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Introduction to the Planimetry of the Quasi-Hyperbolic Plane

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ABSTRACT

The quasi-hyperbolic plane is one of nine projective-metric planes where the absolute figure is the ordered triple $\{j_1, j_2, F\}$, consisting of a pair of real lines j_1 and j_2 through the real point F. In this paper some basic geometric notions of the quasi-hyperbolic plane are introduced. Also the classification of qh-conics in the quasi-hyperbolic plane with respect to their position to the absolute figure is given. The notions concerning the qh-conic are introduced and some selected constructions for qh-conics are presented.

Key words: quasi-hyperbolic plane, perpendicular points, central line, qh-conics classification, osculating qh-circle

MSC2010: 51A05, 51M10, 51M15

Uvod u planimetriju kvazi-hiperboličke ravnine

SAŽETAK

Kvazihiperbolička ravnina je jedna od devet projektivno metričkih ravnina kojoj je apsolutna figura uređena trojka $\{j_1, j_2, F\}$, gdje su j_1 i j_2 realni pravci koji se sijeku u realnoj točki F. U ovom članku uvodimo neke osnovne pojmove za kvazihiperboličku ravninu, te dajemo klasifikaciju konika u odnosu na njihov položaj prema apsolutnoj figuri. Nadalje, uvesti ćemo pojmove vezane uz konike u kvazihiperboličkoj ravnini i pokazati nekoliko izabranih konstrukcija vezanih uz konike.

Ključne riječi: kvazihiperbolička ravnina, okomite točke, centrala, klasifikacija qh-konika, oskulacijske qh-kružnice

1 Introduction

In the second half of the 19th century F. Klein opened a new field for geometers with his famous Erlangen program which is the study of the properties of a space which are invariant under a given group of transformations. Klein was influenced by some earlier research of A. Cayley, so today it is known that there exist nine geometries in plane with projective metric on a line and on a pencil of lines which are denoted as Cayley-Klein projective metrics. Hence, these plane geometries differ according to the type of the measure of distance between points and measure of angles which can be parabolic, hyperbolic, or elliptic. Furthermore, each of these geometries can be embedded in the real projective plane $\mathbb{P}_2(\mathbb{R})$ where an absolute figure is given as non-degenerated or degenerated conic [4], [5], [12] (for space and n-dimension see [11]).
In this article the geometry, denoted as *quasi-hyperbolic*, with hyperbolic measure of distance and parabolic measure of angle will be presented.

2 Basic notation in the quasi-hyperbolic plane

In the quasi-hyperbolic plane (further in text qh-plane) the metric is induced by a real degenerated conic i.e. a pair of real lines j_1 and j_2 incidental with the real point F. The lines j_1 and j_2 are called the *absolute lines*, while the point F is called the *absolute point*. In the Cayley-Klein model of the qh-plane only the points, lines and segments inside of one projective angle between the absolute lines are observed. In this article all points and lines of the qh-plane embedded in the real projective plane $\mathbb{P}_2(\mathbb{R})$ are observed.

There are three different positions for the absolute triple $\{j_1, j_2, F\}$: neither of the absolute elements are at infinity, only the absolute point is at infinity and the absolute point and one absolute line are at infinity (see Fig. 1). The first position of the absolute triple is used for constructions in this article.

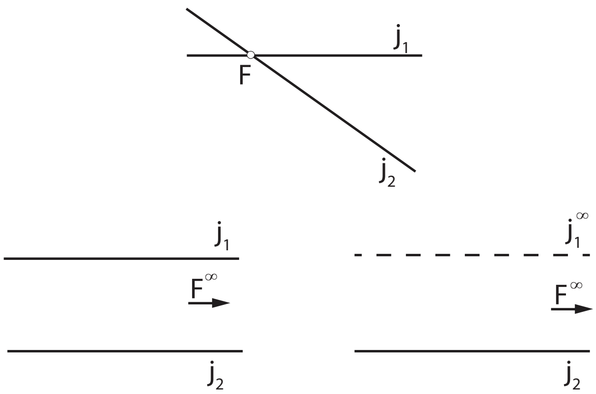

Figure 1

For the points and the lines in the qh-plane the following terms are defined:

• *isotropic lines* - the lines incidental with the absolute point F,

• *isotropic points* - the points incidental with one of the absolute lines j_1 or j_2,

• *parallel lines* - two lines which intersect at an isotropic point,

• *parallel points* - two points incidental with an isotropic line,

• *perpendicular lines* - if at least one of two lines is an isotropic line,

• *perpendicular points* - two points (A and B) that lie on a pair of isotropic lines (a and b) that are in harmonic relation with the absolute lines j_1 and j_2.

Furthermore, an involution of pencil of lines (F) having the absolute lines for double lines is called the *absolute involution*, denoted as I_{QH}. This is a hyperbolic involution on the pencil (F) where every pair of corresponding lines is in a harmonic relation with the double lines j_1 and j_2 ([1], p.244-245, [6], p.46). Notice that every pair of perpendicular points lie on a pair of I_{QH} corresponding lines. Hence, the perpendicularity of points in qh-plane is determined by the absolute involution, therefore I_{QH} is a circular involution in the qh-plane ([7], p.75).

Remark. Any two isotropic points on the same absolute line are perpendicular and parallel. Any two lines from a pencil (F) are perpendicular and parallel.

3 Qh-conics classification

There are nine types of regular qh-conics classified according to their position with respect to the absolute figure:

• *qh-hyperbola* - a qh-conic which has a pair of real tangent lines from the absolute point,

- *hyperbola of type 1* (h_1) - intersects each absolute line in a pair of real and distinct points,
- *hyperbola of type 2* (h_2) - intersects one absolute line in a pair of real and distinct points and another absolute line in a pair of imaginary points,
- *hyperbola of type 3* (h_3) - intersects each absolute line in a pair of imaginary points,
- *special hyperbola of type 1* (h_{s1}) - one absolute line is a tangent line and another absolute line intersects the qh-conic in a pair of real and distinct points,
- *special hyperbola of type 2* (h_{s2}) - one absolute line is a tangent line and another absolute line intersects the qh-conic in a pair of imaginary points,

• *qh-ellipse* (e) - a qh-conic (imaginary or real) which has a pair of imaginary tangent lines from the absolute point,

• *qh-parabola* (p) - a qh-conic passing through the absolute point i.e. both isotropic tangent lines coincide,

- *special parabola* (p_s) - a qh-parabola whose isotropic tangent is an absolute line,

• *qh-circle* (k) - a qh-conic for which the tangents from the absolute point are the absolute lines.

In the projective model of the qh-plane every type of a qh-conic can be represented with the Euclidean circle without loss of generality (see Fig. 2). This fact simplifies the constructions in the qh-plane.

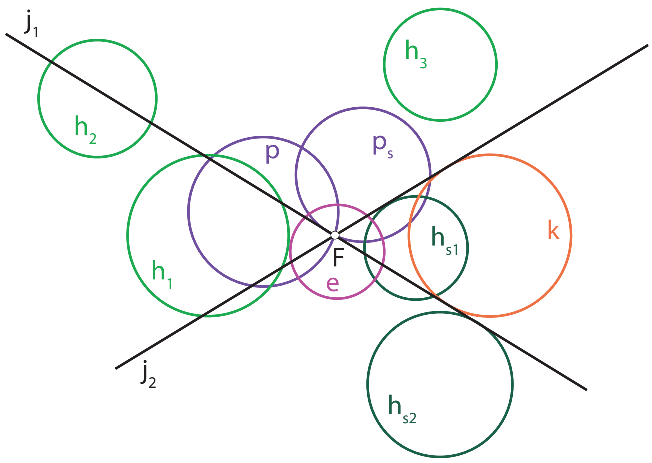

Figure 2

Furthermore every qh-conic q, except qh-parabolae, induces an involution ϕ_q on the pencil (F) where the double lines are the isotropic tangents of the qh-conic q, and the corresponding lines of the involution ϕ_q are called *conjugate lines*. Notice that every qh-ellipse induces an elliptic involution, every qh-hyperbola induces a hyperbolic involution and every qh-circle induces an involution that coincides with the absolute involution I_{QH}.

Remark. A qh-conic is called *equiform* if the isotropic tangent lines of the qh-conic are in harmonic relation with the absolute lines j_1 and j_2. In terms of the above mentioned involutions a qh-conic q is equiform if the absolute involution I_{QH} is commutative with the involution ϕ_q induced by the qh-conic q. Notice that only qh-ellipses, qh-hyperbolae of type 2 and qh-circles can be equiform [2], [3].

In the following some basic notions related to a qh-conic in the qh-plane are defined:

• The polar line of the absolute point F with respect to a qh-conic is called the *central line* c or the *major diameter* of the qh-conic (see Fig. 3). All qh-conics, except qh-parabolas, have a non-isotropic central line. The central line of a qh-parabola is its isotropic tangent line, while for the special parabola it is an absolute line.

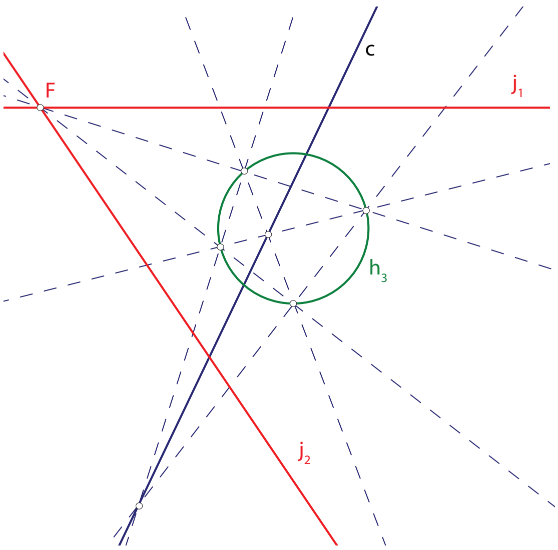

Figure 3

• The *directrices* of a qh-conic are (non-absolute) lines incident with the isotropic points of the qh-conic, i.e. lines incidental with the intersection points of the qh-conic with the absolute lines j_1 and j_2. A qh-conic can have none, one, two or four directrices f_i, $i \in \{1,2,3,4\}$ (see Fig. 4).

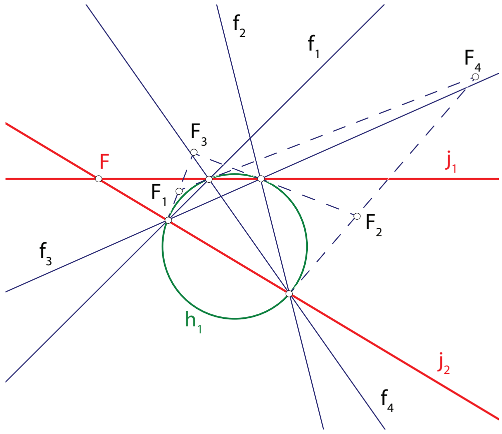

Figure 4

• The pole of the directrix with respect to a qh-conic is called a *focus* of the qh-conic. The number of foci F_i, $i \in \{1,2,3,4\}$, is equal to the number of directrices (see Fig. 4).

• The lines that are incident with the opposite foci are called *isotropic diameters* of a qh-conic (see Fig. 5). Especially for the qh-circles, which have one focus, the isotropic diameters are the lines of the pencil (F). Hence a qh-conic can have none, one, two or infinitely many isotropic diameters o_i, $i \in \{1,2\}$.

• The *qh-centers* of a qh-conic are the points of intersection of the isotropic diameters and the central line of the qh-conic. A qh-conic can have none, one, two or infinitely many qh-centers S_i, $i \in \{1,2\}$ (see Fig. 5).

• The intersection points of a qh-conic with its isotropic diameters are called *vertices* of the qh-conic (see Fig. 5). A qh-conic can have four, two, one or none vertices T_i, $i \in \{1,2,3,4\}$.

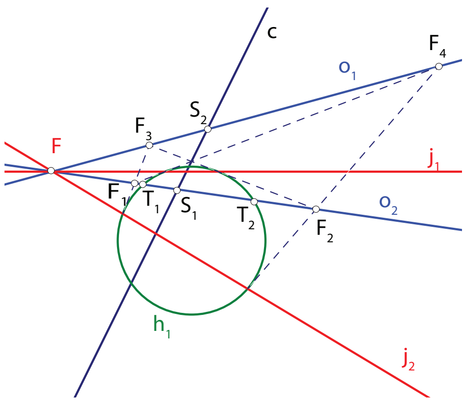

Figure 5

The absolute involution I_{QH} can be observed as a point range involution on any non-isotropic line, hence it can be observed on the central line of a qh-conic, except for qh-parabolae. Also the involution ϕ_q on a pencil (F) induced by a qh-conic q can be observed as the involution φ_q of a point range on the central line c of the qh-conic q, and two corresponding points of involution φ_q are called conjugate points. Therefore, the qh-centers for the qh-ellipses and qh-hyperbolae can be found as a pair of perpendicular and conjugate points on the central line, and the isotropic diameters as the perpendicular and conjugate lines of the pencil (F). The construction will be shown later. Notice that because the involution induced by a qh-circle coincides with the absolute involution all pairs of conjugate points on the central line of the qh-circle are perpendicular points. Hence any point on the central line is its center and every line of the pencil (F) is its isotropic diameter.

Aforementioned qh-conics and notions can be summarized in the following table:

Qh-Conic	Directrix	Focus	Isotropic diameter	Center	Vertex
Ellipse e	4 real	4 real	2 real	2 real	4 real
Hyperbola h_1	4 real	4 real	2 real	2 real	2 real + 2 imaginary
Hyperbola h_2	4 imaginary	4 imaginary	2 imaginary	2 imaginary	4 imaginary
Hyperbola h_3	4 imaginary	4 imaginary	2 real	2 real	2 real + 2 imaginary
Parabola p	2 real	2 real	1 real	1 real	2 real
Special parabola p_s	0	0	0	0	0
Special hyperbola h_{s1}	2 real	2 real	1 real	1 real	1 real
Special hyperbola h_{s2}	2 imaginary	2 imaginary	1 real	1 real	1 real
Circle c	1 real	1 real	infinite	infinite	0

Table 1

For parabolae and special hyperbolae see figure 6.

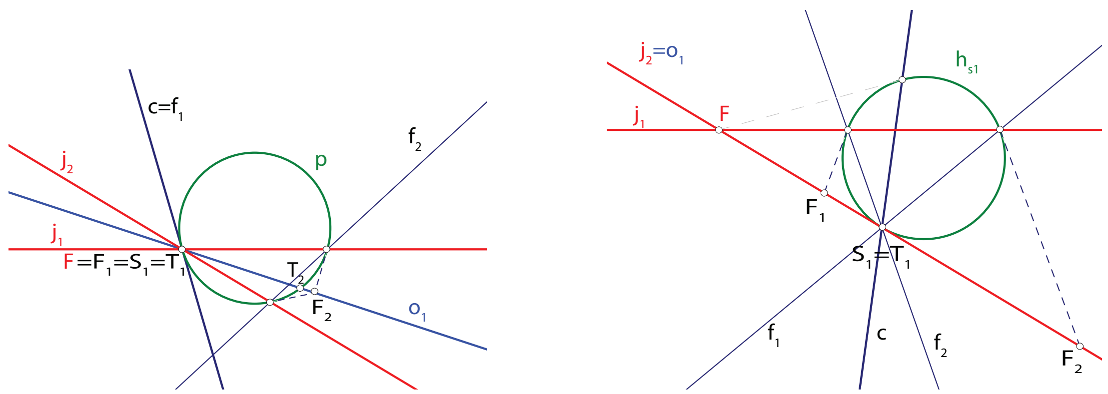

Figure 6

Remark. The qh-plane is dual to the pseudo-Euclidean (Minkowski) plane where the metric is induced by a real line and two real points incident with it. Therefore the notions defined above can be explained as duals of the Minkowski plane.

The conics in pseudo-Euclidean plane (pe-plane) are classified in nine subtypes, hence the classification of qh-conics was based on [3], [9], [10]. Furthermore, the aforementioned elements for qh-conics can be presented as follows:

- the central line is a dual of the center of a conic in the pe-plane,
- the directrices are a dual of the foci of a conic in the pe-plane,
- the foci are a dual of the directrices of a conic in the pe-plane,
- the qh-centers are dual to the axes of a conic in the pe-plane.

The dual of the isotropic diameters are the intersections of the axes with the absolute line, but they were not of special interest in the pe-plane. Also the dual of the vertices in qh-plane are the tangents to the conic in pe-plane from the above mentioned intersections. It should be emphasized that the dual of the vertices in pe-plane are tangents to the qh-plane from the qh-centers. Since the axes in pe-plane and qh-centers in qh-plane are dual, therefore it was not chosen in this article to observe the vertices of a qh-conic as a line.

Furthermore, the pairs of conjugate points on the central line of the involution φ_q induced by a qh-conic q in the qh-plane are dual to the pairs of lines on which lie the conjugate diameters of a conic in the pe-plane. Consequently, the aforementioned property of qh-centers for a qh-circle is dual to the fact that all pairs of conjugate diameters of a pseudo-Euclidean circle are perpendicular.

4 Some construction assignments

4.1 Qh-centers and isotropic diameters of the qh-ellipses and qh-hyperbolae

Let a qh-conic q be given, that is not a qh-parabola. As already mentioned, a pair of conjugate and perpendicular points on the central line c will be qh-centers of a qh-conic. In order to construct these qh-centers for the given qh-conic q we observe the involution ϕ_q induced by the qh-conic q and the absolute involution I_{QH}. These pencils will be supplemented by the same Steiner's conic s, which is an arbitrary chosen conic through F. Let a pair of isotropic lines n and n_1 be the double lines of the involution ϕ_q. The involutions I_{QH} and ϕ_q determine two involutions on the conic s. Let the points O_1 and O_2 be denoted as the centers of these involutions. The line O_1O_2 intersects the conic s at two points I_1 and I_2. Isotropic lines ($o_1 = FI_1$, $o_2 = FI_2$) through these points are a common pair of these two involutions. Hence, lines o_1 and o_2 are isotropic diameters for the given qh-conic q. The intersection points S_1 and S_2 of o_1 and o_2 with the central line c are qh-centers of the given qh-conic. Figure 7 shows the described construction for hyperbola of type 3.

The construction is based on the Steiner's construction ([6], p.26, [7], p.74-75).

Notice that for the hyperbola of type 2 the line O_1O_2 in the construction will not intersect the conic s, and therefore it has a pair of imaginary isotropic diameters. In general, two involutions on a same pencil (line) have a common pair of real corresponding lines (points) if at least one of them is an elliptic involution. If both of the involutions are hyperbolic then they have a common pair of real corresponding lines (points) if both double lines of one involution are between the double lines (points) of the other involution. In the other case the common pair is a pair of imaginary lines ([6], p.60).

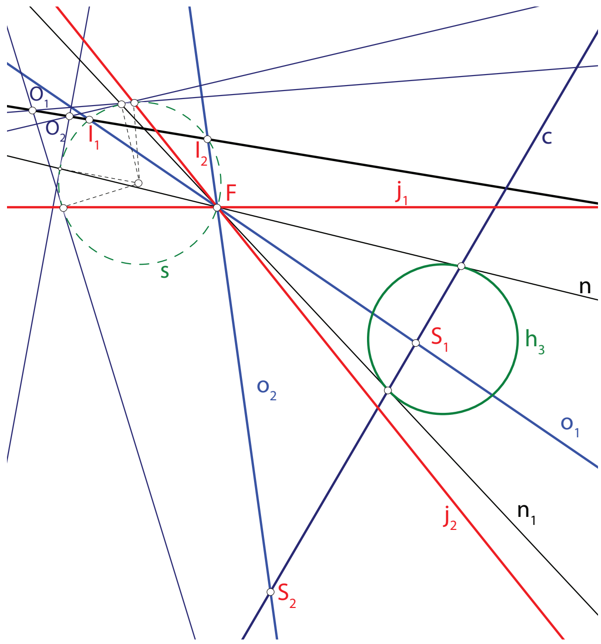

Figure 7

4.2 Osculating qh-circle of a qh-conic

Generally, it is know that two arbitrary conics have four common tangents, therefore the same applies for a conic and a circle. Furthermore, if three of this common tangents coincide then the circle is called a osculating circle of the conic at the point which is the point of tangency of the triple tangent. Hence, there is an osculating circle at any point of a conic.

Let a qh-conic be given, and a tangent t_A at an arbitrary point A of the qh-conic. Figure 8 shows the construction of the qh-circle osculating a qh-conic at the point A by using the elation $(C, t_A, D_1, D_1^{'})$ [8]. Let points J_1 and J_2 be the isotropic points of the tangent t_A. The tangents d_1 and d_2 from the points J_1 and J_2, respectively, to the given qh-conic intersect at the point $F^{'}$ which corresponds to the absolute point F. The ray $F^{'}F$ intersects the tangent t_A, which is the axis of the elation, at the center C of the elation. Hence the tangent lines j_1 and j_2 (absolute lines) of the osculating qh-circle correspond to the tangent lines d_1 and d_2 of a given qh-conic. Let the points of tangency of a qh-circle and j_1, j_2 be denoted as D_1 and D_2, respectively. Let the point of tangency of a qh-conic and d_1, d_2 be denoted as $D_1^{'}$ and $D_2^{'}$. Therefore $D_1^{'}$, D_1 and $D_2^{'}$, D_2 are the pairs of corresponding points of the elation. Similar construction principle is given in [13].

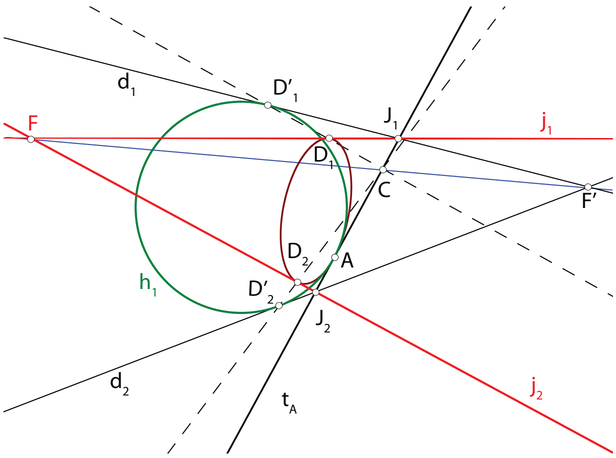

Figure 8

Remark. It should be emphasized that in a qh-plane it is possible to construct infinitely many osculating qh-circles at the isotropic tangency point if the given qh-conic is a qh-circle. The qh-circle osculating the given qh-circle k at its isotropic point J_i, $(i = 1,2)$ can be constructed by using the elation $(F, j_i, A, A^{'})$, $(i = 1,2)$. The point F is the center of the elation, the absolute line j_i its axis, A an arbitrary chosen point on qh-circle and $A^{'}$ an arbitrary chosen point on the ray AF (see Fig. 9).

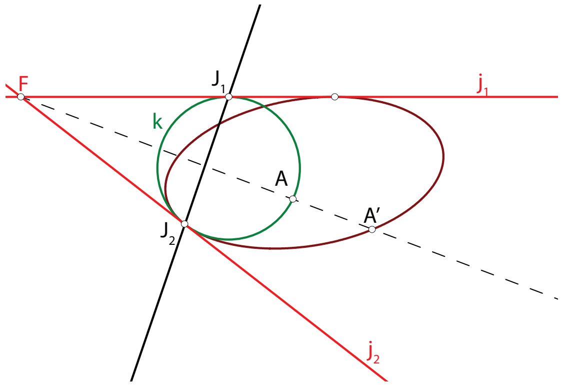

Figure 9

4.3 Hyperosculating qh-circle of qh-conics

A hyperosculating circle of a conic has a common quadruple tangent with the conic, hence it can be constructed only at the vertices of a conic. The similar construction principle as for the osculating circle can be performed to construct the hyperosculating qh-circle at the vertex of a qh-conic.

Let the hyperbola h_1 be given. The intersection points T_1 and T_2 of the qh-conic h_1 with its isotropic diameter are the vertices of the hyperbola. The hyperosculating qh-circle at the vertex T_2 is completely determined with the elation $(T_2, t_2, D_i, D_i^{'})$ $(i = 1,2)$ where T_2 is the center and tangent t_2 at T_2 its axis. The tangent lines j_1 and j_2 of the hyperosculating qh-circle correspond to the tangent lines d_1 and d_2 of the h_1. Let the point of tangency of a qh-conic h_1 and d_1, d_2 be denoted as $D_1^{'}$ and $D_2^{'}$, respectively. Let the points of tangency of a qh-circle and j_1, j_2 be denoted as D_1 and D_2, respectively. $D_1^{'}$, D_1 and $D_2^{'}$, D_2 are the pairs of corresponding points of the elation (see Fig. 10).

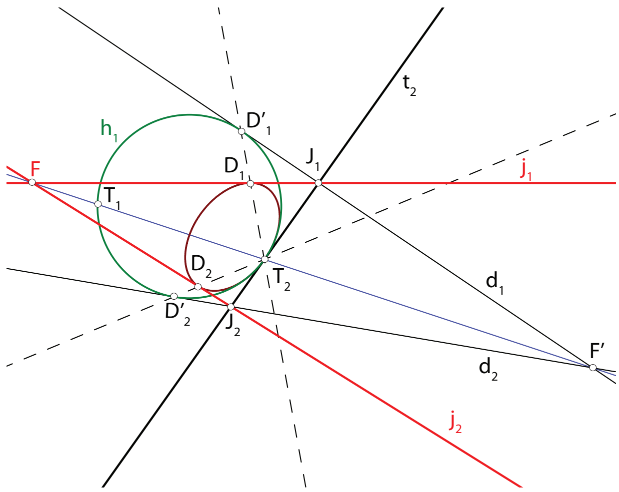

Figure 10

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