangle may have four circumcircles.

Theorem 34 (Two tangents circumcircle) *Suppose that the two points $p_0 \equiv p(t)$ and $q_0 \equiv p(u)$ on a parabola $\mathcal{P}_0$ have respective altitude base points t_1, t_2 and t_1', t_2' on F_1, F_2 respectively, and that their tangents meet at the polar point z. Then z is a circumcenter of both the triangles $\overline{t_1 f_2 t_1'}$ and $\overline{t_2 f_1 t_2'}$. In particular $q(t_1, z) = q(t_1', z) = q(z, f_2)$ and $q(t_2, z) = q(t_2', z) = q(z, f_1)$.*

Proof. Suppose that $p_0 \equiv [t^2 : t : 1]$ and $q_0 \equiv [u^2 : u : 1]$ are on the parabola $\mathcal{P}_0$, then,

$$\begin{aligned} q(z, f_2) &= q([2tu : t+u : 2], [1-\alpha : 0 : \alpha(\alpha+1)]) \\ &= \frac{\Delta_4(t)\Delta_4(u)}{\alpha\left(\left(4t^2u^2-(t+u)^2\right)\alpha^2+(t+u)^2-4\right)} \\ &= q(z, t_1) = q(z, t_1'). \end{aligned}$$

Hence z is a circumcenter of the triangle $\overline{t_1 f_2 t_1'}$. Similarly, z is the circumcenter of the triangle $\overline{t_2 f_1 t_2'}$ since

$$\begin{aligned} q(z, f_1) &= q([2tu : t+u : 2], [\alpha+1 : 0 : \alpha(\alpha-1)]) \\ &= -\frac{\Delta_3(t)\Delta_3(u)}{\alpha\left(\left(4t^2u^2-(t+u)^2\right)\alpha^2+(t+u)^2-4\right)} \\ &= q(z, t_2) = q(z, t_2'). \end{aligned}$$

□

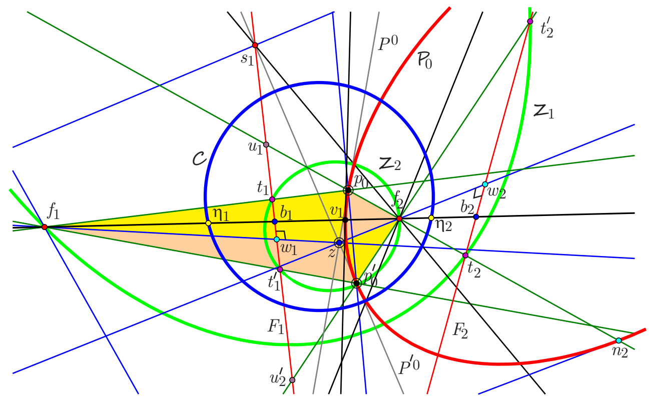

Figure 24: *Two points and polar circles*

In Figure 24 we see the polar point of $\overline{p_0p_0'}$ together with the two **polar circles** centered at z through the foci.

Corollary 2 *If the tangents at $p_0 \equiv p(t)$ and $q_0 \equiv p(u)$ on $\mathcal{P}_0$ meet at z then the line f_1z is a midline of the side $\overline{t_1t_1'}$ and similarly f_2z is a midline of the side $\overline{t_2t_2'}$.*

Proof. This follows immediately from the previous theorem, since f_1z is the altitude from z to the directrix F_1, so it bisects the chord $\overline{t_1t_1'}$. □

Theorem 35 (Opposite triangle spreads) *If the tangents at $p_0 \equiv p(t)$ and $q_0 \equiv p(u)$ on $\mathcal{P}_0$ meet at z, then $S(zp_0, zf_1) = S(zq_0, zf_2)$ and $S(zp_0, zf_2) = S(zq_0, zf_1)$.*

Proof. Using the Spread formula, we obtain

$$\begin{aligned} S(zp_0, zf_2) &= -\frac{(\alpha^2-1)\left(\left(4t^2u^2-(t+u)^2\right)\alpha^2+(t+u)^2-4\right)}{4\Delta_4(u)\Delta_3(t)} \\ &= S(zq_0, zf_1) \end{aligned}$$

and

$$\begin{aligned} S(zp_0, zf_1) &= -\frac{(\alpha^2-1)\left(\left(4t^2u^2-(t+u)^2\right)\alpha^2+(t+u)^2-4\right)}{4\Delta_3(u)\Delta_4(t)} \\ &= S(zq_0, zf_2). \end{aligned}$$

□

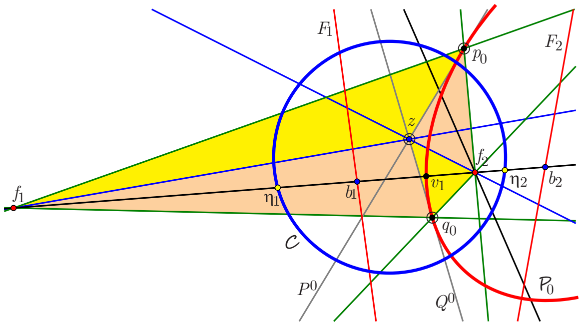

Figure 25: *Opposite triangle spreads*

5 Normals to the parabola $\mathcal{P}_0$

In the Euclidean case, it is well known that the evolute of the parabola, which is defined as the locus of the center of curvature of the curve—namely the meet of adjacent normals, as Huygens or Newton would have said—is a *semi-cubical parabola*. For the curve $y = x^2$, shown in Figure 26, the evolute has equation

$$\left(y - \frac{1}{2}\right)^3 = \frac{27}{16}x^2.$$

This formula suggests that there is no Euclidean ruler and compass construction for the center of curvature C_0 of the parabola for a general point P_0 on it. We will see that in the hyperbolic case, the situation is in some ways simpler, and indeed we will show how to give a straightedge construction for the center of curvature!

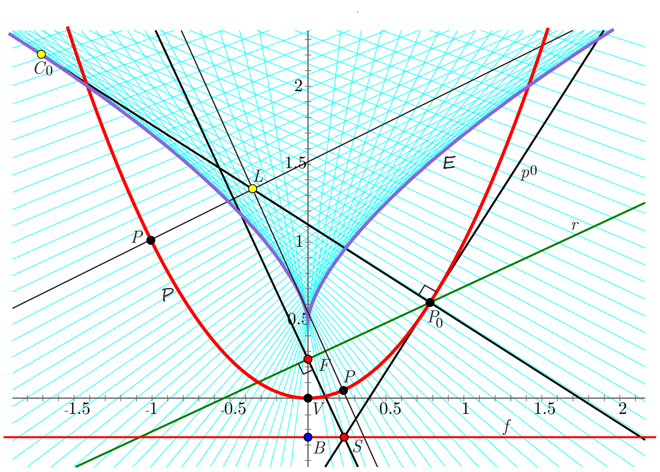

Figure 26: *Evolute of a Euclidean parabola*

In Figure 26 we see a point P_0 on the Euclidean parabola, with its tangent p^0, obtained by finding the meet S of the directrix f with the altitude to the focal line $r = FP_0$ through the focus F. The center of curvature is the point C_0 on the evolute $\mathcal{E}$. The figure shows also that for points L above the evolute, there are three normals that meet there; we exhibit also the other two points marked P whose normals also pass through L. Below the evolute only one normal passes through any fixed point.

For a point p_0 on the hyperbolic parabola $\mathcal{P}_0$, the altitude line P to the tangent P^0 through p_0 is called the **normal line** at p_0.

Since the dual of P^0 is the twin point p^0, we see that

$$\begin{aligned} P &\equiv p_0 p^0 = \left[t^2 : t : 1\right] \times \left[\alpha^2 - 1 : 2t\alpha^2 : -t^2\alpha^2\left(\alpha^2 - 1\right)\right] \\ &= \left\langle -t\alpha^2\left(t^2\alpha^2 - t^2 + 2\right) : \left(\alpha^2 - 1\right)\left(t^4\alpha^2 + 1\right)\right. \\ &\qquad \left. : t\left(2t^2\alpha^2 - \alpha^2 + 1\right)\right\rangle. \end{aligned} \tag{15}$$

By symmetry, this means that P is both the normal line to the parabola $\mathcal{P}_0$ at p_0 as well as the normal line to the twin parabola $\mathcal{P}^0$ at p^0.

The meet of P and the axis A is the point

$$\begin{aligned} n &\equiv PA \\ &= \left\langle -t\alpha^2\left(t^2\alpha^2 - t^2 + 2\right) : \left(\alpha^2 - 1\right)\left(t^4\alpha^2 + 1\right)\right. \\ &\qquad \left. : t\left(2t^2\alpha^2 - \alpha^2 + 1\right)\right\rangle \times \langle 0 : 1 : 0 \rangle \\ &= \left[t\left(2t^2\alpha^2 - \alpha^2 + 1\right) : 0 : t\alpha^2\left(t^2\alpha^2 - t^2 + 2\right)\right] \\ &= \left[2t^2\alpha^2 - \alpha^2 + 1 : 0 : \alpha^2\left(t^2\alpha^2 - t^2 + 2\right)\right] \end{aligned}$$

provided that $t \neq 0$. Since the normal P of is perpendicular to the tangent P^0, and since P^0 is a biline of the vertex $\overline{R_1R_2}$, the normal P is the other biline for the vertex $\overline{R_1R_2}$. In fact we may calculate that

$$S(R_1, P) = S(P, R_2) = \frac{t^2\left(\alpha^2 + 1\right)^2}{-\Delta_3\Delta_4}.$$

5.1 Conjugate normals and conics

Recall that the conjugate points n_1, n_2 of p_0 are the second meets of the focal lines $R_1 \equiv f_1 p_0$ and $R_2 \equiv f_2 p_0$ with the parabola $\mathcal{P}_0$ respectively. They are given in (14). The normal lines to $\mathcal{P}_0$ at the conjugate points n_1 and n_2 can then be computed using the formula (15):

$$\begin{aligned} P_1 \equiv &\left\langle t\alpha(\alpha - 1)\left(2\alpha^2(\alpha - 1)t^2 + (\alpha + 1)^3\right)\right. \\ &: \alpha^2(\alpha - 1)^4 t^4 + (\alpha + 1)^4 \\ &\left. : -t\alpha(\alpha + 1)\left(2(\alpha + 1) - (\alpha - 1)^3 t^2\right)\right\rangle \\ P_2 \equiv &\left\langle -t\alpha(\alpha + 1)\left(2\alpha^2(\alpha + 1)t^2 + (\alpha - 1)^3\right)\right. \\ &: \alpha^2(\alpha + 1)^4 t^4 + (\alpha - 1)^4 \\ &\left. : t\alpha(\alpha - 1)\left(2(\alpha - 1) - (\alpha + 1)^3 t^2\right)\right\rangle. \end{aligned}$$

We will call these the **conjugate normal lines** of p_0.

Theorem 36 (Conjugate normal conics) *There are two conics $\mathcal{H}_1$ and $\mathcal{H}_2$ with the following properties. Let h_1 be the meet of the normal P and the conjugate normal P_1 of a point p_0 on $\mathcal{P}_0$. Then h_1 lies on $\mathcal{H}_1$, which passes through f_2 and is tangent to B_1 there. Similarly if h_2 is the meet of P and P_2 at p_0, then h_2 lies on $\mathcal{H}_2$, which passes through f_1 and is tangent to B_2 there. Furthermore we have collinearities $[[f_1s_2h_2]]$ as well as $[[f_2s_1h_1]]$. In addition $\mathcal{H}_1$ passes through the points d_0 and $\overline{d_0}$.*

Proof. The conjugate normal P_1 will meet the normal P at

$$h_1 \equiv PP_1 = \Big[-\alpha^2(\alpha-1)^3t^4+4\alpha^2(\alpha+1)t^2-(\alpha-1)(\alpha+1)^2 : t\alpha\left(\alpha^2+1\right)\Delta_1$$
$$: \alpha\left(\alpha^2(\alpha+1)(\alpha-1)^2t^4+4\alpha^2(\alpha-1)t^2+(\alpha+1)^3\right)\Big]$$

A computation shows this point always lies on the conic $\mathcal{H}_1$ with equation

$$\alpha^2\left(\alpha^2-1\right)\left(1+4\alpha+\alpha^2\right)x^2$$
$$+2\alpha\left(1-2\alpha-\alpha^2\right)\left(1+2\alpha-\alpha^2\right)xz$$
$$+32\alpha^3y^2+\left(\alpha^2-1\right)\left(1-4\alpha+\alpha^2\right)z^2=0.$$

The conjugate normal P_2 will meet the normal P at

$$h_2 \equiv PP_2 = \Big[\alpha^2(\alpha+1)^3t^4-4\alpha^2(\alpha-1)t^2+(\alpha+1)(\alpha-1)^2$$
$$: t\alpha\left(\alpha^2+1\right)\Delta_2$$
$$: \alpha\left(\alpha^2(\alpha-1)(\alpha+1)^2t^4+4\alpha^2(\alpha+1)t^2+(\alpha-1)^3\right)\Big].$$

This point always lies on the conic $\mathcal{H}_2$ with equation

$$\alpha^2\left(\alpha^2-1\right)\left(1-4\alpha+\alpha^2\right)x^2$$
$$-2\alpha\left(2\alpha+\alpha^2-1\right)\left(-2\alpha+\alpha^2-1\right)xz$$
$$-32\alpha^3y^2+\left(\alpha^2-1\right)\left(1+4\alpha+\alpha^2\right)z^2=0.$$

The collinearity $[[f_1s_1h_2]]$ is established by checking that the determinant formed by the respective vectors is indeed 0 (it is!), and similarly for the collinearity $[[f_2s_2h_1]]$. We can also check (with a computer package) that both of the points d_0 and $\overline{d_0}$ identically satisfy the equation of $\mathcal{H}_1$. $\square$

The normal P at p_0 meets the parabola $\mathcal{P}_0$ again at a second point

$$p_0' = \left[\left(2t^2\alpha^2-\alpha^2+1\right)^2 : t\alpha^2\left(-t^2\alpha^2+t^2-2\right)\left(2t^2\alpha^2-\alpha^2+1\right)\right.$$
$$\left.: t^2\alpha^4\left(t^2\alpha^2-t^2+2\right)^2\right]$$

and similarly the conjugate normals P_1, P_2 at n_1, n_2 meet $\mathcal{P}_0$ respectively also at

$$n_1' = \Big[t^2\alpha^2\left(\left(\alpha-1\right)^3t^2-2\left(\alpha+1\right)\right)^2$$
$$: t\alpha\left(2(\alpha+1)-(\alpha-1)^3t^2\right)\left(2\alpha^2(\alpha-1)t^2+(\alpha+1)^3\right)$$
$$: \left(2\alpha^2(\alpha-1)t^2+(\alpha+1)^3\right)^2\Big]$$

$$n_2' = \Big[t^2\alpha^2\left(\left(\alpha+1\right)^3t^2+2\left(1-\alpha\right)\right)^2$$
$$: \left(2\alpha^2(\alpha+1)t^2+(\alpha-1)^3\right)\left(\alpha(\alpha+1)^3t^3-2\alpha(\alpha-1)t\right)$$
$$: \left(2\alpha^2(\alpha+1)t^2+(\alpha-1)^3\right)^2\Big].$$

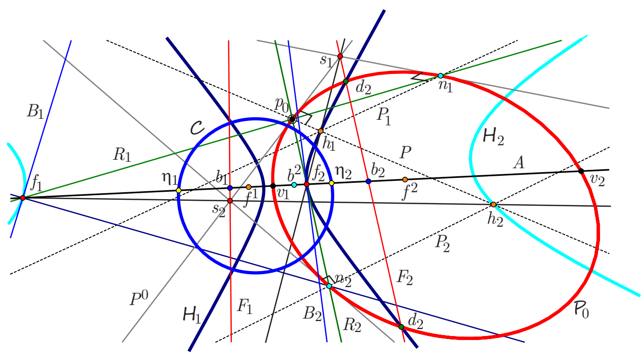

Figure 27: *Conjugate normal meets h_1 and h_2 and conics*

Theorem 37 (Normal conjugate colliearities) *Let p_0', n_1' and n_2' be the second meets of the normals and conjugate normals P, P_1 and P_2 of p_0 with the parabola $\mathcal{P}_0$ respectively, and t_1, t_2 the altitude base points of p_0. Then we have collinearities $[[p_0'n_1't_1]]$ and $[[p_0'n_2't_2]]$.*

Proof. Since the forms of all the points involved are known, it is straightforward (with a computer package) to verify that the corresponding determinants for both collinearities do evaluate identically to 0. $\square$

These collinearities are illustrated in Figure 28.

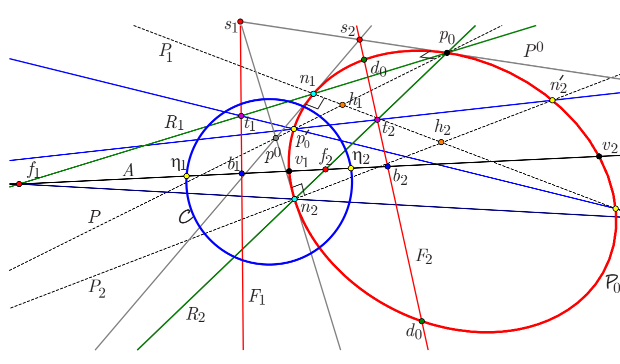

Figure 28: *Normal conjugate collinearities*

5.2 Four points with concurrent normals

In the Euclidean case, finding the three points P on the parabola whose normals pass through a given point L above the evolute is not straightforward [8]. We will show that in the hyperbolic case there is an interesting conic, related to the elementary symmetric functions of four variables t_1, t_2, t_3, t_4, that allows us to find *four* such points.

Theorem 38 (Four parabola normals) *If l is a point in the hyperbolic plane, then there are at most four points p on the parabola $\mathcal{P}_0$ whose normals pass through l.*

Proof. We know that the normal to $p_0 = [t^2 : t : 1]$ is the line

$$P = \langle t\alpha^2 \left(-t^2\alpha^2 + t^2 - 2\right) : \left(\alpha^2 - 1\right)\left(t^4\alpha^2 + 1\right)$$
$$: t\left(2t^2\alpha^2 - \alpha^2 + 1\right)\rangle.$$

If P passes through a point $l = [x_0 : y_0 : z_0]$, then $lP = 0$, which after rearranging is the equation

$$\alpha^2\left(\alpha^2 - 1\right) y_0 t^4 + \alpha^2\left(\left(1 - \alpha^2\right) x_0 + 2z_0\right) t^3$$
$$+ \left(\left(1 - \alpha^2\right) z_0 - 2\alpha^2 x_0\right) t + \left(\alpha^2 - 1\right) y_0 = 0. \qquad (16)$$

This is a polynomial of degree four in t, so it has at most four solutions. □

Theorem 39 (Quadratic normal meets) *Suppose $p_0 = p(t)$ and $q_0 = p(u)$ are two points on the parabola, whose respective normals P and Q meet at a point l, and suppose $\alpha^2 + 1 \neq 0$. Then there are $0, 1$ or 2 other points on the parabola whose normals pass through l precisely when $\nabla = \left(t^2u^2\alpha^2 + 1\right)^2 - 4tu\alpha^2(t+u)^2$ is not a square, is zero, or is a non-zero square respectively.*

Proof. The meet of the two normals is

$$l \equiv PQ =$$
$$\left[\left(\alpha^2 - 1\right)\left(\left(tu\left(2t^2u^2 - tu - t^2 - u^2\right)\right)\alpha^4 + \left((tu-2)\left(tu + t^2 + u^2\right) + 1\right)\alpha^2 - 1\right)\right.$$
$$: -tu\alpha^2\left(\alpha^2 + 1\right)^2 (t+u)$$
$$\left. : \alpha^2\left(\alpha^2 - 1\right)\left(t^3u^3\alpha^4 + \left((2tu-1)\left(tu + t^2 + u^2\right) - t^3u^3\right)\alpha^2 + \left(t^2 + tu + u^2 - 2\right)\right)\right]$$

and we need to check when a third point $r_0 \equiv p(v)$ on $\mathcal{P}_0$ has a normal R also passing through l. This is equivalent to $lR = 0$ which yields, after remarkable simplification,

$$-\alpha^2\left(\alpha^2 - 1\right)\left(\alpha^2 + 1\right)^2 (u - v)(t - v)$$
$$\cdot\left(t + u + v + tu^2v^2\alpha^2 + t^2uv^2\alpha^2 + t^2u^2v\alpha^2\right) = 0.$$

Since $\alpha \neq 0, \pm 1$ and u, t, v are disjoint, this condition reduces to the quadratic equation $tu\alpha^2(t+u)v^2 + \left(t^2u^2\alpha^2 + 1\right)v + (t+u) = 0$ in v with discriminant

$$\nabla = \left(t^2u^2\alpha^2 + 1\right)^2 - 4tu\alpha^2(t+u)^2. \qquad \square$$

The question of the existence of four points on the parabola $\mathcal{P}_0$ with a common normal point is closely related to an interesting conic associated to four points on the parabola; namely the conic $\mathcal{A}$ through those four points and the axis point a, which has independent interest due to its form. We call this conic $\mathcal{A}(p_1, p_2, p_3, p_4)$ the **four-point conic** through p_1, p_2, p_3 and p_4.

Theorem 40 (Four point conic) *For any four points $p_1 \equiv p(t), p_2 \equiv p(u), p_3 \equiv p(v)$ and $p_4 \equiv p(w)$ lying on $\mathcal{P}_0$, the four-point conic $\mathcal{A}(p_1, p_2, p_3, p_4)$ has equation*

$$0 = x^2 - (t+u+v+w)xy + (tu+tv+tw+uv+uw+vw)xz$$
$$-(tuv+tuw+tvw+uvw)yz + tuvwz^2. \qquad (17)$$

Proof. We use a standard technique for computing a conic through five given points: by taking a combination of the degenerate line products formed by pairs of four points p_1, p_2, p_3 and p_4. Now

$$p_1p_2 = (1 : -(t+u) : tu) \qquad p_3p_4 = (1 : -(v+w) : vw)$$
$$p_1p_3 = (1 : -(t+v) : tv) \qquad p_2p_4 = (1 : -(t+w) : tw)$$

so the general conic in the pencil through p_1, p_2, p_3 and p_4, has the form

$$0 = p(x,y,z) = (x - (t+u)y + tuz)(x - (v+w)y + vwz)$$
$$+ \lambda(x - (t+v)y + tvz)(x - (u+w)y + uwz).$$

Now since also $p(0,1,0) = 0$, we can solve for λ to get

$$\lambda = -\frac{(t+u)(v+w)}{(t+v)(u+w)}.$$

Substituting back and simplifying, we find that the equation of the required conic is (17). □

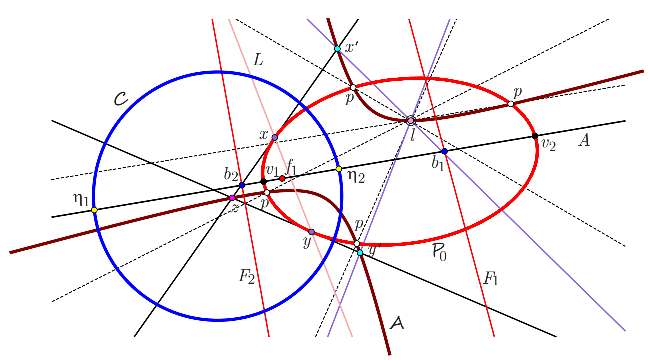

Figure 29: *Four points p with normals through l and associated conic $\mathcal{A}$*

There is a clear similarity between the form of this conic and the familiar identity

$$\begin{aligned}(x-t_1)(x-t_2)(x-t_3)(x-t_4) &= x^4-(t_1+t_2+t_3+t_4)x^3\\ &+(t_1t_2+t_1t_3+t_1t_4+t_2t_3+t_2t_4+t_3t_4)x^2\\ &-(t_1t_2t_3+t_1t_2t_4+t_1t_3t_4+t_2t_3t_4)x+t_1t_2t_3t_4\end{aligned}$$

relating the coefficients of a degree four polynomial and the elementary symmetric functions of its zeros. This may be explained by noting that if $p=[x:y:z]=\left[t^2:t:1\right]$ is a point on the parabola, then the quantities x^2, xy, xz, yz and z^2 are respectively exactly t^4, t^3, t^2, t and 1, while the condition that the conic passes through a ensures that the coefficient of y^2 is necessarily 0.

5.3 The conic $\mathcal{A}_n$ and finding normals

Theorem 41 (Four normal conic) *Suppose that the normal lines at four points p_1, p_2, p_3, p_4 lying on $\mathcal{P}_0$ are concurrent at a point $l=[x_0, y_0, z_0]$ not lying on the axis A. Then the conic $\mathcal{A}_l$ with equation*

$$\begin{aligned}&\alpha^2\left(\alpha^2-1\right)y_0x^2+\alpha^2\left(x_0+2z_0-x_0\alpha^2\right)xy\\ &+\left(z_0-z_0\alpha^2-2x_0\alpha^2\right)yz+\left(\alpha^2-1\right)y_0z^2=0 \qquad (18)\end{aligned}$$

passes through the six points p_1, p_2, p_3, p_4, a and l, so in particular $\mathcal{A}_l=\mathcal{A}(p_1,p_2,p_3,p_4)$.

Proof. The condition (16) on t for $p=\left[t^2:t:1\right]$ on $\mathcal{P}_0$ to have a normal line passing through $l\equiv[x_0, y_0, z_0]$ may be rewritten, since $y_0\neq 0$, as

$$t^4+\frac{\alpha^2\left(x_0\left(1-\alpha^2\right)+2z_0\right)}{\alpha^2\left(\alpha^2-1\right)y_0}t^3+\frac{\left(z_0\left(1-\alpha^2\right)-2x_0\alpha^2\right)}{\alpha^2\left(\alpha^2-1\right)y_0}t+\frac{1}{\alpha^2}=0.$$

If we have four distinct solutions t, u, v, w of this equation, then

$$\begin{aligned}t+u+v+w &= -\frac{\alpha^2\left(x_0\left(1-\alpha^2\right)+2z_0\right)}{\alpha^2y_0\left(\alpha^2-1\right)}\\ tu+tv+tw+uv+uw+vw &= 0\\ tuv+tuw+tvw+uvw &= -\frac{z_0\left(1-\alpha^2\right)-2x_0\alpha^2}{\alpha^2y_0\left(\alpha^2-1\right)}\\ tuvw &= \frac{1}{\alpha^2}.\end{aligned}$$

From the previous theorem, the conic passing through the five points $p_1=p(t), p_2=p(u), p_3=p(v), p_4=p(w)$ and a then has the form

$$x^2+\frac{\alpha^2\left(x_0+2z_0-x_0\alpha^2\right)}{\alpha^2\left(\alpha^2-1\right)y_0}xy+\frac{\left(z_0-2x_0\alpha^2-z_0\alpha^2\right)}{\alpha^2\left(\alpha^2-1\right)y_0}yz+\frac{1}{\alpha^2}z^2=0$$

which we can rewrite as the conic $\mathcal{A}_l$ (18). But now we can check that also l lies on this conic, since identically

$$\begin{aligned}&\alpha^2\left(\alpha^2-1\right)y_0x_0^2+\alpha^2\left(x_0\left(1-\alpha^2\right)+2z_0\right)x_0y_0\\ &+\left(z_0\left(1-\alpha^2\right)-2x_0\alpha^2\right)y_0z_0+\left(\alpha^2-1\right)y_0z_0^2=0. \qquad \square\end{aligned}$$

Theorem 42 (Conic construction of common normals) *Let l be a point of the hyperbolic plane with the property that the dual line L of l meets $\mathcal{P}_0$ at two points x and y. Then the meet z of the tangent lines to $\mathcal{P}_0$ at x and y, the meet x' of the tangent line at x and the dual line of x, and the meet y' of the tangent line at y and the dual line of y, all line on the conic $\mathcal{A}_l$.*

Proof. Suppose that the dual line L of l meets $\mathcal{P}_0$ at two points $x=\left[t^2:t:1\right]$ and $y=\left[u^2:u:1\right]$. Then the meets of the tangent lines is $z=[2tu:t+u:2]$ from the Tangent meets theorem. Also $L=\langle 1:-(t+u):tu\rangle$ and

$$l=\left[\alpha^2-1:\alpha^2(t+u):-\alpha^2tu\left(\alpha^2-1\right)\right].$$

In this case the equation (18) for the conic $\mathcal{A}_l$ simplifies, after some cancellation, to

$$\begin{aligned}&\alpha^2(t+u)x^2+\left(1-2tu\alpha^2-\alpha^2\right)xy\\ &+\left(tu\alpha^2-tu-2\right)yz+(t+u)z^2=0. \qquad (19)\end{aligned}$$

The dual line of x meets the tangent line at x at

$$x'=\left[t\left(\alpha^2t^2-t^2+2\right):\alpha^2t^4+1:t\left(2\alpha^2t^2-\alpha^2+1\right)\right]$$

and the dual line of y meets the tangent line at y at

$$y'=\left[u\left(\alpha^2u^2-u^2+2\right):\alpha^2u^4+1:u\left(2\alpha^2u^2-\alpha^2+1\right)\right].$$

We check that both of these points identically satisfy the equation (19). $\square$

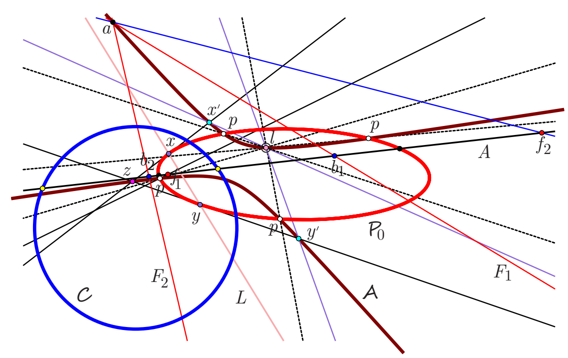

Figure 30: *Construction of points p on $\mathcal{P}_0$ with normals through n*

This also provides us with an elegant method to find all normals through a given point l. Firstly, find the dual line L of the point l and then find the meets x, y of this line L with the parabola $\mathcal{P}_0$. Construct the tangents P_x, P_y to $\mathcal{P}_0$ at x and y and find their meet z. Construct the dual lines X and Y of x and y, then find the meet of the tangent at x and the dual line of x, that is $x'=P_xX$ and the meet of the tangent at y and the dual line of y, that is $y' = P_y Y$. According to the above theorem, the five points l, x', y', z, a lie on a conic A_l which may meet the parabola $\mathcal{P}_0$ in at most four points which have the property that their normals meet at l. We see that the number of normals passing through l is determined by the meet of the conic A_l with the parabola $\mathcal{P}_0$. So if we can find the meets of these two conics, we have the normals which pass through l.

This construction shows that some aspects of hyperbolic geometry are surprisingly more simple than in Euclidean geometry. In the latter, finding normals to points on a parabola from a particular point is quite cumbersome, as shown in [8].

Furthermore, the four normals drawn from a particular points are also the normals to four points on the twin parabola $\mathcal{P}^0$. These points are the dual points of the tangents to four points on the original parabola $\mathcal{P}_0$. This observation is the result of duality between lines and points.

5.4 Normal conjugate points

If p_0 is a point on $\mathcal{P}_0$ with tangent line P^0 and normal line P, then the other meet of P with the parabola gives a point p_0', which we call the **normal conjugate point** of p_0. Then the tangent line $P^{0\prime}$ to p_0' meets with P^0 at the point

$$\begin{aligned}k_0 &= P^0 P^{0\prime}\\ &= \left\langle t^2\alpha^4\left(t^2\alpha^2 - t^2 + 2\right)^2 : 2t\alpha^2\left(t^2\alpha^2 - t^2 + 2\right)\left(2t^2\alpha^2 - \alpha^2 + 1\right)\right.\\ &\quad \left. : \left(2t^2\alpha^2 - \alpha^2 + 1\right)^2\right\rangle \times \left\langle 1 : -2t : t^2\right\rangle\\ &= \left[-2t\left(2t^2\alpha^2 - \alpha^2 + 1\right) : \left(\alpha^2 - 1\right)\left(t^4\alpha^2 + 1\right) : 2t\alpha^2\left(t^2\alpha^2 - t^2 + 2\right)\right].\end{aligned}$$

Figure 31 shows the **normal conjugate curve** $\mathcal{K}_0$: the locus of k_0 as p_0 moves. This a higher degree curve which passes through a as well as d_0 and $\overline{d_0}$, and is tangent to $\mathcal{P}_0$ at those latter two points. It seems an interesting future direction to investigate more fully such associated algebraic curves connected with $\mathcal{P}_0$.

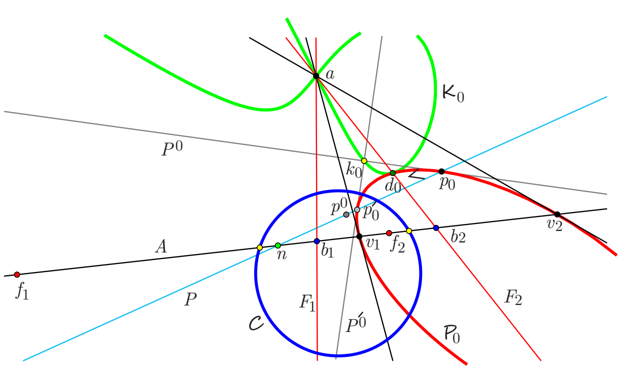

Figure 31: *The normal conjugate conic $\mathcal{K}_0$*

5.5 The evolute and centers of curvature

Recall that the **evolute** of a curve is the envelope of the normals to that curve, or equivalently the locus of the centers of curvature. Following the technique described in [4], here is a pleasant construction of the center of curvature c_0 to the hyperbolic parabola $\mathcal{P}_0$ at the point p_0.

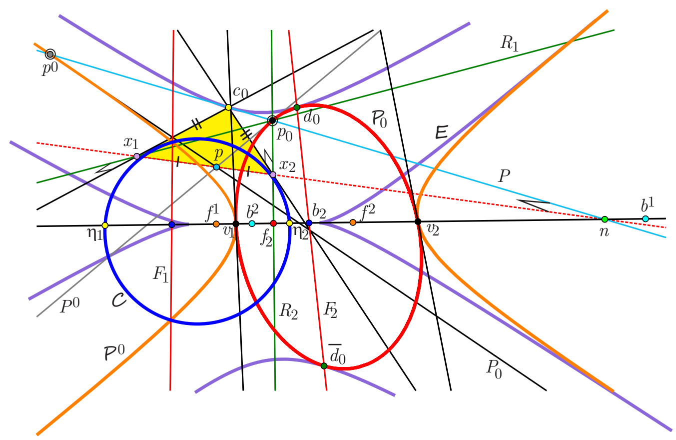

Figure 32: *Evolute of a parabola*

Theorem 43 (Center of curvature construction) *Let P be the normal at p_0 to the parabola $\mathcal{P}_0$, and construct the altitude line Q to P through $n = AP$. Suppose that the meets of Q with the focal lines R_1 and R_2 are respectively x_1 and x_2. Then the meet of the perpendicular line to R_1 through x_1 and the perpendicular line to R_2 through x_2 is the required center of curvature c_0 to $\mathcal{P}_0$ at the point p_0.*

Proof. Let $p_0 = \left[t^2 : t : 1\right]$ and $n = \left[2t^2\alpha^2 - \alpha^2 + 1 : 0 : \alpha^2\left(t^2\alpha^2 - t^2 + 2\right)\right]$, then the perpendicular to P through $l = n$ is

$$\begin{aligned}Q \equiv pn = \Big[&\alpha^2\left(t^4\alpha^2 + 1\right)\left(t^2\alpha^2 - t^2 + 2\right)\\ &: t\left(2\alpha - t^2\alpha - 2t^2\alpha^2 + t^2\alpha^3 + \alpha^2 - 1\right)\\ &\quad \cdot\left(-2\alpha + t^2\alpha - 2t^2\alpha^2 - t^2\alpha^3 + \alpha^2 - 1\right)\\ &: \left(t^4\alpha^2 + 1\right)\left(-2t^2\alpha^2 + \alpha^2 - 1\right)\Big]\end{aligned}$$

This line will meet the line R_1 at

$$\begin{aligned}x_1 = \Big[&-2\alpha^4t^6 + \left(\alpha^5 + 3\alpha^4 - 3\alpha^2 - \alpha\right)t^4\\ &+ \left(2\alpha^3 - \alpha^4 + 4\alpha^2 + 2\alpha - 1\right)t^2 + \left(1 - \alpha^2\right)\\ &: t\alpha\left(\alpha^2 + 1\right)\left(t^4\alpha^2 + 1\right)\\ &: \alpha\left(-\alpha^3\left(\alpha^2 - 1\right)t^6 + \alpha\left(2\alpha - 4\alpha^2 + 2\alpha^3 + \alpha^4 + 1\right)t^4\right.\\ &\quad \left. - \left(\alpha^2 - 1\right)\left(-3\alpha + \alpha^2 + 1\right)t^2 + 2\alpha\right)\Big]\end{aligned}$$

and the line R_2 at

$$\begin{aligned} x_2 = \Big[& \left(2\alpha^4\right) t^6 + \left(\alpha^5 - 3\alpha^4 + 3\alpha^2 - \alpha\right) t^4 \\ & + \left(\alpha^4 + 2\alpha^3 - 4\alpha^2 + 2\alpha + 1\right) t^2 + \left(\alpha^2 - 1\right) \\ & : t\alpha\left(\alpha^2+1\right)\left(t^4\alpha^2+1\right) \\ & : \alpha\left(\alpha^3\left(\alpha^2-1\right) t^6 + \left(2\alpha^4 - \alpha^5 + 4\alpha^3 + 2\alpha^2 - \alpha\right) t^4 \right. \\ & \left. + \left(3\alpha - 3\alpha^3 - \alpha^4 + 1\right) t^2 - 2\alpha\right) \Big]. \end{aligned}$$

The perpendicular line to R_1 through x_1 is $X_1 = x_1 r_1$ and the perpendicular line to R_2 through x_2 is $X_2 = x_2 r_2$ which meet at

$$\begin{aligned} & c_0 = X_1 X_2 = \\ & \Big[\left(\alpha^2-1\right)\left(2\alpha^4 t^6 + 3\alpha^2\left(1-\alpha^2\right) t^4 - 6\alpha^2 t^2 + \left(\alpha^2-1\right)\right) \\ & \quad : -2t^3\alpha^2\left(\alpha^2+1\right)^2 \\ & \quad : \alpha^2\left(\alpha^2-1\right)\left(\alpha^2\left(\alpha^2-1\right) t^6 + 6\alpha^2 t^4 + 3\left(1-\alpha^2\right) t^2 - 2\right) \Big]. \end{aligned}$$

To evaluate the center of curvature, we note that adjacent normals, say at $p(t)$ and $p(r)$, meet at

$$\begin{aligned} & f(t,r) = \\ & \Big[\left(\alpha^2-1\right)\left(-2r^3t^3\alpha^4 + r^3t\alpha^4 - r^3t\alpha^2 + r^2t^2\alpha^4 - r^2t^2\alpha^2 \right. \\ & \qquad \left. + 2r^2\alpha^2 + rt^3\alpha^4 - rt^3\alpha^2 + 2rt\alpha^2 + 2t^2\alpha^2 - \alpha^2 + 1\right) \\ & \quad : rt\alpha^2(r+t)\left(\alpha^2+1\right)^2 \\ & \quad : -\alpha^2\left(\alpha^2-1\right)\left(r^3t^3\alpha^4 - r^3t^3\alpha^2 + 2r^3t\alpha^2 + 2r^2t^2\alpha^2 \right. \\ & \qquad \left. - r^2\alpha^2 + r^2 + 2rt^3\alpha^2 - rt\alpha^2 + rt - t^2\alpha^2 + t^2 - 2\right) \Big] \end{aligned}$$

where we have removed a common factor of $r-t$. Now let $r = t$ to find that $f(t,t) = c_0$. □

5.6 Formula for the evolute

Can we get a formula for the evolute? Working with affine coordinates (setting $z = 1$), we need eliminate t from the equations

$$x = \frac{\left(2t^6\alpha^4 - 3t^4\alpha^4 + 3t^4\alpha^2 - 6t^2\alpha^2 + \alpha^2 - 1\right)}{\alpha^2\left(t^6\alpha^4 - t^6\alpha^2 + 6t^4\alpha^2 - 3t^2\alpha^2 + 3t^2 - 2\right)}$$

$$y = \frac{-2t^3\left(\alpha^2+1\right)^2}{\left(\alpha^2-1\right)\left(t^6\alpha^4 - t^6\alpha^2 + 6t^4\alpha^2 - 3t^2\alpha^2 + 3t^2 - 2\right)}.$$

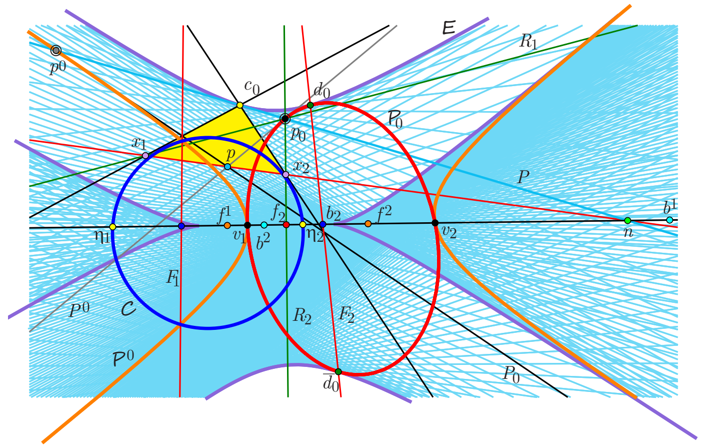

Figure 33: *Normals to a parabola*

We could use a Gröbner basis to calculate this, but the polynomials are small enough to do it by hand with classical elimination. We get, after some calculation, that x and y satisfy the affine equation

$$\begin{aligned} 0 = h(x,y) = & \; 32\alpha^8\left(\alpha^2-1\right)^3 x^6 - 256\alpha^2\left(\alpha^2-1\right)^6 y^6 \\ & + 3\alpha^4\left(8\alpha + 6\alpha^2 - 8\alpha^3 + 3\alpha^4 + 3\right)\left(-8\alpha + 6\alpha^2 + 8\alpha^3 + 3\alpha^4 + 3\right) \\ & \quad \cdot (\alpha-1)^2(\alpha+1)^2 x^4y^2 \\ & + 384\alpha^4\left(\alpha^2-1\right)^5 x^2y^4 + 48\alpha^6\left(-2\alpha+\alpha^2-1\right)\left(2\alpha+\alpha^2-1\right)\left(\alpha^2-1\right)^2 x^5 \\ & - 192\alpha^4\left(-2\alpha+\alpha^2-1\right)\left(2\alpha+\alpha^2-1\right)\left(\alpha^2-1\right)^3 x^3y^2 \\ & + 192\alpha^2\left(-2\alpha+\alpha^2-1\right)\left(2\alpha+\alpha^2-1\right)\left(\alpha^2-1\right)^4 xy^4 \\ & + 24\alpha^4\left(\alpha^2-1\right)\left(-2\alpha-6\alpha^2+2\alpha^3+\alpha^4+1\right)\left(2\alpha-6\alpha^2-2\alpha^3+\alpha^4+1\right) x^4 \\ & - 384\alpha^2\left(\alpha^2-1\right)^5 y^4 \\ & + 6\alpha^2\left(196\alpha^2 - 378\alpha^4 + 196\alpha^6 + \alpha^8 + 1\right)\left(\alpha^2-1\right)^2 x^2y^2 \\ & + 4\alpha^2\left(2\alpha+\alpha^2-1\right)\left(-2\alpha+\alpha^2-1\right)\left(-36\alpha^2 + 86\alpha^4 - 36\alpha^6 + \alpha^8 + 1\right) x^3 \\ & + 192\alpha^2\left(-2\alpha+\alpha^2-1\right)\left(2\alpha+\alpha^2-1\right)\left(\alpha^2-1\right)^3 xy^2 \\ & - 24\alpha^2\left(\alpha^2-1\right)\left(2\alpha-6\alpha^2-2\alpha^3+\alpha^4+1\right)\left(-2\alpha-6\alpha^2+2\alpha^3+\alpha^4+1\right) x^2 \\ & + 3\left(-8\alpha+6\alpha^2+8\alpha^3+3\alpha^4+3\right)\left(8\alpha+6\alpha^2-8\alpha^3+3\alpha^4+3\right)\left(\alpha^2-1\right)^2 y^2 \\ & + 48\alpha^2\left(-2\alpha+\alpha^2-1\right)\left(2\alpha+\alpha^2-1\right)\left(\alpha^2-1\right)^2 x - 32\alpha^2\left(\alpha^2-1\right)^3. \end{aligned}$$

So the evolute is a six degree curve, with coefficients that depend in a pleasant way on α. Note that all the coefficients are divisible by $\alpha^2 - 1$, with the exception of the coefficient of x^3.

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Conchoids on the Sphere

Conchoids on the Sphere

ABSTRACT

The construction of planar conchoids can be carried over to the Euclidean unit sphere. We study the case of conchoids of (spherical) lines and circles. Some elementary constructions of tangents and osculating circles are stil valid on the sphere. Further, we aim at the illustration and a precise description of the algebraic properties of the principal views of spherical conchoids, *i.e.*, the conchoid's images under orthogonal projections onto their symmetry planes.

Key words: spherical curves, conchoids, algebraic curves, tangent, osculating circle, singularities, orthogonal projection

MSC2010: 51N20, 14H99, 70B99

Konhoide na sferi

SAŽETAK

Konstrukcija ravninskih konhoida može se prenijeti na euklidsku jediničnu sferu. Promatramo slučaj konhoida generiranih sfernim pravacima i kružnicama. Neke elementarne konstrukcije tangenata i kružnica zakrivljenosti vrijede i za sferne konhoide. Nadalje, naš je cilj ilustracija i precizan opis algebarskih svojstava glavnih pogleda sfernih konhoida, tj. slika konhoida pri ortogonalnom projiciranju na njihove ravnine simetrije.

Ključne riječi: krivulje na sferi, konhoide, algebarske krivulje, tangenta, kružnica zakrivljenosti, singulariteti, ortogonalna projekcija

1 Introduction

The construction of conchoids goes back to the early Greek mathematicians [5, 13]. Assume we are given a point F, called *focus* and a line l called *directrix* one can ask for the set c of all points in the Euclidean plane at fixed distance d from l measured on all lines through F, cf. Figure 1.

The set c turns out to be an algebraic curve of degree 4, namely the *conchoid* of the line l with respect to F at distance $d \in \mathbb{R}$. The conchoid c can be described by the equation

$$(x^2 - d^2)(f - x)^2 + x^2y^2 = 0$$

provided that a Cartesian coordinate system is chosen as depicted in Figure 1 with $F = (f, 0)$, $f \in \mathbb{R}$ and $l:\ x = 0$. The conchoid has two branches, one corresponding to the distance $+d$, while the other corresponds to the distance $-d$. The algebraic variety contains both branches.

The conchoid c has an ordinary double point at $F = (f, 0)$ if $|d| > |f|$ (or an isolated double point if $|d| < |f|$). In the case of $|d| = |f|$, F is a cusp of the first kind, *i.e.*, with the local expansion $(u^2 + o(u^3), u^3 + o(u^4))$, see [2, 3]. The cusped curve can also be seen in Figure 2.

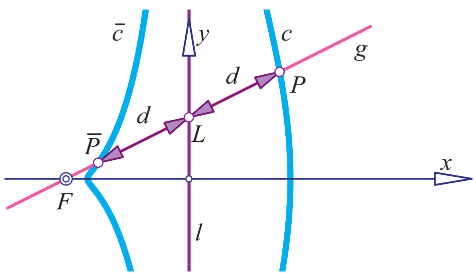

Figure 1: *The construction of the conchoid c of a line l in the plane.*

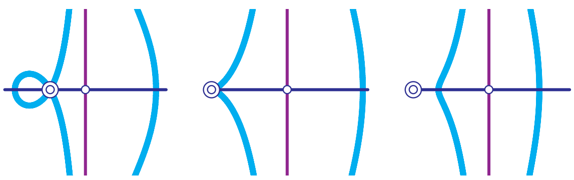

Figure 2: *The planar conchoid of a line has an ordinary double point if $|d| > |f|$ (left), a cusp if $|d| = |f|$ (in the middle), and an isolated double point if $|d| < |f|$ (right).*

Independent of the choice of d and f the curve c considered as a curve in the projective plane (cf. Figure 3) has a tacnode at the ideal point of the y-axis. There, two linear branches with the same tangent emanate. Therefore, the conchoid is of genus 0, and thus, it is a rational curve.

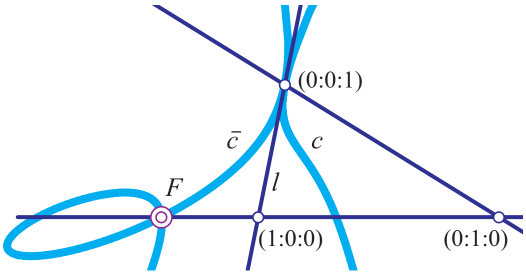

Figure 3: *The singularities of the conchoid considered as a curve in the projective plane.*

The name *conchoid* is due to the fact that its shape somehow reminds of a conch. The conchoid of a line (the directrix l is a line) is frequently called conchoid of Nikomedes, see [4, 5, 13]. The line l can be replaced by an arbitrary curve.

In former years, mathematicians developed elementary constructions of points, tangents, and osculating circles for some kinds of conchoids such as those of lines and circles. The kinematic point of view allows us to see the conchoids as traces of moving particles, and thus, further constructions of tangents and osculating circles can be deduced, see for example [6, 14].

In the last few years conchoids became popular in CAGD, see [1, 8, 9, 10, 11]. This is mainly due to the fact that under certain circumstances conchoids can be parametrized by means of rational functions which is mainly the content of [8, 9]. Thus, a huge class of possibly new surfaces is available for CAGD. The conchoids of spheres and ruled surfaces are not spheres or ruled surfaces anymore, except in some special cases. In order to overcome this flaw, an intrinsic construction of conchoids for some geometries is presented in [7].

It is somehow surprising that conchoids on the sphere have not attracted the researchers' interest. Many constructions that are valid in the Euclidean plane can easily be adapted for the Euclidean unit sphere. In this article, we shall demonstrate this at hand of the spherical analoga to conchoids of lines and circles. The spherical conchoids of lines are conchoids of greatcircles on the sphere. However, the spherical conchoids of circles are stil conchoids of circles but on the sphere.

We shall describe spherical conchoids of lines and circles and study their algebraic properties at hand of their equations. Then, we discuss the shape of the principal views of the spherical conchoids. The principal views are obtained as orthogonal projections to a triple of mutually orthogonal planes where at least one of these planes is a plane of symmetry of the spherical curve. The resulting image curves are at most of degree 8 as is the case for the space curves. For some image curves the degree reduces to 4. Further, we describe the singularities showing up on the principal views of the spherical conchoids.

2 Conchoids of a line

Assume Σ is the Euclidean unit sphere with the equation

$$\Sigma:\ x^2+y^2+z^2=1 \tag{1}$$

and let further l be a *line on* Σ, *i.e.*, a greatcircle of Σ. Without loss of generality, we can asssume that l is the equator of Σ in the plane $z=0$ (see Figure 4). Thus, a parametrization of l reads

$$L(\lambda)=(c_\lambda,s_\lambda,0)\quad \text{with } \lambda\in[0,2\pi[\tag{2}$$

where we have used the abbreviations $c_\lambda:=\cos\lambda$ and $s_\lambda:=\sin\lambda$.

The focus F of the conchoid shall be at spherical distance $\phi\in\,]0,\pi/2[$ from l. Therefore, its coordinates are

$$F=(c_\phi,0,s_\phi) \tag{3}$$

(with $c_\phi:=\cos\phi$ and $s_\phi:=\sin\phi$) since it means no restriction to assume that the greatcircle orthogonal to l through F lies in the plane $y=0$.

The points on the spherical conchoid c of l with respect to F at distance $\delta\in\,]0,\frac{\pi}{2}[$ are found via the analogous construction on the sphere: Choose a point L on the equator l, join it with F by a greatcircle, and determine the points P at spherical distance δ from L.

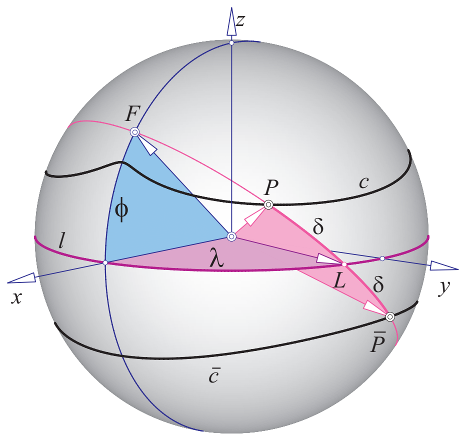

Figure 4: *Construction of a conchoid on the unit sphere and the choice of a coordinate system.*

We exclude the case $\phi=\frac{\pi}{2}$ which yields a pair of *distance curves* provided that $\delta\neq 0$. These distance curves are circles on Σ with spherical radius $\frac{\pi}{2}-\delta$ in planes parallel to the equator plane. The choice $\delta=0$ shows that the equator can be seen as a trivial conchoid $c=l$. The case $\phi=\frac{\pi}{2}$ also yields circles as spherical conchoids of l.

Now we are going to derive an analytical description of the spherical conchoid. Assume that (x,y,z) are the Cartesian coordinates of a point X on the conchoid of l at the spherical distance $\delta \in]0,\frac{\pi}{2}[$ with respect to the point F. These coordinates satisfy Eq. (1). Since $[L,F]$ is a greatcircle of Σ, the points F, L, and the point X on the conchoid are coplanar with the center $(0,0,0)$ of Σ. This is equivalent to

$$s_\lambda s_\phi\, x - c_\lambda s_\phi\, y - s_\lambda c_\phi\, z = 0. \tag{4}$$

Further, we have $\widehat{LX} = \delta$ which is measured along the greatcircle $[L,X]$. Thus, the canonical scalar product of the unit vectors $X = (x,y,z)$ and $L = (c_\lambda, s_\lambda, 0)$ yields the cosine of the angle subtained by $\widehat{LX}$, and therefore, we have

$$c_\lambda\, x + s_\lambda\, y = \cos\delta. \tag{5}$$

We can eliminate λ from Eqs. (4) and (5): These equations are linear in c_λ and s_λ, and thus, we can solve this system for c_λ and s_λ which gives

$$\begin{aligned} c_\lambda &= \frac{\cos\delta(s_\phi\, x - c_\phi\, z)}{s_\phi(x^2+y^2) - c_\phi\, xz}, \\ s_\lambda &= \frac{\cos\delta s_\phi\, y}{s_\phi(x^2+y^2) - c_\phi\, xz}. \end{aligned}$$

Since $c_\lambda{}^2 + s_\lambda{}^2 = 1$ holds for any $\lambda \in \mathbb{C}$, we arrive at an implicit equation of the spherical conchoids c of a (spherical) line l:

$$c: \left\{ \begin{aligned} \cos^2\delta\left((s_\phi\, x - c_\phi\, z)^2 + s_\phi{}^2 y^2\right) & \\ -(s_\phi(x^2+y^2) - c_\phi\, xz)^2 &= 0, \\ x^2+y^2+z^2 &= 1. \end{aligned} \right. \tag{6}$$

Obviously, c is a space curve of degree 8, since it is the intersection of a quartic surface Φ (an example of which is displayed in Figure 5) with the unit sphere. Thus, we can say:

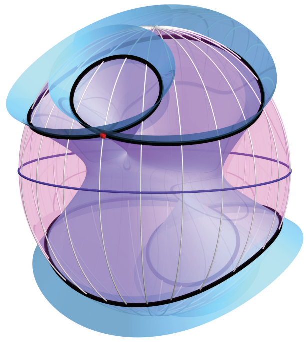

Figure 5: *A spherical conchoid is the intersection of the unit sphere with a quartic surface.*

Theorem 1. *The spherical conchoid c of a (spherical) line l with respect to the focus F at (spherical) distance $\delta \in]0,\frac{\pi}{2}[$ is an algebraic space curve of degree 8 and can be given by the two equations (6).*

It is clear that these curves are spherical so that it is not worth to be mentioned that Eq. (1) is fulfilled by the coordinates (x,y,z) of a generic point on the conchoid. Therefore, only the first equation of (6) matters. Thus, such curves are often called of *spherical degree four*.

The three different shapes of conchoids of a line that can be observed in a plane also appear on the sphere as can be seen in Figure 6. There are conchoids with loops, *i.e.*, they have a spherical double point (actually a pair of opposite double points) with real tangents at the double point F if $\delta > \phi$. The conchoids with spherical cusps (a pair of opposite cusps) appear if, and only if, $\delta = \phi$. In the case of $\delta < \phi$, we observe that F is an isolated (spherical double) point on the conchoid.

As can be seen from Figures 4 and 6 the spherical conchoids always consist of two branches. This is caused by the fact that points in spherical geometry are actually a pair of antipodal points on the sphere. Therefore, any singular point on a conchoid also shows up twice. Even the spherical singularity is a pair of antipodal points.

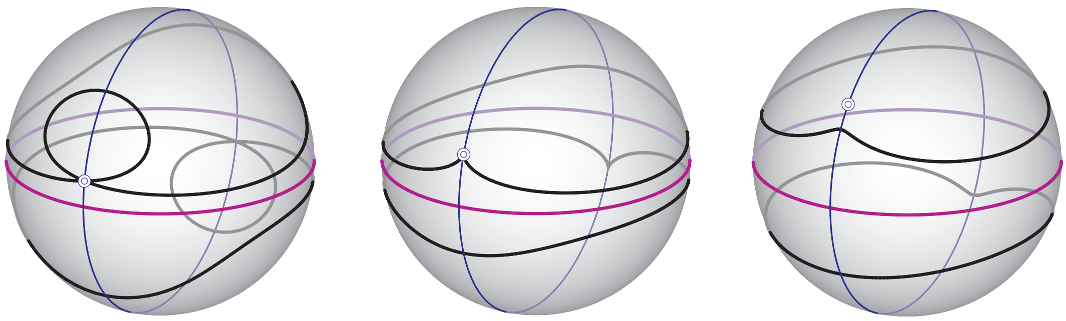

Figure 6: *Three different appearances of spherical conchoids of a the equator: $\delta > \phi$ (left), $\delta = \phi$ (middle), $\delta < \phi$ (right).*

2.1 Principal views of spherical conchoids

The orthogonal projections of c onto the three planes $z=0$, $x=0$, and $y=0$ shall be called top view, front view, and (right) side view. We can state:

Theorem 2. *The front and top view of a spherical conchoid given by Eq.* (6) *with* $\delta \in]0, \frac{\pi}{2}[$ *are of algebraic degree* 8 *and of genus* 1, i.e., *they are elliptic. The right side view is a rational quartic.*

Proof. The equations of c's principal views can be obtained from (6) by simply eliminating z, x, or y. Since c is of degree 8, the principal views of c are at most of degree 8. Reductions of the degree occur only in cases where the image plane is a plane of symmetry of each branch, *i.e.*, each point of the image curve is the image of two points on c. Because of the special choice of the coordinate system, we see that c is symmetric with respect to the plane $y=0$, and therefore, the side view is covered twice. Hence, it is of degree 4. When computing the resultants of both equations in Eq. (6) with respect to y, we find the square of

$$q:\ (c_\lambda x + s_\lambda z)^2 z^2 - 2 s_\lambda c_\lambda \sin^2\delta xz$$
$$-(c_{2\lambda}\cos^2\delta + 2{s_\lambda}^2) z^2 + {s_\lambda}^2 \sin^2\delta = 0$$

as the equation of the right side view of the spherical conchoid.

The computations can be carried out by Maple. The *algcurves* package allows us to compute the singularities and the genus of an algebraic curve. We summarize the results in tables: Besides the degree we give the singularities in terms of homogeneous coordinates (with the homogenizing factor always in the first position), the invariants $[m,d,b]$, where m is the multiplicity, d is the δ-invariant, and b is the branching number.

Note that for an ordinary m-fold point the equation $m=b$ holds. In any other case we have $m>d$. The genus g of a planar algebraic curve c of degree n is the integer

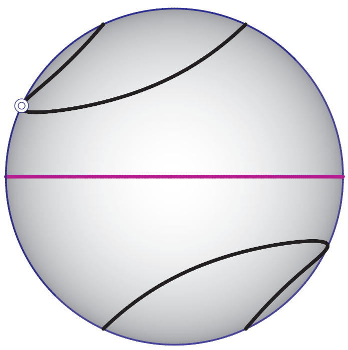

Figure 7: *Right side view of the spherical conchoid shows no singularity in the affine part. Note that the image of the focus is not singular.*

$$g = \frac{1}{2}(n-1)(n-2) - \sum_{\mathcal{S}} d_{\mathcal{S}},$$

where $\mathcal{S}$ is the set of singular points on c and $d_{\mathcal{S}}$ are the δ-invariants of all singularities on c. According to the Milnor-Jung formula, the δ-invariant d can be computed from the Milnor number μ and the branching number b of a singularity as $d=\frac{1}{2}(\mu+b-1)$. Thus, an ordinary k-fold point has invariants $[k, \frac{1}{2}k(k-1), k]$, see [2, 3].

We have to distinguish between two cases whether $\phi \neq \delta$ or $\phi = \delta$.

(1) Let us first assume that $\phi \neq \delta$:

The singularities of the right side view are given in Table 1. Since the genus equals zero, the curve showing up in the right side view is rational. Note that both singularities are ideal points of the $[x,z]$-plane. The point $(0:1:0)$ is an isolated tacnode, *i.e.*, a point where a pair of complex conjugate linear branches touches a real tangent at the real point $(0:1:0)$. The remaining singularity is an ordinary double point. The right side view of the spherical conchoid is displayed in Figure 7.

right side view		
	$\deg(c)=4$	
S_1	$(0:1:0)$	[2,2,2]
S_2	$(0:1:-\cot\phi)$	[2,1,2]
	$\text{genus}(c)=0$	

Table 1: *Singularities on the right side view.*

In Figure 8 we can observe another phenomenon which may not only appear in connection with spherical conchoids. The algebraic image curve carries points that are outside the silhouette of the unit sphere. Thus, these points cannot be the images of points on the spherical curve. The points on these parts of the curve are called *parasitic*.

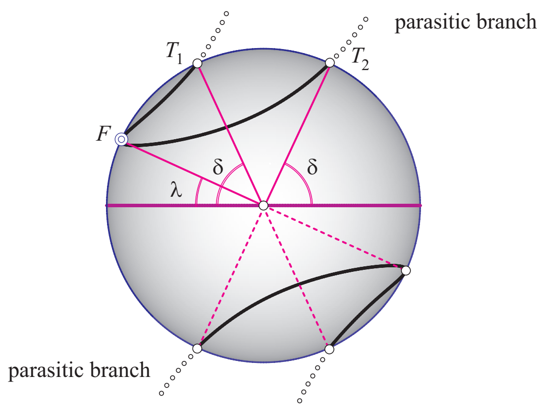

Figure 8: *Singularities on the principal views of spherical conchoids of lines.*

The front view shows a curve of degree eight (shown in Figure 9). It has a pair of complex conjugate ordinary double points $(0 : \pm i : c_\phi)$ at the ideal line of the $[y,z]$-plane. Further, there is an ideal 4-fold point with δ-invariant $d = 12$. Among the four singularities in the affine part of the curve (the part we can see in Figure 9) there are two tacnodes $(1 : 0 : \pm\sin\delta)$ which are the images of the top most points T_1 and T_2 of the conchoid on the front and back side of the sphere (cf. Figure 8). The fact that the two linear branches are in contact at the common image of the top most point is caused by the fact that the spherical conchoid has horizontal tangents at both points, T_1 and T_2. The image of the spherical focus F (antipodal pair) completes the list of singular points, cf. Table 2.

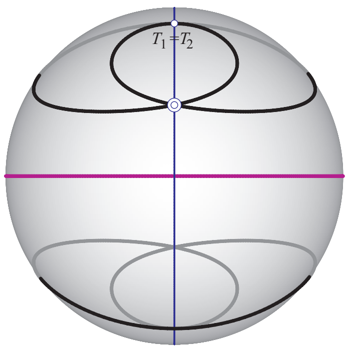

Figure 9: *The front view of the spherical conchoid shows up to four singularties.*

front view		
	$\deg(c) = 8$	
$S_{1,2}$	$(1 : 0 : \pm s_\phi)$	[2,1,2]
$T_{1,2}$	$(1 : 0 : \pm\sin\delta)$	[2,2,2]
S_5	$(0 : 1 : 0)$	[4,12,4]
$S_{6,7}$	$(0 : \pm i : c_\phi)$	[2,1,2]
	$\mathrm{genus}(c) = 1$	

Table 2: *Singularities on the front view.*

The top view has six real ordinary double points (see Figure 10). These are the image points $(\pm c_\phi, 0)$ of F and its antipode. Further, there are four ordinary double points at $(0, w)$ where w is a solution of the quartic equation

$$t^4 {s_\phi}^2 + t^2 \cos^2\delta({c_\phi}^2 - {s_\phi}^2) - {c_\phi}^2 \cos^2\delta = 0.$$

Two of these double points are real, two are complex conjugate. The ideal points $(0 : 1 : \pm i)$ of the $[x,y]$-plane are double points on the top view of the spherical conchoid. However, they are not ordinary double points, for their δ-invariant equals four. At these points the curve hyperosculates itself. Further, we find tacnodes at $(1 : \pm\cos\delta : 0)$ being the images of the front and back most points of the conchoid on the upper and lower hemisphere, see Figures 8 and 10. The singularities of the spherical conchoid's top view are listed in Table 3.

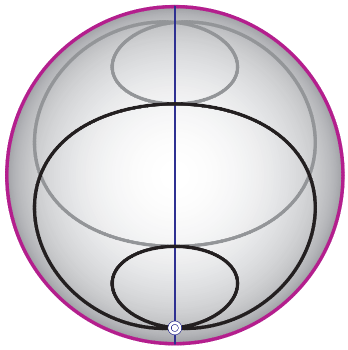

Figure 10: *The top view of the spherical conchoid shows up to six singular points.*

top view		
	$\deg(c) = 8$	
$S_{1,2}$	$(1 : \pm\cos\delta : 0)$	[2,2,2]
$S_{3,4}$	$(1 : \pm c_\phi : 0)$	[2,1,2]
$S_{5,6,7,8}$	$(1 : 0 : w)$	[2,1,2]
$S_{9,10}$	$(0 : 1 : \pm i)$	[2,4,2]
$S_{11,12}$	$(0 : \pm s_\phi : 1)$	[2,1,2]
	$\mathrm{genus}(c) = 1$	

Table 3: *Singularities on the top view.*

(2) Finally, we deal with the case $\phi = \delta$, *i.e.*, the curves with cusps.

We do not have to go through all the details. There are some minor changes in the types of some singularitiers showing up on the different views. Figure 11 shows the right side view, the front view, and the top view.

right side view		
	$\deg(c) = 4$	
S_1	$(0 : 1 : 0)$	[2,2,2]
	$\mathrm{genus}(c) = 1$	

Table 4: *Singularities of the right side view of the curve with cusp.*

The right side view of the spherical conchoid with cusp shows no singularity in the affine part. There is only one ideal point which is a tacnode, cf. Table 4. In this case the curve is of degree four, but nevertheless, it has genus 1 and is, therefore, elliptic since the only singularity has δ-invariant $d = 2$.

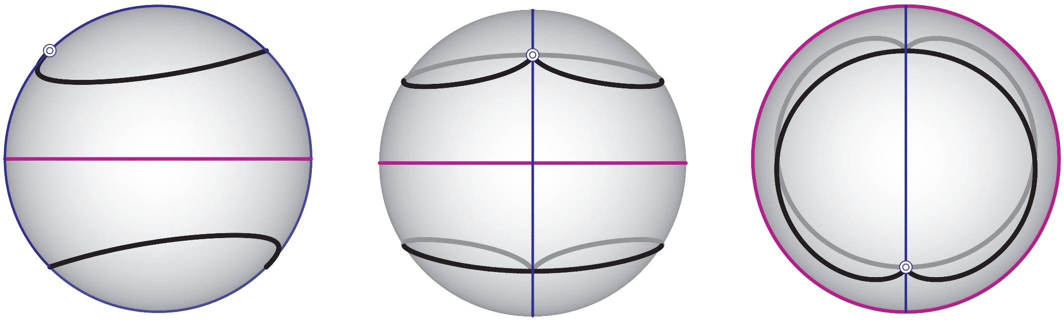

Figure 11: *From left to right: the right side view, the front view, and the top view of the spherical conchoid with cusp. The front and top view show triple points that are composed of cusps and linear branches.*

front view		
	$\deg(c) = 8$	
$S_{1,2}$	$(1 : \pm\sin\delta : 0)$	[3,3,2]
S_3	$(0 : 1 : 0)$	[4,12,4]
$S_{4,5}$	$(0 : \pm i : \cos\delta)$	[2,1,2]
	$\text{genus}(c) = 1$	

Table 5: *Singularities of the front view of the curve with cusp.*

The front view shows a pair of triple points. Here, the images of the top most points and the image of the focus F coincide. These triple points have δ-invariant $d = 3$ and branching number $b = 2$, cf. Table 5. Thus, these triple points are composed singularities, consisting of an ordinary cusp sitting on a linear branch. Further, there are two complex conjugate ideal singular points on the front view.

top view		
	$\deg(c) = 8$	
$S_{1,2}$	$(1 : \pm\cos\delta : 0)$	[3,3,2]
$S_{3,4}$	$(0 : 1 : \pm i)$	[2,1,2]
$S_{5,6}$	$(0 : \pm i\sin\delta : 1)$	[2,1,2]
$S_{7,8,9,10}$	$(1 : 0 : w)$	[2,1,2]
	$\text{genus}(c) = 1$	

Table 6: *Singularities of the top view of the curve with cusp.*

Again, the top view shows more singularities then any other view. The two triple points (see Table 6) showing up are composed singularities of the same type as those in the front view. Furthermore, there are four ordinary double points (two real ones and a pair of complex conjugate) at $(1 : 0 : w)$ where w is a solution of the quartic equation

$$t^4 {s_\phi}^2 - t^2\cos^2\delta(2-\cos^2\delta) - \cos^4\delta = 0.$$

According to the genus formula the front and top view are of genus 1, and thus, elliptic. □

There is a special type of spherical conchoid if we choose $\delta = \frac{\pi}{2}$. In this case the conchoid construction assigns to each point $L \in l$ the absolute polar point, *i.e.*, the *orthogonal point*. Hence, the two branches to $\delta = -\frac{\pi}{2}$ and to $\delta = \frac{\pi}{2}$ are identic since opposite points represent the same point. All the three principal views of *orthogonal conchoids* are curves of degree four. Figure 12 shows an axonometric view of some orthogonal conchoids together with the three principal views of them.

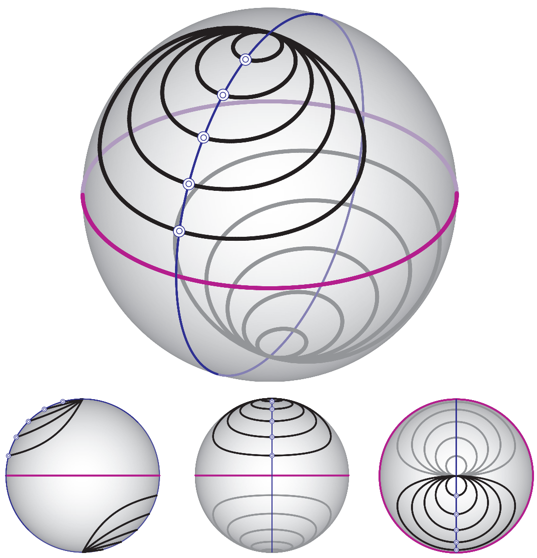

Figure 12: *Above: Some orthogonal conchoids of the equator. Below: Right side view, front view, and top view of some orthogonal conchoids.*

The curves in the right side view are two-fold hyperbolae in a pencil of the second kind with the images of the north and south pole as well as the ideal point of the x-axis for the base points.

2.2 Constructive approach

2.2.1 Planar and spherical tangents

The kinematic generation of conchoids allows us to construct tangents to conchoids in the plane, see for example [14]. The same holds true in the spherical case, cf. [6, 12].

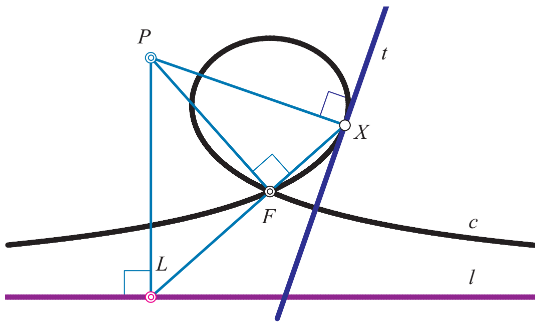

Figure 13: *The instantaneous pole P of the motion of the line $[L,X]$ with respect to the fixed system is found as the intersection of two normals.*

In Figure 13, the construction of the tangent to the planar conchoid c at some point X is shown. The kinematic generation of the curve shows the way: In order to find the instantaneous pole P of the motion of the line $[L,F]$ we observe that L is gliding on the line l, and thus, the pole of the motion of $[L,F]$ with respect to the fixed system l is the ideal point of the lines orthogonal to l. Since $[L,F]$ is gliding through F and rotating about F at the same time the instantaneous pole P is also contained in the line orthogonal to $[L,F]$ through F, see [14]. The construction also works at the double point since this is a singularity of the algebraic curve but not for the trace of X. The tangent t of c at X is orthogonal to $[P,X]$.

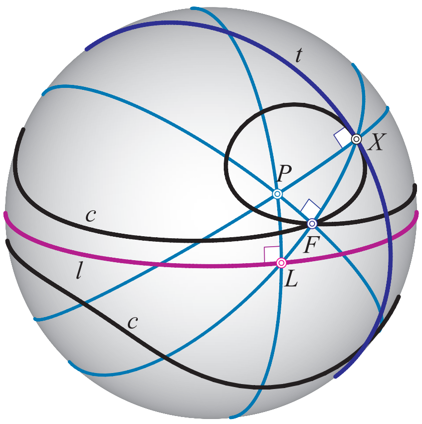

Figure 14: *The construction of the instantaneous pole P and the tangent t on the sphere.*

Figure 14 illustrates the construction of the tangent t to the spherical conchoid at some point X. Actually, the planar construction has to be translated into the spherical setting: We intersect the greatcircle orthogonal to the equator l through the point L with that greatcircle through F that is orthogonal to the greatcircle joining L and F and obtain the instantaneous spherical pole P (actually a pair of antipodal points). The spherical normal of the conchoid at X is the great circle joining X and P. Finally, the spherical tangent t is the greatcircle orthogonal to the spherical normal through the point X.

2.2.2 Planar and spherical osculating circles

Figure 15 shows the construction of the osculating circle o at a generic point X on a planar conchoid c. We use Bobillier's construction (see [14]). For that purpose we have to find two pairs of assigned points of the quadratic transformation that maps a point U to its center of curvature $U^{\star}$. The point L is moving on a straight line l, and thus, the center of its path is the ideal point $L^{\star}$ of all lines orthogonal to l. Further, we observe that the line $[L,F]$ is rotating about F while gliding through F. Thus, F is the envelope of $[L,F]$ and $F = A^{\star}$ is the center of curvature for the trace of the ideal point $A = [L,F]^{\perp}$ of all lines orthogonal to $[L,F]$. The two pairs $(L,L^{\star})$ and $(A,A^{\star})$ uniquely define the *quadratic curvature mapping*.

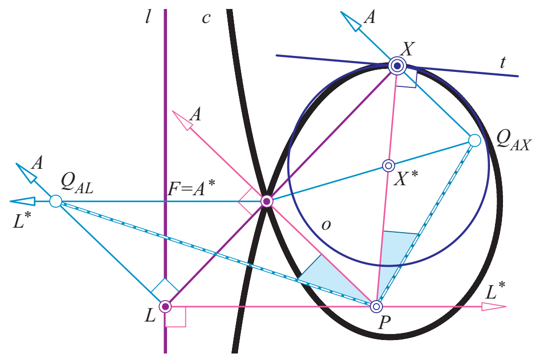

Figure 15: *Bobilier's construction simplifies in the case of the conchoid.*

Now, we can apply Bobbilier's construction to any of the pairs $(L,L^{\star})$ or $(A,A^{\star})$ in order to complete $(X,X^{\star})$ with the yet unknown point $X^{\star}$. Note that $[L,A]\cap[L^{\star},A^{\star}] =: Q_{AL}$ defines an auxiliary line $q_{AL} := [Q_{AL},P]$ with the property $\measuredangle(q_{AL},p) = \measuredangle(q_{AX},p)$ (after proper orientation), see [14], where p is the pole tangent, *i.e.*, the common tangent to the two polhodes at P.

In the case of the conchoid it is not necessary to construct the pole tangent p since we only have to add an angle as shown in Figure 15. On the auxiliary line q_{AX} we find the point $Q_{AX} := [A,X]\cap q_{AX}$, and finally, $X^{\star} = [X,P]\cap[A^{\star},Q_{AX}]$.

In order to find the spherical osculating circle o (as shown in Figure 16) we translate all the constructions done in the planar case to the sphere. We are allowed to do this since the quadratic curvature mapping can be lifted to the sphere. We consider the Euclidean unit sphere to be placed such that it touches the Euclidean plane (carrying the planar figure) at the instantaneous pole P. Then, we perform a gnomonic projection from the plane to the sphere. The center of the projection is the center of the sphere, and thus, the projectively extended Euclidean plane is mapped to the sphere model of projective geometry. The gnomonic projection is locally (around P) conformal, and therefore, the quadratic curvature mapping is lifted to that on the sphere.

Figure 16 shows the construction of the spherical center of curvature. At this point we shall remark that the spherical osculating circle o is not a greatcircle on Σ, except in those cases where X is a spherical point of inflection. The spherical radius of curvature equals the spherical distance of X and ist center of curvature $X^\star$.

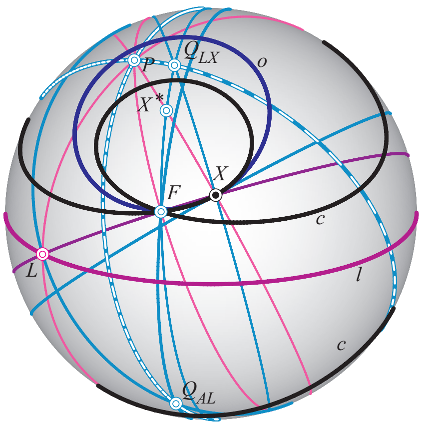

Figure 16: *The spherical version of Bobillier's construction yields the spherical center of curvature $X^\star$ for an arbitrary point X on the spherical conchoid.*

3 Conchoids of a circle

The construction of a conchoid is independent of the choice of the directrix curve. If we replace the line l by a circle, we obtain the conchoids of circles. The analytic as well as the constructive treatment of conchoids of circles does not differ that much from the affore mentioned types of conchoids. Since circles can also be found on a sphere, we can also find conchoids of circles on the sphere. We will not discuss the conchoids of a circle in the plane and on the sphere in all details. We shall just show that the equations of these special spherical curves can be derived in a similar way.

Conchoids of a circle in the Euclidean plane are of algebraic degree 6. Surprsingly, their spherical counter parts are of algebraic degree 8 (or, equivalently, of spherical degree 4), although we would expect them to be of degree 12. Some spherical conchoids of a circle are displayed in Figure 17.

The computation of an equation of spherical conchoids slightly differs from that of spherical conchoids of (spherical) lines.

Again, we assume that the focus F lies in $y = 0$ at latitude $\phi \in [0, \frac{\pi}{2}[$. It means no restriction to assume that F is a point on the upper hemisphere. There is a change in the directrix l which shall henceforth be the circle of latitiude $\beta \neq 0, \frac{\pi}{2}$. Thus, the directrix is given by

$$L(\lambda) = (c_\beta c_\lambda, c_\beta s_\lambda, s_\beta) \text{ with } \lambda \in [0, 2\pi[\quad (7)$$

(with $c_\beta := \cos\beta$ and $s_\beta := \sin\beta$). Here, we should remark that this restricts the class of spherical conchoids of a circle. In this case, there exists a greatcircle through F in a plane parallel to the plane of l which, in general, needs not be true. However, we deal with the simpler type.

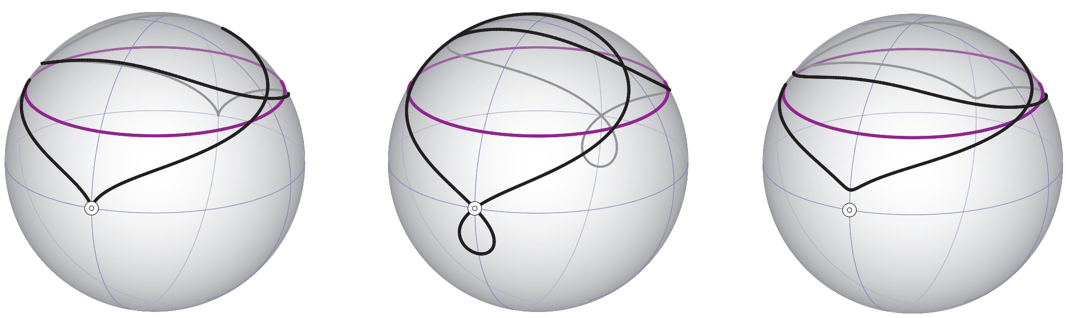

Figure 17: *Spherical conchoids of a circle show cusps, and two types of double points.*

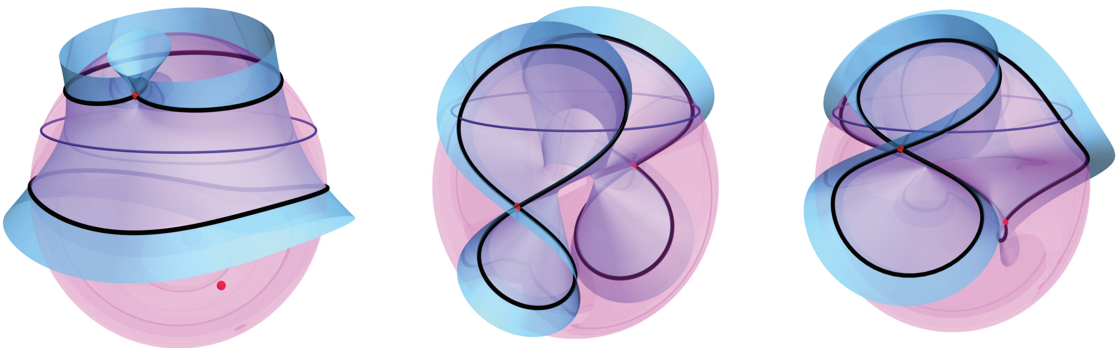

Figure 18: *Spherical conchoids as intersections of a quartic and the unit sphere.*

Let $X = (x,y,z)$ be the point on the conchoid of l with respect to F at spherical distance $\delta \in [0, \frac{\pi}{2}[$. Note that X is also a point on the unit sphere, and therefore, $x^2+y^2+z^2 = 1$ holds. The collinearity condition of F, X, and L from Eq. (4) now changes to

$$s_\phi s_\lambda\, x + (c_\phi t_\beta - c_\lambda s_\phi) y - c_\phi s_\lambda\, z = 0 \tag{8}$$

with $t_\beta := \tan\beta$. Between the point $l(t)$ on the directrix and the point X on the conchoid we measure the spherical distance δ which is a value with sign. Consequently, Eq. (5) modifies to

$$c_\lambda c_\beta\, x + s_\lambda c_\beta\, y + s_\beta\, z = \cos\delta. \tag{9}$$

Like in the case of the spherical conchoids of lines, we solve the system of linear equations (8), (9) with respect to c_λ and s_λ. Since ${c_\lambda}^2 + {s_\lambda}^2 = 1$ for all $\lambda \in \mathbb{C}$, we have the following two equations that have to be satisfied by the coordinates of a point on the spherical conchoid c of a circle l:

$$c:\left\{\begin{aligned} &(2{c_\phi}^2-1)x^2 - {s_\phi}^2 y^4\\ &+({c_\phi}^2-2{s_\phi}^2)x^2y^2\\ &+2c_\phi s_\phi(x^2z+y^2)x\\ &-4c_\phi s_\phi s_\beta \cos\delta(y+x)y^2\\ &+2s_\beta\cos\delta(2{c_\phi}^2-1)x^2z\\ &-2{s_\phi}^2 s_\beta\cos\delta y^2 z\\ &+((\cos^2\delta+{s_\beta}^2)(1-2{c_\phi}^2)-{c_\phi}^2)x^2\\ &+(\cos^2\delta(1+2{c_\phi}^2)+{s_\phi}^2{s_\beta}^2)y^2\\ &-2c_\phi s_\phi(\cos\delta^2+{s_\beta}^2)xz\\ &+2c_\phi s_\beta\cos\delta(2s_\phi x - c_\phi z)\\ &{c_\phi}^2(\cos^2\delta+{s_\beta}^2) = 0,\\ &x^2+y^2+z^2 = 1.\end{aligned}\right. \tag{10}$$

From that we can infer in analogy to Theorem 1:

Theorem 3. *The spherical conchoids of a circle at latitude* β *with respect to a point* F *is an algebraic curve of degree* 8 *or of spherical degree* 4*. The coordinates of all points on the spherical conchoid fulfill Equation* (10)*.*

The spherical conchoid of a circle is the intersection of a quartic surface with the sphere Σ. Some examples of the quartic surface are displayed in Figure 18. Like in the case of spherical and planar conchoids of lines, the spherical conchoids of circles can have cusps, isolated, and ordinary double points, see Figure 17.

Equations of the principal views (right side view, front view, top view) can be easily derived by eliminating coordinates (y, x, z) from the two equations given in Eq. (10). It is not necessary to go into all the details of the computations and discussions. They are similar to those in the previous section. Now, we can state (cf. Theorem 2):

Theorem 4. *The front and top view of spherical conchoids of circle are algebraic curves of degree* 8 *and genus* 1*, i.e., they are elliptic. The right side view is an elliptic quartic.*

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On the Isoptic Hypersurfaces in the n-Dimensional Euclidean Space

On the Isoptic Hypersurfaces in the n-Dimensional Euclidean Space

ABSTRACT

The theory of the isoptic curves is widely studied in the Euclidean plane $\mathbf{E}^2$ (see [1] and [13] and the references given there). The analogous question was investigated by the authors in the hyperbolic $\mathbf{H}^2$ and elliptic $\mathcal{E}^2$ planes (see [3], [4]), but in the higher dimensional spaces there is no result according to this topic.

In this paper we give a natural extension of the notion of the isoptic curves to the n-dimensional Euclidean space $\mathbf{E}^n$ $(n \geq 3)$ which are called isoptic hypersurfaces. We develope an algorithm to determine the isoptic hypersurface $\mathcal{H}_{\mathcal{D}}$ of an arbitrary $(n-1)$ dimensional compact parametric domain $\mathcal{D}$ lying in a hyperplane in the Euclidean n-space. We will determine the equation of the isoptic hypersurfaces of rectangles $\mathcal{D} \subset \mathbf{E}^2$ and visualize them with Wolfram Mathematica. Moreover, we will show some possible applications of the isoptic hypersurfaces.

Key words: isoptic curves, hypersurfaces, differential geometry, elliptic geometry

MSC2010: 53A05, 51N20, 68A05

O izooptičkim hiperplohama u n-dimenzionalnom euklidskom prostoru

SAŽETAK

Teorija o izooptičkim krivuljama dosta se proučava u euklidskoj ravnini $\mathbf{E}^2$ (vidi [1] i [13] te u referencama koje se tamo mogu naći). Autori su proučavali analogno pitanje u hiperboličkoj $\mathbf{H}^2$ i eliptičkoj ravnini $\mathcal{E}^2$ (vidi [3], [4]), međutim u višedimenzionalnim prostorima nema rezultata vezanih za ovu temu.

U ovom članku dajemo prirodno proširenje pojma izooptičkih krivulja na n-dimenzionalni euklidski prostor $\mathbf{E}^n$ $(n \geq 3)$ koje zovemo izooptičke hiperplohe. Razvijamo algoritam kojim određujemo izooptičke hiperplohe $\mathcal{H}_{\mathcal{D}}$ proizvoljne $(n-1)$-dimenzionalne kompaktne parametarske domene $\mathcal{D}$ koja leži u hiperravnini u n-dimenzionalnom euklidskom prostoru.

Odredit ćemo jednadžbu izooptičkih hiperploha pravokutnika $\mathcal{D} \subset \mathbf{E}^2$ i vizualizirati ih koristeći program Wolfram Mathematica. Štoviše, pokazat ćemo neke moguće primjene izoptičkih hiperploha.

Ključne riječi: izooptičke krivulje, hiperplohe, diferencijalna geometrija, eliptička geometrija

1 Introduction

Definition 1 *Let X be one of the constant curvature plane geometries $\mathbf{E}^2$, $\mathbf{H}^2$, $\mathcal{E}^2$. The isoptic curve $\mathcal{C}^\alpha$ of an arbitrary given plane curve $\mathcal{C}$ of X is the locus of points P where $\mathcal{C}$ is seen under a given fixed angle α $(0 < \alpha < \pi)$.*

An isoptic curve formed from the locus of two tangents meeting at right angle $(\alpha = \frac{\pi}{2})$ are called orthoptic curve. The name isoptic curve was suggested by C. Taylor in his work [12] in 1884.

In the Euclidean plane $\mathbf{E}^2$ the easiest case if $\mathcal{C}$ is a line segment then the set of all points (locus) for which a line segment can be seen under angle α contains two arcs in both half-plane of the line segment, each are with central angle 2α. In the special case $\alpha = \frac{\pi}{2}$, we get exactly the so-called Thales circle (without the endpoints of the given segment) with center the middle of the line segment.

Further curves appearing as isoptic curves are well studied in the Euclidean plane geometry $\mathbf{E}^2$, see e.g. [8],[13]. In [1] and [2] can be seen the isoptic curves of the closed, strictly convex curves, using their support function. The papers [14] and [15] deal with curves having a circle or an ellipse for an isoptic curve. Isoptic curves of conic sections have been studied in [6], [8] and [11]. A lot of papers concentrate on the properties of the isoptics e.g. [9], [7], [10] and the references given there.

In the hyperbolic and elliptic planes $\mathbf{H}^2$ and $\mathcal{E}^2$ the isoptic curves of segments and proper conic sections are determined by the authors ([3], [4], [5]).

In the higher dimensions by our best knowledge there are no results in this topic thus in this paper we give a natural extension of the Definition 1 in the n-dimensional Euclidean space $\mathbf{E}^n$. Moreover, we develope a procedure to determine the isoptic hypersurface $\mathcal{H}_{\mathcal{D}}$ of an arbitrary $(n-1)$ dimensional compact parametric domain $\mathcal{D}$ lying in a hyperplane in the Euclidean space. We will determine the equation of the isoptic hypersurfaces (see Theorem 1) of rectangles $\mathcal{D} \subset \mathbf{E}^2$ and visualize them with Wolfram Mathematica (see Fig. 2-3). Moreover, we will show some possible applications of the isoptic hypersurfaces.

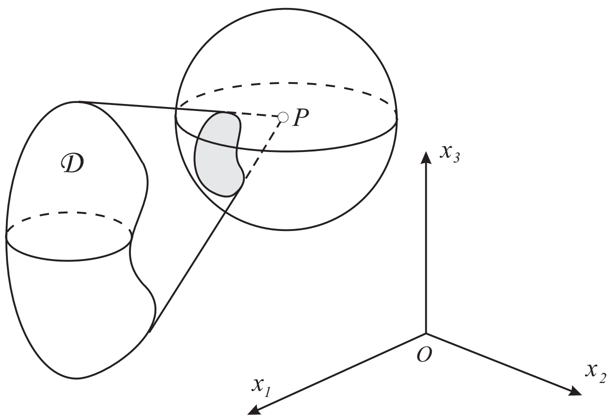

Figure 1: *Projection of a compact domain $\mathcal{D}$ to unit sphere in $\mathbf{E}^3$*

2 Isoptic hypersurface of a compact domain lying in a hyperplane of $\mathbf{E}^n$

In Definition 1 we have considered that, the angle can be measured by the arc length on the unit circle around the point. From this statement, Definition 1 can be extended to the n-dimensional Euclidean space $\mathbf{E}^n$.

Definition 2 *The isoptic hypersurface $\mathcal{H}_{\mathcal{D}}^{\alpha}$ in $\mathbf{E}^n$ $(n \geq 3)$ of an arbitrary d dimensional compact parametric domain $\mathcal{D}(2 \leq d \leq n)$ is the locus of points P where the measure of the projection of $\mathcal{D}$ onto the unit $(n-1)$-sphere around P is a given fixed value α $(0 < \alpha < \frac{\pi^{\frac{n-1}{2}}}{\Gamma(\frac{n-1}{2})})$ $\left[\Gamma(t) = \int_0^\infty x^{t-1}e^{-x}dt\right]$ (see Fig. 1).*

We consider a compact parametric $(n-1)$ $(n \geq 3)$-dimensional domain $\mathcal{D}$ lying in a hyperplane of $\mathbf{E}^n$. We can suppose the next form of parametrization:
$\phi(x,y)$ plane surface

$$\widetilde{\boldsymbol{\phi}}(u_1,u_2,\dots,u_{n-1}) = \begin{pmatrix} \widetilde{f}_1(u_1,u_2,\dots,u_{n-1}) \\ \widetilde{f}_2(u_1,u_2,\dots,u_{n-1}) \\ \vdots \\ \widetilde{f}_{n-1}(u_1,u_2,\dots,u_{n-1}) \\ 0 \end{pmatrix}, \tag{2.1}$$

where $u_i \in [a_i,b_i]$, $(a_i,b_i \in \mathbb{R})$, $(i = 1,\dots,n-1)$.

For the point $P(x_1,x_2,\dots,x_n) = P(\mathbf{x})$ the inequality $x_n > 0$ will be assumed. Projecting the surface onto the unit sphere with centre P, we have the following parametrization:

$$\boldsymbol{\phi}(u_1,u_2,\dots,u_{n-1}) =$$
$$= \begin{pmatrix} f_1(u_1,u_2,\dots,u_{n-1}) \\ f_2(u_1,u_2,\dots,u_{n-1}) \\ \vdots \\ f_n(u_1,u_2,\dots,u_{n-1}) \end{pmatrix} = \begin{pmatrix} f_1(\mathbf{u}) \\ f_2(\mathbf{u}) \\ \vdots \\ f_n(\mathbf{u}) \end{pmatrix}. \tag{2.2}$$

Here, if $i \neq n$ we have

$$f_i(\mathbf{u}) = \frac{\widetilde{f}_i(u_1,\dots,u_{n-1}) - x_1}{\sqrt{(\widetilde{f}_1(u_1,\dots,u_{n-1}) - x_1)^2 + \dots + (\widetilde{f}_{n-1}(u_1,\dots,u_{n-1}) - x_{n-1})^2 + (x_n)^2}},$$

else $(i = n)$

$$f_n(\mathbf{u}) = \frac{-x_n}{\sqrt{(\widetilde{f}_1(u_1,\dots,u_{n-1}) - x_1)^2 + \dots + (\widetilde{f}_{n-1}(u_1,\dots,u_{n-1}) - x_{n-1})^2 + (x_n)^2}}.$$

Now, it is well known, that the measure of the $n-1$-surface can be calculated using the forumla below:

$$S(x_1,x_2,\dots,x_n) =$$
$$= \int_{a_1}^{b_1}\int_{a_2}^{b_2}\dots\int_{a_{n-1}}^{b_{n-1}} \sqrt{\det G}\ \mathrm{d}u_{n-1}\,\mathrm{d}u_{n-2}\dots\mathrm{d}u_1 \tag{2.3}$$

by successive integration, where

$$G = J^T J =$$
$$= \begin{pmatrix} \frac{\partial f_1}{\partial u_1} & \frac{\partial f_1}{\partial u_2} & \dots & \frac{\partial f_1}{\partial u_{n-1}} \\ \frac{\partial f_2}{\partial u_1} & \frac{\partial f_1}{\partial u_2} & \dots & \frac{\partial f_1}{\partial u_{n-1}} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial f_{n-1}}{\partial u_1} & \frac{\partial f_{n-1}}{\partial u_2} & \dots & \frac{\partial f_{n-1}}{\partial u_{n-1}} \end{pmatrix}^T \cdot$$
$$\cdot \begin{pmatrix} \frac{\partial f_1}{\partial u_1} & \frac{\partial f_1}{\partial u_2} & \dots & \frac{\partial f_1}{\partial u_{n-1}} \\ \frac{\partial f_2}{\partial u_1} & \frac{\partial f_1}{\partial u_2} & \dots & \frac{\partial f_1}{\partial u_{n-1}} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial f_{n-1}}{\partial u_1} & \frac{\partial f_{n-1}}{\partial u_2} & \dots & \frac{\partial f_{n-1}}{\partial u_{n-1}} \end{pmatrix}.$$

The isoptic hypersurface $\mathcal{H}_{\mathcal{D}}^{\alpha}$ by the Definition 2 is the following:

$$\mathcal{H}_{\mathcal{D}}^{\alpha} = \{\mathbf{x} \in \mathbf{E}^n | \alpha = S(x_1,x_2,\dots,x_n)\}$$

In the general case, the isoptic hypersurface can be determined only by numerical computations. In the next section we show an explicite application of our algorithm.

3 Isoptic surface of the rectangle

Now, let suppose that $n = 3$ and $\mathcal{D} \subset \mathbf{E}^2$ is a rectangle lying in the $[x,y]$ plane in a given Cartesian coordinate system. Moreover, we can assume, that it is centered, so the parametrization is the following:

$$\widetilde{\phi}(x,y) = \begin{pmatrix} x \\ y \\ 0 \end{pmatrix}, \tag{3.1}$$

where $x \in [-a,a]$ and $y \in [-b,b]$ $(a,b \in \mathbf{R})$. And the parametrization of the projection from $P(x_0,y_0,z_0)$ can be seen below:

$$\phi(x,y) = \begin{pmatrix} \frac{x-x_0}{\sqrt{(x-x_0)^2+(y-y_0)^2+z_0^2}} \\ \frac{y-y_0}{\sqrt{(x-x_0)^2+(y-y_0)^2+z_0^2}} \\ \frac{-z_0}{\sqrt{(x-x_0)^2+(y-y_0)^2+z_0^2}} \end{pmatrix}. \tag{3.2}$$

Remark 1 *It is clear, that the computations is similar if $\mathcal{D}$ is a normal domain concerning to x or y on the plane. The difference is appered only on the boundaries of the integrals.*

Now, we need the partial derivatives, to calculate the surface area:

$$\phi_x(x,y) = \begin{pmatrix} \frac{(y-y_0)^2+z_0^2}{\left((x-x_0)^2+(y-y_0)^2+z_0^2\right)^{3/2}} \\ \frac{-(x-x_0)(y-y_0)}{\left((x-x_0)^2+(y-y_0)^2+z_0^2\right)^{3/2}} \\ \frac{z_0(x-x_0)}{\left((x-x_0)^2+(y-y_0)^2+z_0^2\right)^{3/2}} \end{pmatrix},$$

$$\phi_y(x,y) = \begin{pmatrix} \frac{-(x-x_0)(y-y_0)}{\left((x-x_0)^2+(y-y_0)^2+z_0^2\right)^{3/2}} \\ \frac{(x-x_0)^2+z_0^2}{\left((x-x_0)^2+(y-y_0)^2+z_0^2\right)^{3/2}} \\ \frac{z_0(y-y_0)}{\left((x-x_0)^2+(y-y_0)^2+z_0^2\right)^{3/2}} \end{pmatrix}$$

/medskip
Taking the cross product of the vectors above, we obtain:

$$\phi_x(x,y) \times \phi_y(x,y) = \begin{pmatrix} \frac{z_0(x_0-x)}{\left((x-x_0)^2+(y-y_0)^2+z_0^2\right)^2} \\ \frac{z_0(y_0-y)}{\left((x-x_0)^2+(y-y_0)^2+z_0^2\right)^2} \\ \frac{z_0^2}{\left((x-x_0)^2+(y-y_0)^2+z_0^2\right)^2} \end{pmatrix}$$

Now we can substitute $\left|\phi_x(x,y) \times \phi_y(x,y)\right|$ into formula (2.3) to get the spatial angle:

$S(x_0,y_0,z_0) =$

$$\int_{-a}^{+a}\int_{-b}^{+b} \frac{|z_0|}{\left((x-x_0)^2+(y-y_0)^2+z_0^2\right)^{\frac{3}{2}}} \mathrm{d}y\,\mathrm{d}x = \tag{3.3}$$

$$\arctan\left(\frac{(a-x_0)(b-y_0)}{z_0\sqrt{(a-x_0)^2+(b-y_0)^2+z_0^2}}\right) +$$
$$\arctan\left(\frac{(a+x_0)(b-y_0)}{z_0\sqrt{(a+x_0)^2+(b-y_0)^2+z_0^2}}\right) +$$
$$\arctan\left(\frac{(a-x_0)(b+y_0)}{z_0\sqrt{(a-x_0)^2+(b+y_0)^2+z_0^2}}\right) +$$
$$\arctan\left(\frac{(a+x_0)(b+y_0)}{z_0\sqrt{(a+x_0)^2+(b+y_0)^2+z_0^2}}\right).$$

Remark 2 *It is easy to see, if $a \to \infty$ and $b \to \infty$, then the angle tendst to 2π for every z_0. This implies some kind of elliptic properties. The normalised cross pruduct of the two partial derivatives can be interpreted as a weight function on this elliptic plane. Now, if we have a domain on the plane, we can integrate this function over the domain, to obtain the angle. But the symbolic integral for a given domain almost never works, so in this case, it is suggested also, to use numerical approach.*

Using the results abowe, we can claim the following theorem:

Theorem 1 *Let us given a rectangle $\mathcal{D} \subset \mathbf{E}^2$ lying in the $[x,y]$ plane in a given Cartesian coordinate system. Moreover, we can assume, that it is centered at the origin with sides $(2a,2b)$. Then the isoptic surface for a given spatial angle α $(0 < \alpha < 2\pi)$ is determined by the following equation:*

$$\alpha = \arctan\left(\frac{(a-x)(b-y)}{z\sqrt{(a-x)^2+(b-y)^2+z^2}}\right) +$$
$$\arctan\left(\frac{(a+x)(b-y)}{z\sqrt{(a-x)^2+(b-y)^2+z^2}}\right) +$$
$$\arctan\left(\frac{(a-x)(b+y)}{z\sqrt{(a-x)^2+(b-y)^2+z^2}}\right) +$$
$$\arctan\left(\frac{(a+x)(b+y)}{z\sqrt{(a-x)^2+(b-y)^2+z^2}}\right).$$

In the following figures, there can be seen the isoptic surface of the rectangle:

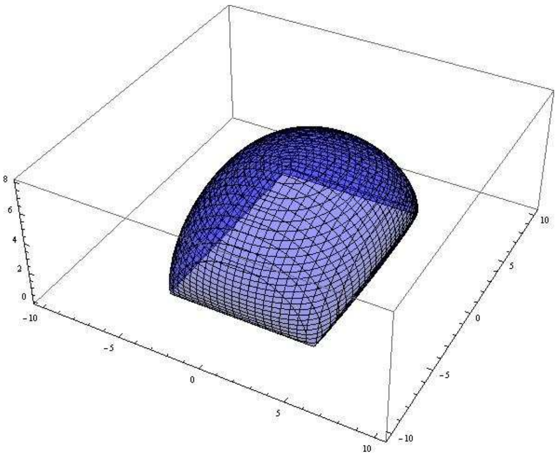

Figure 2: $2a = 9$, $2b = 13$, $\alpha = \frac{\pi}{2}$

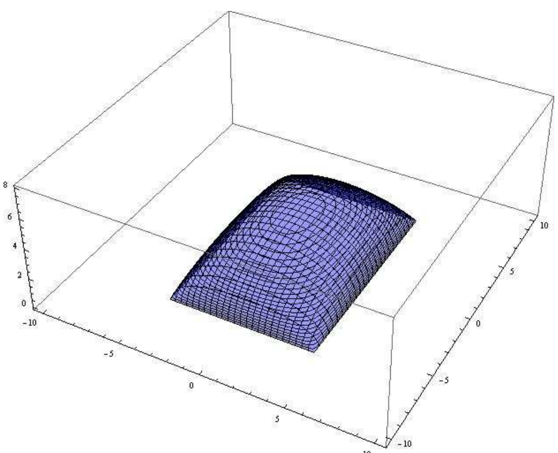

Figure 3: $2a = 9$, $2b = 13$, $\alpha = \pi$

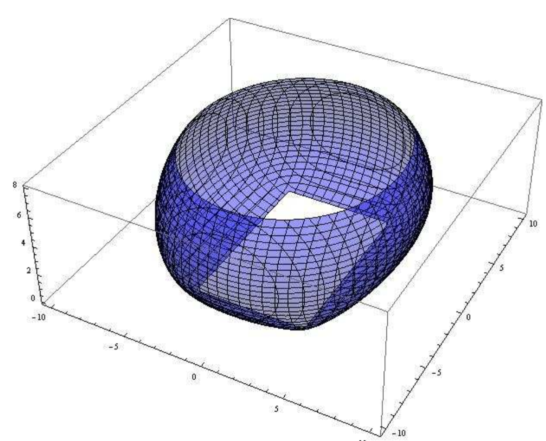

Figure 4: $2a = 7$, $2b = 11$, $\alpha = \frac{\pi}{6}$

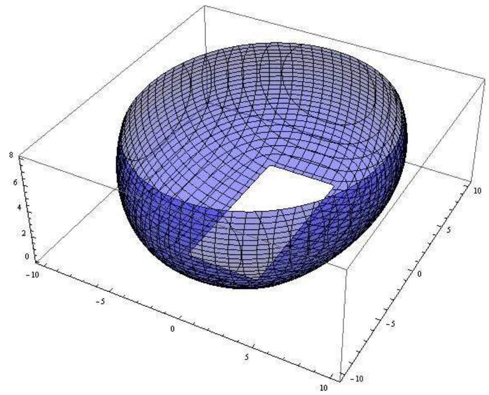

Figure 5: $2a = 11$, $2b = 5$, $\alpha = \frac{\pi}{12}$

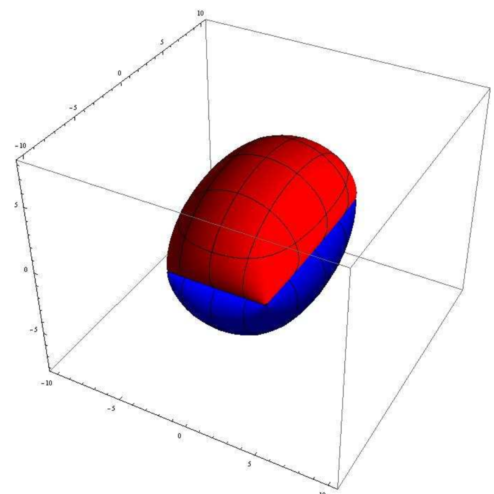

Figure 6: $2a = 7$, $2b = 13$, $\alpha = \frac{\pi}{2}$ *(both half-spaces)*

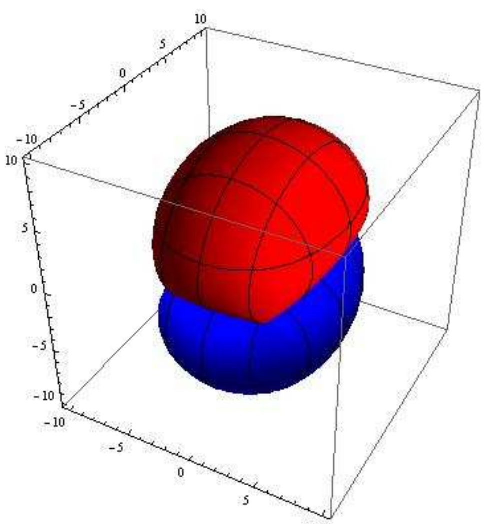

Figure 7: $2a = 7$, $2b = 13$, $\alpha = \frac{\pi}{4}$ *(both half-spaces)*

Remark 3 *The figures show us, that this topic has severeal applications, for example designing stadiums, theaters or cinemas. It can be interesting, if we have a stadium, which has the property, that from every seat on the grandstand, the field can be seen under a same angle.*

Designing a lecture hall, it is important, that the screen or the blackboard is clearly visible from every seat. In this case, the isoptic lecture hall is not feasible, but it can be optimized.

Figure 8: *MetLife Stadium:* `http://www.bonjovi.pl/forum/topics58/25-27072013-east-rutherford-vt3278.htm`

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Introduction to the Planimetry of the Quasi-Hyperbolic Plane

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ABSTRACT

The quasi-hyperbolic plane is one of nine projective-metric planes where the absolute figure is the ordered triple $\{j_1, j_2, F\}$, consisting of a pair of real lines j_1 and j_2 through the real point F. In this paper some basic geometric notions of the quasi-hyperbolic plane are introduced. Also the classification of qh-conics in the quasi-hyperbolic plane with respect to their position to the absolute figure is given. The notions concerning the qh-conic are introduced and some selected constructions for qh-conics are presented.

Key words: quasi-hyperbolic plane, perpendicular points, central line, qh-conics classification, osculating qh-circle

MSC2010: 51A05, 51M10, 51M15

Uvod u planimetriju kvazi-hiperboličke ravnine

SAŽETAK

Kvazihiperbolička ravnina je jedna od devet projektivno metričkih ravnina kojoj je apsolutna figura uređena trojka $\{j_1, j_2, F\}$, gdje su j_1 i j_2 realni pravci koji se sijeku u realnoj točki F. U ovom članku uvodimo neke osnovne pojmove za kvazihiperboličku ravninu, te dajemo klasifikaciju konika u odnosu na njihov položaj prema apsolutnoj figuri. Nadalje, uvesti ćemo pojmove vezane uz konike u kvazihiperboličkoj ravnini i pokazati nekoliko izabranih konstrukcija vezanih uz konike.

Ključne riječi: kvazihiperbolička ravnina, okomite točke, centrala, klasifikacija qh-konika, oskulacijske qh-kružnice

1 Introduction

In the second half of the 19th century F. Klein opened a new field for geometers with his famous Erlangen program which is the study of the properties of a space which are invariant under a given group of transformations. Klein was influenced by some earlier research of A. Cayley, so today it is known that there exist nine geometries in plane with projective metric on a line and on a pencil of lines which are denoted as Cayley-Klein projective metrics. Hence, these plane geometries differ according to the type of the measure of distance between points and measure of angles which can be parabolic, hyperbolic, or elliptic. Furthermore, each of these geometries can be embedded in the real projective plane $\mathbb{P}_2(\mathbb{R})$ where an absolute figure is given as non-degenerated or degenerated conic [4], [5], [12] (for space and n-dimension see [11]).
In this article the geometry, denoted as *quasi-hyperbolic*, with hyperbolic measure of distance and parabolic measure of angle will be presented.

2 Basic notation in the quasi-hyperbolic plane

In the quasi-hyperbolic plane (further in text qh-plane) the metric is induced by a real degenerated conic i.e. a pair of real lines j_1 and j_2 incidental with the real point F. The lines j_1 and j_2 are called the *absolute lines*, while the point F is called the *absolute point*. In the Cayley-Klein model of the qh-plane only the points, lines and segments inside of one projective angle between the absolute lines are observed. In this article all points and lines of the qh-plane embedded in the real projective plane $\mathbb{P}_2(\mathbb{R})$ are observed.

There are three different positions for the absolute triple $\{j_1, j_2, F\}$: neither of the absolute elements are at infinity, only the absolute point is at infinity and the absolute point and one absolute line are at infinity (see Fig. 1). The first position of the absolute triple is used for constructions in this article.

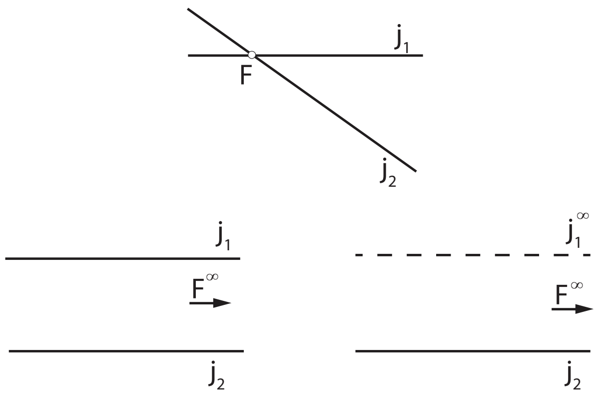

Figure 1

For the points and the lines in the qh-plane the following terms are defined:

• *isotropic lines* - the lines incidental with the absolute point F,

• *isotropic points* - the points incidental with one of the absolute lines j_1 or j_2,

• *parallel lines* - two lines which intersect at an isotropic point,

• *parallel points* - two points incidental with an isotropic line,

• *perpendicular lines* - if at least one of two lines is an isotropic line,

• *perpendicular points* - two points (A and B) that lie on a pair of isotropic lines (a and b) that are in harmonic relation with the absolute lines j_1 and j_2.

Furthermore, an involution of pencil of lines (F) having the absolute lines for double lines is called the *absolute involution*, denoted as I_{QH}. This is a hyperbolic involution on the pencil (F) where every pair of corresponding lines is in a harmonic relation with the double lines j_1 and j_2 ([1], p.244-245, [6], p.46). Notice that every pair of perpendicular points lie on a pair of I_{QH} corresponding lines. Hence, the perpendicularity of points in qh-plane is determined by the absolute involution, therefore I_{QH} is a circular involution in the qh-plane ([7], p.75).

Remark. Any two isotropic points on the same absolute line are perpendicular and parallel. Any two lines from a pencil (F) are perpendicular and parallel.

3 Qh-conics classification

There are nine types of regular qh-conics classified according to their position with respect to the absolute figure:

• *qh-hyperbola* - a qh-conic which has a pair of real tangent lines from the absolute point,

- *hyperbola of type 1* (h_1) - intersects each absolute line in a pair of real and distinct points,
- *hyperbola of type 2* (h_2) - intersects one absolute line in a pair of real and distinct points and another absolute line in a pair of imaginary points,
- *hyperbola of type 3* (h_3) - intersects each absolute line in a pair of imaginary points,
- *special hyperbola of type 1* (h_{s1}) - one absolute line is a tangent line and another absolute line intersects the qh-conic in a pair of real and distinct points,
- *special hyperbola of type 2* (h_{s2}) - one absolute line is a tangent line and another absolute line intersects the qh-conic in a pair of imaginary points,

• *qh-ellipse* (e) - a qh-conic (imaginary or real) which has a pair of imaginary tangent lines from the absolute point,

• *qh-parabola* (p) - a qh-conic passing through the absolute point i.e. both isotropic tangent lines coincide,

- *special parabola* (p_s) - a qh-parabola whose isotropic tangent is an absolute line,

• *qh-circle* (k) - a qh-conic for which the tangents from the absolute point are the absolute lines.

In the projective model of the qh-plane every type of a qh-conic can be represented with the Euclidean circle without loss of generality (see Fig. 2). This fact simplifies the constructions in the qh-plane.

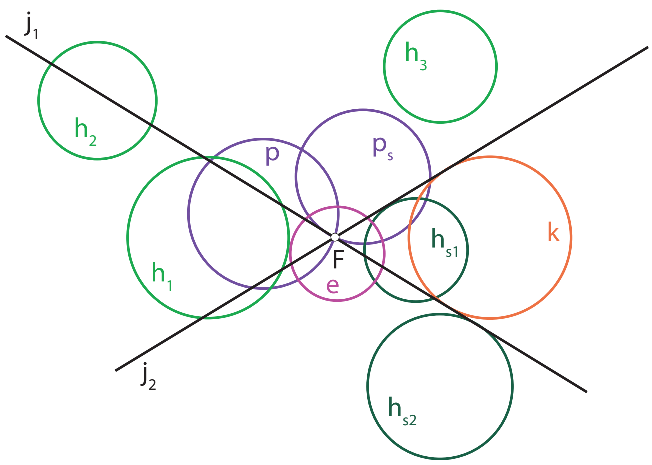

Figure 2

Furthermore every qh-conic q, except qh-parabolae, induces an involution ϕ_q on the pencil (F) where the double lines are the isotropic tangents of the qh-conic q, and the corresponding lines of the involution ϕ_q are called *conjugate lines*. Notice that every qh-ellipse induces an elliptic involution, every qh-hyperbola induces a hyperbolic involution and every qh-circle induces an involution that coincides with the absolute involution I_{QH}.

Remark. A qh-conic is called *equiform* if the isotropic tangent lines of the qh-conic are in harmonic relation with the absolute lines j_1 and j_2. In terms of the above mentioned involutions a qh-conic q is equiform if the absolute involution I_{QH} is commutative with the involution ϕ_q induced by the qh-conic q. Notice that only qh-ellipses, qh-hyperbolae of type 2 and qh-circles can be equiform [2], [3].

In the following some basic notions related to a qh-conic in the qh-plane are defined:

• The polar line of the absolute point F with respect to a qh-conic is called the *central line* c or the *major diameter* of the qh-conic (see Fig. 3). All qh-conics, except qh-parabolas, have a non-isotropic central line. The central line of a qh-parabola is its isotropic tangent line, while for the special parabola it is an absolute line.

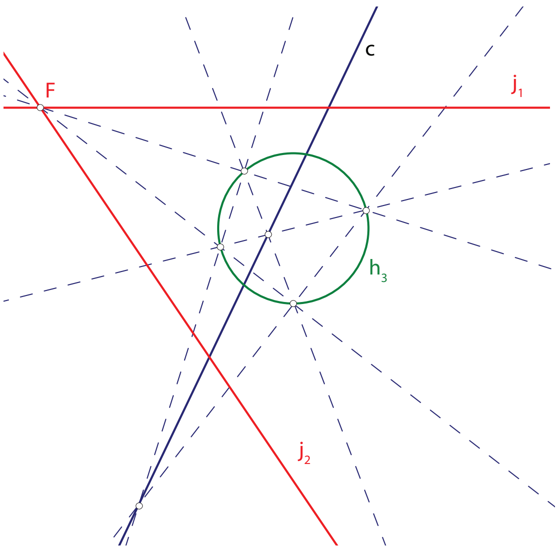

Figure 3

• The *directrices* of a qh-conic are (non-absolute) lines incident with the isotropic points of the qh-conic, i.e. lines incidental with the intersection points of the qh-conic with the absolute lines j_1 and j_2. A qh-conic can have none, one, two or four directrices f_i, $i \in \{1,2,3,4\}$ (see Fig. 4).

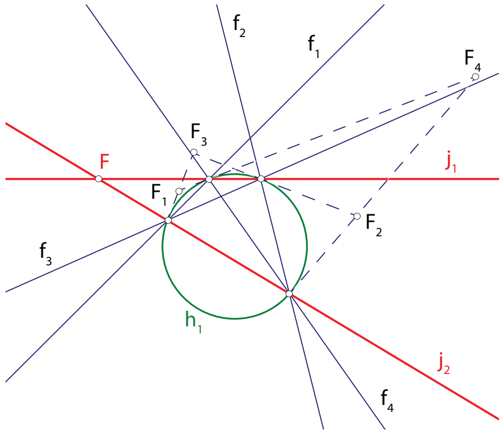

Figure 4

• The pole of the directrix with respect to a qh-conic is called a *focus* of the qh-conic. The number of foci F_i, $i \in \{1,2,3,4\}$, is equal to the number of directrices (see Fig. 4).

• The lines that are incident with the opposite foci are called *isotropic diameters* of a qh-conic (see Fig. 5). Especially for the qh-circles, which have one focus, the isotropic diameters are the lines of the pencil (F). Hence a qh-conic can have none, one, two or infinitely many isotropic diameters o_i, $i \in \{1,2\}$.

• The *qh-centers* of a qh-conic are the points of intersection of the isotropic diameters and the central line of the qh-conic. A qh-conic can have none, one, two or infinitely many qh-centers S_i, $i \in \{1,2\}$ (see Fig. 5).

• The intersection points of a qh-conic with its isotropic diameters are called *vertices* of the qh-conic (see Fig. 5). A qh-conic can have four, two, one or none vertices T_i, $i \in \{1,2,3,4\}$.

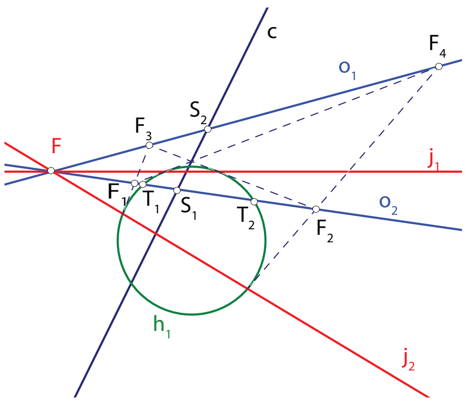

Figure 5

The absolute involution I_{QH} can be observed as a point range involution on any non-isotropic line, hence it can be observed on the central line of a qh-conic, except for qh-parabolae. Also the involution ϕ_q on a pencil (F) induced by a qh-conic q can be observed as the involution φ_q of a point range on the central line c of the qh-conic q, and two corresponding points of involution φ_q are called conjugate points. Therefore, the qh-centers for the qh-ellipses and qh-hyperbolae can be found as a pair of perpendicular and conjugate points on the central line, and the isotropic diameters as the perpendicular and conjugate lines of the pencil (F). The construction will be shown later. Notice that because the involution induced by a qh-circle coincides with the absolute involution all pairs of conjugate points on the central line of the qh-circle are perpendicular points. Hence any point on the central line is its center and every line of the pencil (F) is its isotropic diameter.

Aforementioned qh-conics and notions can be summarized in the following table:

Qh-Conic	Directrix	Focus	Isotropic diameter	Center	Vertex
Ellipse e	4 real	4 real	2 real	2 real	4 real
Hyperbola h_1	4 real	4 real	2 real	2 real	2 real + 2 imaginary
Hyperbola h_2	4 imaginary	4 imaginary	2 imaginary	2 imaginary	4 imaginary
Hyperbola h_3	4 imaginary	4 imaginary	2 real	2 real	2 real + 2 imaginary
Parabola p	2 real	2 real	1 real	1 real	2 real
Special parabola p_s	0	0	0	0	0
Special hyperbola h_{s1}	2 real	2 real	1 real	1 real	1 real
Special hyperbola h_{s2}	2 imaginary	2 imaginary	1 real	1 real	1 real
Circle c	1 real	1 real	infinite	infinite	0

Table 1

For parabolae and special hyperbolae see figure 6.

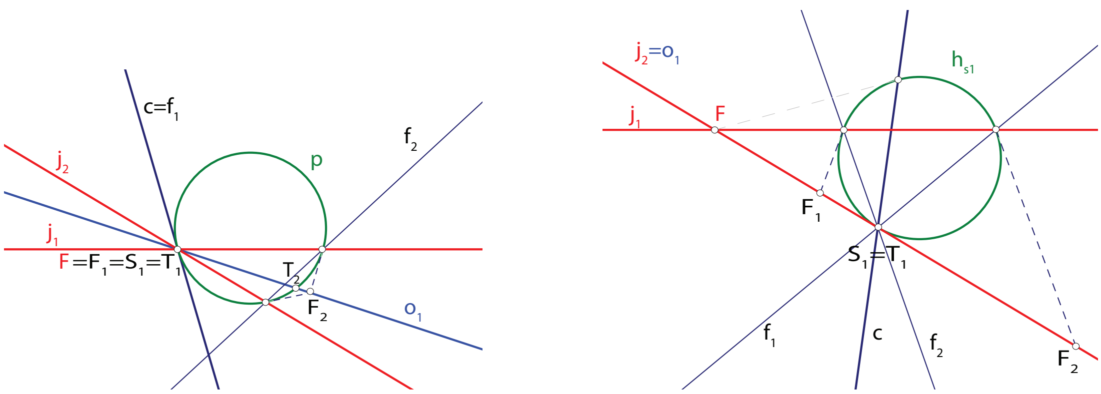

Figure 6

Remark. The qh-plane is dual to the pseudo-Euclidean (Minkowski) plane where the metric is induced by a real line and two real points incident with it. Therefore the notions defined above can be explained as duals of the Minkowski plane.

The conics in pseudo-Euclidean plane (pe-plane) are classified in nine subtypes, hence the classification of qh-conics was based on [3], [9], [10]. Furthermore, the aforementioned elements for qh-conics can be presented as follows:

- the central line is a dual of the center of a conic in the pe-plane,
- the directrices are a dual of the foci of a conic in the pe-plane,
- the foci are a dual of the directrices of a conic in the pe-plane,
- the qh-centers are dual to the axes of a conic in the pe-plane.

The dual of the isotropic diameters are the intersections of the axes with the absolute line, but they were not of special interest in the pe-plane. Also the dual of the vertices in qh-plane are the tangents to the conic in pe-plane from the above mentioned intersections. It should be emphasized that the dual of the vertices in pe-plane are tangents to the qh-plane from the qh-centers. Since the axes in pe-plane and qh-centers in qh-plane are dual, therefore it was not chosen in this article to observe the vertices of a qh-conic as a line.

Furthermore, the pairs of conjugate points on the central line of the involution φ_q induced by a qh-conic q in the qh-plane are dual to the pairs of lines on which lie the conjugate diameters of a conic in the pe-plane. Consequently, the aforementioned property of qh-centers for a qh-circle is dual to the fact that all pairs of conjugate diameters of a pseudo-Euclidean circle are perpendicular.

4 Some construction assignments

4.1 Qh-centers and isotropic diameters of the qh-ellipses and qh-hyperbolae

Let a qh-conic q be given, that is not a qh-parabola. As already mentioned, a pair of conjugate and perpendicular points on the central line c will be qh-centers of a qh-conic. In order to construct these qh-centers for the given qh-conic q we observe the involution ϕ_q induced by the qh-conic q and the absolute involution I_{QH}. These pencils will be supplemented by the same Steiner's conic s, which is an arbitrary chosen conic through F. Let a pair of isotropic lines n and n_1 be the double lines of the involution ϕ_q. The involutions I_{QH} and ϕ_q determine two involutions on the conic s. Let the points O_1 and O_2 be denoted as the centers of these involutions. The line O_1O_2 intersects the conic s at two points I_1 and I_2. Isotropic lines ($o_1 = FI_1$, $o_2 = FI_2$) through these points are a common pair of these two involutions. Hence, lines o_1 and o_2 are isotropic diameters for the given qh-conic q. The intersection points S_1 and S_2 of o_1 and o_2 with the central line c are qh-centers of the given qh-conic. Figure 7 shows the described construction for hyperbola of type 3.

The construction is based on the Steiner's construction ([6], p.26, [7], p.74-75).

Notice that for the hyperbola of type 2 the line O_1O_2 in the construction will not intersect the conic s, and therefore it has a pair of imaginary isotropic diameters. In general, two involutions on a same pencil (line) have a common pair of real corresponding lines (points) if at least one of them is an elliptic involution. If both of the involutions are hyperbolic then they have a common pair of real corresponding lines (points) if both double lines of one involution are between the double lines (points) of the other involution. In the other case the common pair is a pair of imaginary lines ([6], p.60).

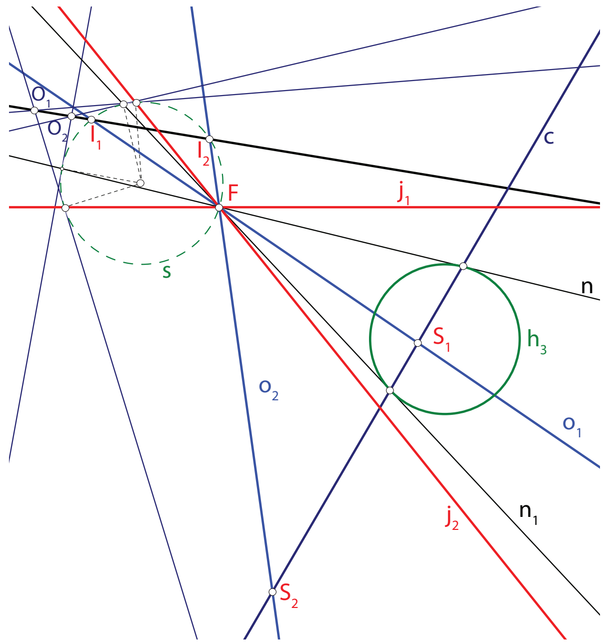

Figure 7

4.2 Osculating qh-circle of a qh-conic

Generally, it is know that two arbitrary conics have four common tangents, therefore the same applies for a conic and a circle. Furthermore, if three of this common tangents coincide then the circle is called a osculating circle of the conic at the point which is the point of tangency of the triple tangent. Hence, there is an osculating circle at any point of a conic.

Let a qh-conic be given, and a tangent t_A at an arbitrary point A of the qh-conic. Figure 8 shows the construction of the qh-circle osculating a qh-conic at the point A by using the elation $(C,t_A,D_1,D_1^{'})$ [8]. Let points J_1 and J_2 be the isotropic points of the tangent t_A. The tangents d_1 and d_2 from the points J_1 and J_2, respectively, to the given qh-conic intersect at the point $F^{'}$ which corresponds to the absolute point F. The ray $F^{'}F$ intersects the tangent t_A, which is the axis of the elation, at the center C of the elation. Hence the tangent lines j_1 and j_2 (absolute lines) of the osculating qh-circle correspond to the tangent lines d_1 and d_2 of a given qh-conic. Let the points of tangency of a qh-circle and j_1, j_2 be denoted as D_1 and D_2, respectively. Let the point of tangency of a qh-conic and d_1, d_2 be denoted as $D_1^{'}$ and $D_2^{'}$. Therefore $D_1^{'}$, D_1 and $D_2^{'}$, D_2 are the pairs of corresponding points of the elation. Similar construction principle is given in [13].

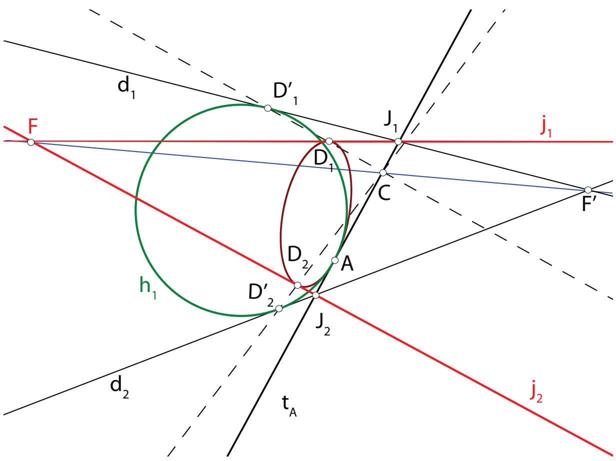

Figure 8

Remark. It should be emphasized that in a qh-plane it is possible to construct infinitely many osculating qh-circles at the isotropic tangency point if the given qh-conic is a qh-circle. The qh-circle osculating the given qh-circle k at its isotropic point J_i, $(i = 1,2)$ can be constructed by using the elation $(F, j_i,A,A^{'})$, $(i = 1,2)$. The point F is the center of the elation, the absolute line j_i its axis, A an arbitrary chosen point on qh-circle and $A^{'}$ an arbitrary chosen point on the ray AF (see Fig. 9).

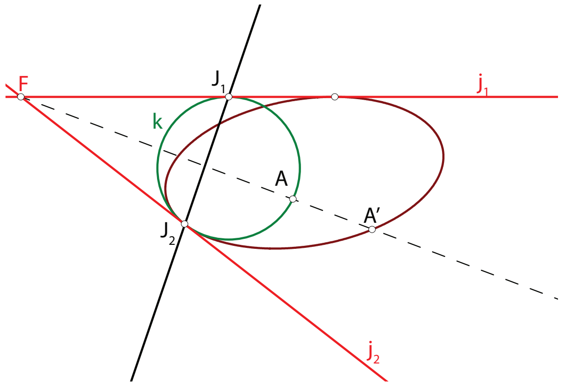

Figure 9

4.3 Hyperosculating qh-circle of qh-conics

A hyperosculating circle of a conic has a common quadruple tangent with the conic, hence it can be constructed only at the vertices of a conic. The similar construction principle as for the osculating circle can be performed to construct the hyperosculating qh-circle at the vertex of a qh-conic.
Let the hyperbola h_1 be given. The intersection points T_1 and T_2 of the qh-conic h_1 with its isotropic diameter are the vertices of the hyperbola. The hyperosculating qh-circle at the vertex T_2 is completely determined with the elation $(T_2,t_2,D_i,D_i^{'})$ $(i = 1,2)$ where T_2 is the center and tangent t_2 at T_2 its axis. The tangent lines j_1 and j_2 of the hyperosculating qh-circle correspond to the tangent lines d_1 and d_2 of the h_1. Let the point of tangency of a qh-conic h_1 and d_1, d_2 be denoted as $D_1^{'}$ and $D_2^{'}$, respectively. Let the points of tangency of a qh-circle and j_1, j_2 be denoted as D_1 and D_2, respectively. $D_1^{'}$, D_1 and $D_2^{'}$, D_2 are the pairs of corresponding points of the elation (see Fig. 10).

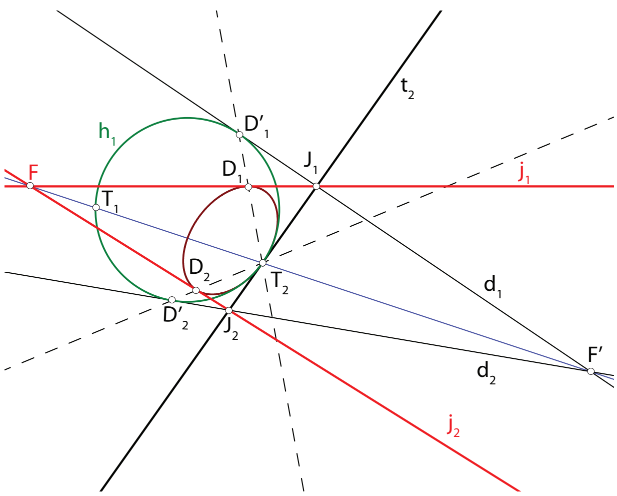

Figure 10

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Izometrije u Escherovim radovima

Isometries in Escher's Work

ABSTRACT

For better understanding of M. C. Escher's tesselation graphics we provide an overview of planar isometries and classification of plane symmetry groups. Some of the plane symmetry groups are explained on prominent Escher's graphics.

Key words: Escher, isometries, tessellation, plane symmetry groups

MSC2010: 00A66, 05B45, 20H15, 51F15, 52C20

Izometrije u Escherovim radovima

SAŽETAK

U ovom članku dan je pregled izometrija ravnine i klasifikacija ravninskih grupa simetrija kao matematička podloga za razumijevanje "trikova" kojima se M. C. Escher služio prilikom stvaranja velikog broja svojih grafika. Razmatrat ćemo grupe simetrija na primjerima nekih od najpoznatijih Escherovih grafika.

Ključne riječi: Escher, izometrije, popločavanje, ravninske grupe simetrija

1 Uvod

Simetrija kao aspekt umjetnosti cijeni se i interpretira već stoljećima. Neki od ranijih umjetničkih radova koji su integrirali simetriju datiraju još iz doba antičkih kultura. Euklid se bavio simetrijom u desetoj knjizi svojih *Elemenata*, gdje definira kada su neke dvije figure simetrične. Nizozemski umjetnik-grafičar Maurits Cornelis Escher (1898.-1972.) ima zadivljujući umjetnički opus te je omiljen među matematičarima. Njegova osnovna inspiracija potiče od arabesknih ukrasa srednjovjekovne palače Alhambre u Španjolskoj. Impresivni opus obuhvaća, između ostalog, i 43 grafike koje Escher jednostavno naziva (npr. *Escher drawing no. 8*). Ta djela nastala su u razdoblju 1936.-1942. nakon čega je Escherova popularnost poprimila svjetske razmjere. Escherove grafike su predmet znanstvenih i stručnih radova matematičara, informatičara, grafičara, a kao vrsta popločavanja ravnine zanimljive su i u kristalografiji ([2]). U ovom radu, koji je nastao na osnovu studentskog seminara, otkrit ćemo tehniku kreiranja nekih njegovih grafika pomoću izometrija u euklidskoj ravnini. Na osnovu klasifikacije ravninskih grupa simetrija prepoznat ćemo grupe simetrija na primjerima. Neke Escherove grafike mogu se promatrati i u neeuklidskoj ravnini ([6]).

2 Definicije i svojstva izometrija

Na početku navodimo osnovne definicije i svojstva izometrija u euklidskoj ravnini E^2 ([3], [4], [5]).

Definicija 1 *Izometrija euklidske ravnine je svaka bijekcija $f : E^2 \to E^2$ ravnine na sebe koja čuva udaljenost točaka, tj. takva da je $d(f(A), f(B)) = d(A,B)$ za sve točke A i B iz E^2.*

Svojstva izometrija u odnosu na kompoziciju funkcija:

Teorem 1

(i) Kompozicija izometrija f i g, $f \circ g$, je također izometrija.

(ii) Neka je f izometrija. Tada je njezin inverz f^{-1} također izometrija.

Definicija 2 *Kažemo da je izometrija involutorna ako je $f \circ f = id$ i $f \neq id$.*

Involutorna izometrija je sama sebi inverz.

Definicija 3 *Figura je svaki podskup od E^2.*

Za figuru F iz Euklidske ravnine E^2 kažemo da je *fiksna figura* izometrije f ako je f preslikava u nju samu, tj. ako je $f(F) = F$.

Svojstva izometrija u odnosu na fiksnu figuru:

Teorem 2 *Neka je f izometrija.*

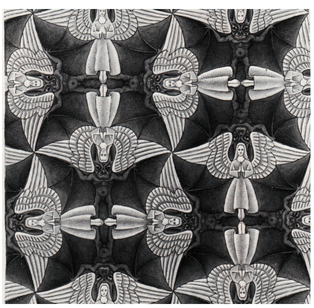
(i) Sjecište, ako postoji, dvaju različitih fiksnih pravaca od f je fiksna točka od f.

(ii) Spojnica dvaju fiksnih točaka od f je fiksni pravac od f.

(iii) Ako je f involutorna izometrija, onda kroz točku koja nije fiksna za f prolazi točno jedan fiksni pravac za f.

Definicija 4 *Involutorna izometrija kojoj su sve točke pravca a fiksne zove se osna simetrija s obzirom na pravac a, u oznaci s_a.*

Escher je u svojim grafikama koristio izometrije: osnu simetriju, translaciju, rotaciju i centralnu simetriju. Svojstvo tih izometrija je da se mogu definirati pomoću osne simetrije.



Slika 1: *Primjer simetrija na Escherovoj grafici (Angel and devil)*

Definicija 5 *Izometriju koja se može prikazati kao kompozicija $s_a \circ s_b$ dviju osnih simetrija s_a i s_b zovemo translacija ako su osi simetrije a i b paralelni pravci.*

Definicija 6 *Izometriju koja se može prikazati kao kompozicija $s_a \circ s_b$ dviju osnih simetrija s_a i s_b zovemo rotacija ako osi simetrije a i b nisu paralelni pravci.*

Definicija 7 *Centralna simetrija je rotacija $s_a \circ s_b$ za koju su osi simetrije a i b okomiti pravci.*

Definicija 8 *Izometrija koja se može prikazati u obliku kompozicije $s_g \circ s_b \circ s_a$, gdje je pravac g okomit na pravce a i b zove se klizna simetrija.*

Klizna simetrija je najzastupljenija u Escherovim grafikama.

Sljedeći teorem daje karakterizaciju nekih izometrija:

Teorem 3

(i) Svaka involutorna izometrija je ili osna ili centralna simetrija.

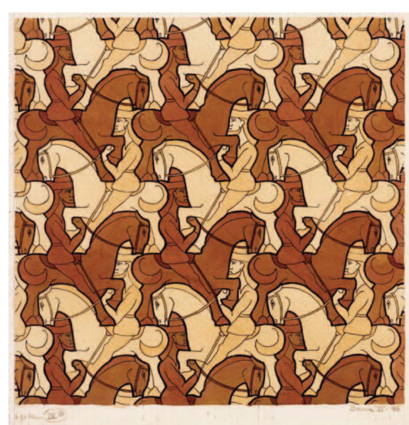
(ii) Kompozicija dviju rotacija je ili rotacija ili translacija.

(iii) Izometrija je klizna simetrija ako i samo ako se može predočiti u obliku kompozicije jedne osne i jedne centralne simetrije ili jedne centralne i jedne osne simetrije.

(iv) Svaka izometrija je ili translacija ili rotacija ili klizna simetrija.

Slika 2: *Primjer translacije na Escherovoj grafici*

Slika 3: *Primjer rotacije na Escherovoj grafici*



Slika 4: *Primjer klizne simetrije na Escherovoj grafici*

3 Grupe simetrija

Neka je $Iz(E^2)$ skup svih izometrija Euklidske ravnine E^2. Poznato je da je $Iz(E^2)$ zajedno s komponiranjem funkcija kao binarnom operacijom grupa izometrija, u oznaci $(Iz(E^2), \circ)$.

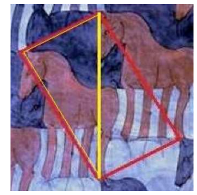
Teorem 4 *Neka je* $Iz(F) = \{f \in Iz(E^2) : f(F) = F\}$, *gdje je F figura Euklidske ravnine E^2. Tada je $(Iz(F), \circ)$ je grupa simetrija figure F.*

Kako bi u Escherovim grafikama prepoznali izometrije definiramo popločavanje.

Definicija 9 *Popločavanje je razdioba (particija) ravnine na disjunktne skupove H_i, $i \in \mathbb{N}$ čija unija daje cijelu ravninu.*

- *Uzorak* u ravnini je figura, koja je u Escherovim grafikama oblika životinje. Uzorak preslikavamo u samog sebe i pomoću izometrija ravnine: rotacija, simetrija, klíznih simetrija, translacija.
- *Osnovni uzorak* je dio uzorka sa svojstvom da skup uzoraka u *grupi izometrija* prekriva ravninu. Drugim riječima, osnovnim uzorkom popločavamo ravninu.
- *Generirajuće područje* je dio osnovnog uzorka čije slike u grupi simetrija uzorka popločavaju ravninu.

Na slici 5 prikazan je uzorak konja, osnovni uzorak (crveni paralelogram) i generirajuće područje (žuti trokut).



Slika 5: *Primjer generirajućeg područja*

Sljedeći teorem daje klasifikaciju ravninskih grupa simetrija ([1], [8]). Dokaz je izostavljen i može se naći u [9].

Teorem 5 ***(Barlow, Fedorov, Schönflies-1891.)***

Postoji samo 17 *mogućih ravninskih grupa simetrija.*

Tih sedamnaest grupa poznate su i kao ravninske grupe kristalografije. Pomoću njih, kristalografi sistematiziraju kristale ([2]). Ravninske grupe simetrija odgovaraju sedamnaest načina popločavanja ravnine. U Escherovim grafikama se mogu razmatrati vrlo vješta i zanimljiva popločavanja.

Napomena 1 *Broj n označava stupanj rotacije. Rotacija za kut $\frac{360^\circ}{n}$ ima stupanj rotacije n.*

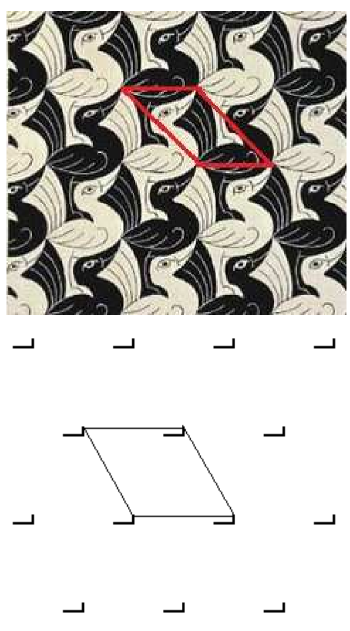
Naziv	**Osnovni uzorak**	**Stupanj rotacije**	**Klizna simetrija**	**Generirajuće područje uzorka**	**Značajke**
p1	paralelogram	1	/	cijela površina	translacija
p2	paralelogram	2	/	1/2 površine	4 rotacije za 180°
pm	četverokut	1	/	1/2	2 osne simetrije
pmm	četverokut	2	/	1/4	2 osne simetrije
pg	četverokut	1	da	1/2	
pgg	četverokut	2	da	1/4	
pmg	četverokut	2	da	1/4	osi simetrije su paralelne
cm	romb	1	da	1/2	
cmm	romb	2	da	1/4	osi simetrije su okomite
p4	kvadrat	4	/	1/4	
p4m	kvadrat	4	da	1/8	centar rotacije je na osi simetrije
p4g	kvadrat	4	da	1/8	centar rotacije nije na osi simetrije
p3	šesterokut	3	/	1/3	
p3m1	šesterokut	3	da	1/6	centar rotacije je na osi simetrije
p31m	šesterokut	3	da	1/6	
p6	šesterokut	6	/	1/6	
p6m	šesterokut	6	da	1/2	

Tablica 1: *Klasifikacija ravninskih grupa simetrija* (preuzeto iz [7])

4 Primjeri Escherovih grafika

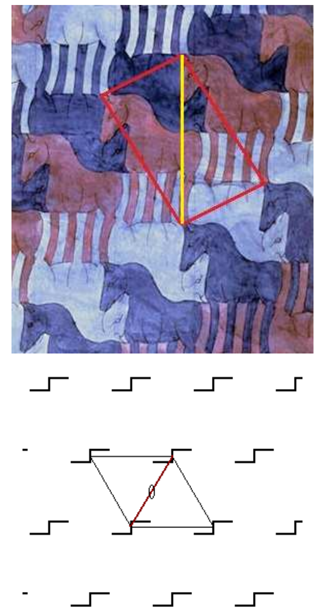
U ovom poglavlju detaljno razmatramo Escherove grafike. Poznavajući sustav 17 ravninskih grupa simetrija, Escher je otkrio svoj sistem ”grupirajućih pločica”. Njegove grafike popločavanja odgovaraju pet od sedamnaest Fedorovih grupa simetrija. Na sljedećim primjerima uzorci su likovi životinja.

Na slici 6, dan je primjer ravninske grupe simetrija $p1$ i detaljan prikaz svojstava ravninske grupe simetrija $p1$. Uzorak grafike je patka. Osnovni uzorak je paralelogram označen crvenom bojom. Generirajuće područje je ekvivalentno osnovnom uzorku. Osnovni uzorak se translatira u smjeru okomitom na stranice paralelograma. Dakle, radi se o translacijama paralelograma koje čine grupu s obzirom na kompoziciju. Kod detaljnog prikaza osnovni uzorak i generirajuće područje su paralelogram pomoću kojeg se generira (popločava ravnina) motiv oblika slova L.



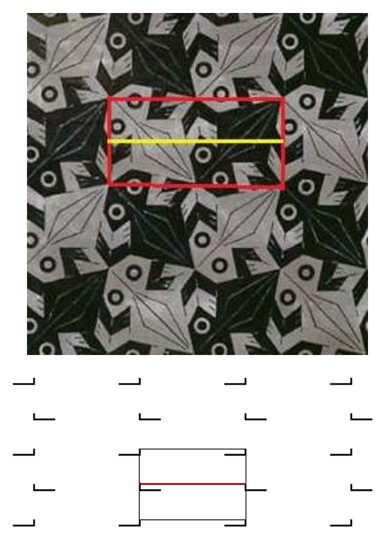
Slika 6: *Escher drawing no.* 128 *i vizualna reprezentacija* $p1$ *grupe simetrija*

Slika 7 predstavlja grupu simetrija $p2$. Osnovni uzorak je paralelogram označen crvenom bojom. Žutom bojom je istaknuto generirajuće područje osnovnog uzorka. Generirajuće područje se transformira rotacijom za 180° i zatim translatira u smjeru rubova osnovnog uzorka. Detaljnija reprezentacija dana je na istoj slici gdje je generiran motiv oblika slova L. Znak elipse označava rotaciju generirajućeg područja (u ovom slučaju radi se o $\frac{1}{2}$ površine osnovnog uzorka). Stupanj rotacije je 2, tj. rotacija za 180°.



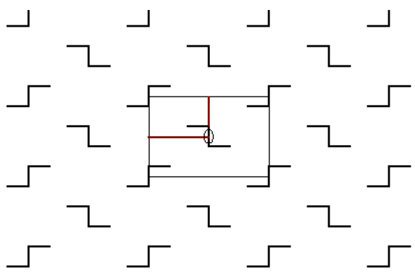
Slika 7: *Escher drawing no.* 8 *i vizualna reprezentacija* $p2$ *grupe simetrija*

Na slici 8 je Escherova grafika *Escher drawing no.* 109 s istaknutim osnovnim uzorkom crvene boje. Generirajuće područje uzorka je označeno žutom bojom. Vertikalno se translatira za $\frac{1}{2}$ duljine kraće stranice te se transformira kliznom simetrijom na lijevu i desnu stranu. Kod ravninske grupe simetrija pg generirajuće područje je $\frac{1}{2}$ površine osnovnog uzorka.



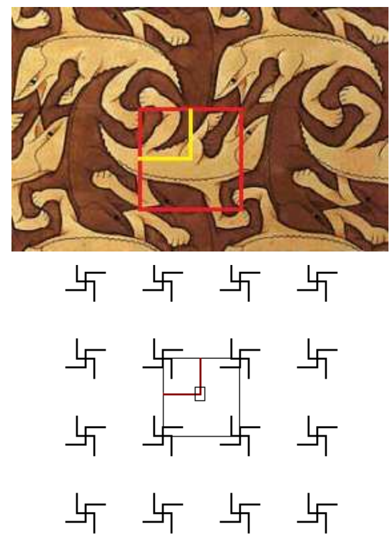
Slika 8: *Escher drawing no.* 109 *i vizualna reprezentacija* pg *grupe simetrija*

Slika 9 daje uvid u grupu simetrija *pgg*. Znak elipse označava rotaciju generirajućeg područja uzorka za 180°. Generirajuće područje je istaknuto crvenom bojom. Grupa *pgg* sadrži izometrije: rotaciju i kliznu simetriju. Generirajuće područje se rotira za 180° i zatim transformira vertikalno, kliznom simetrijom. Ova metoda slijedi iz Definicije 6, Definicije 8 i svojstva izometrija.



Slika 9: *Vizualna reprezentacija pgg grupe simetrija*

Slika 10 predstavlja grupu *p*4. Uzorak grafike je gušter. Crvenom bojom je označen osnovni uzorak, a žutom bojom generirajuće područje uzorka. Ravninska grupa simetrija *p*4 sadrži izometrije: rotaciju i translaciju. Na detaljnijoj reprezentaciji prikazan je manji četverokut koji (uz istaknuto generirajuće područje) označava rotaciju za 90°. Generirajuće područje se transformira rotacijom, tri puta, u smjeru kazaljke na satu. Zatim se translatira, vertikalno i horizontalno, za duljinu stranice osnovnog uzorka (kvadrat). Radi lakšeg razumijevanja ravninske grupe simetrija *p*4 koristi se i alternativni naziv, "grupa simetrija s obzirom na translaciju".



Slika 10: *Escher drawing no.* 15 *i vizualna reprezentacija p*4 *grupe simetrija*

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