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Central Projection of Hyperbolic Space onto a Horosphere

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ABSTRACT

Horosphere is surface in hyperbolic space that is isometric to the Euclidean plane. In order to correctly visualize hyperbolic space we embed flat computer screen as horosphere and investigate geometry of central projection of hyperbolic space onto horosphere. We also discuss realization of hyperbolic isometries. Corresponding algorithms are implemented in Mathematica package *L3toHorospere*. We briefly present the package and obtain some interesting pictures of hyperbolic polyhedra.

Key words: hyperbolic space, horosphere, central projection

MSC 2000: 00A66, 51M10

Centralna projekcija hiperboličkog prostora na horosferu

SAŽETAK

Horosfera je ploha u hiperboličkom prostoru izometrična euklidskoj ravnini. Kako bismo vjerno prikazali hiperbolički prostor, ravni ekran smjestili smo kao horosferu, a zatim istraživali geometriju centralnog projiciranja hiperboličkog prostora na horosferu. Također smo proučavali realizaciju izometrija hiperboličkog prostora. Odgovarajući su algoritmi implementirani u Mathematica paketu *L3toHorospere*. Dan je kratak prikaz tog paketa i dobivene su zanimljive slike hiperboličkih poliedara.

Ključne riječi: hiperbolički prostor, horosfera, centralna projekcija

Introduction

Hyperbolic geometry (or geometry of Bolyai - Lobachevskii) is together with spherical geometry the simplest “curved” geometry. Hyperbolic plane, usually denoted by H^2 , is usually visualized using various models of hyperbolic plane. Poincaré disk, Klein disk and half-plane model are the best known models of hyperbolic plane.

In the Figure 1, obtained using Mathematica package [10], the red triangle, together with three triangles obtained by reflection with respect to its edges are shown in those three models. Note, that in all pictures, all four triangles are mutually congruent in hyperbolic plane. A region of hyperbolic plane is not isometric to a region of “flat”, Euclidean plane. This means that it is not possible to represent, without distortions, a region of hyperbolic plane on a flat computer screen.

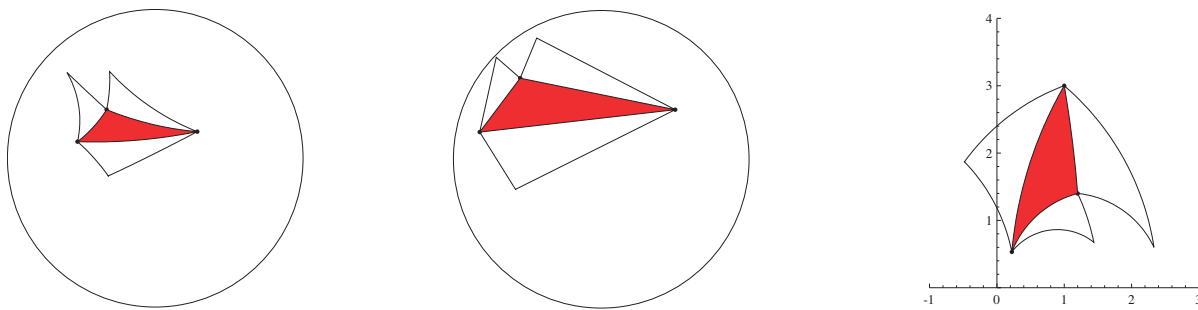


Figure 1: *The same objects in various models of hyperbolic plane*

Therefore, hyperbolic metric of model is not inherited from the Euclidean plane: distances become infinitely big near absolute (unit circle in the first two models and x-axis in the third model). Although to our eyes the absolute is finite, it represents infinity of the hyperbolic plane.

For hyperbolic space H^3 there exist analogous models: Poincaré ball, Klein ball and half-space model. Two approaches for visualization of the hyperbolic space have been used so far, and both approaches at least in one instance use some model.

The first is to represent a geometrical object in some model of hyperbolic space in \mathbb{R}^3 and then to project it onto computer screen by standard central projection of \mathbb{R}^3 . This approach was used in the famous movie *Not knot* ([6]). This is also used a popular way of visualizing large graph objects and the structure of world wide web (see [7] and others). Probably the most famous visualizations of hyperbolic space so far are done by J. R. Weeks ([11, 12]) and use this approach.

The second approach is to fix a hyperbolic plane H^2 in hyperbolic space L^3 , project the space onto H^2 by means of central projection in L^3 and finally visualize the plane H^2 using some model on the computer screen. In the master thesis [1] the author develops ray tracing algorithm for hyperbolic space and uses this approach for visualization.

In this paper we take a different approach. We wonder: how would hyperbolic space look like to us, if we were there? Equivalently, in terms of computer graphics: how would the picture look like if we isometrically embed our flat computer screen into H^3 , project the hyperbolic space on the screen by means of central projection in H^3 and then watch the picture on the screen without any distortions and models?

The mathematical answer was well known to very founders of hyperbolic geometry: there is a peculiar surface in H^3 , called horosphere, which is isometric to Euclidean plane. Therefore, if we isometrically embed a flat, Euclidean screen in hyperbolic space it may become a part of horosphere. In this paper we discuss necessary mathematical background regarding central projection of hyperbolic space onto horosphere, as well as, isometric transformations of the hyperbolic space. The final result is Mathematica package *L3toHorosphere* that allows visualization of H^3 by means of central projection onto horosphere and also visualization of hyperbolic motions. Using this package many interesting pictures and animations are obtained (see [3]).

On our request, Prof. Emil Molnár informed us that Prof. Imre Juhász (the head of Department of Descriptive Geometry of the University of Miskolc) dealt with a similar

topic in his awarded Scientific Student Circle (OTDK) paper (in 1979) and in his diploma work (1978) at the Debrecen University, without any scientific publication on this topic, later on.

It is also worth mentioning that methods of Descriptive geometry in hyperbolic space have also been investigated (see [8, 9]).

Description of hyperbolic isometries is mathematically simple and found in many classical books, but when it comes to practical implementation the paper [5] is usually used. In this work we briefly cover this topic using slightly different approach that someone may find easier to understand.

The paper is organized as follows. In the first section we give a brief overview of models of hyperbolic space. In the second section we study horosphere and central projection onto horosphere. The third section is devoted to implementation of isometries of hyperbolic space. In the fourth section we give some examples of projections and animations obtained using package *L3toHorosphere*. In the last section we compare various approaches in visualizing hyperbolic geometry and give some ideas for future work.

Authors would like to thank Prof. Emil Molnár for useful discussions and for careful reading which significantly improved the final version of the paper.

1 Models of hyperbolic space

1.1 Klein ball and projective model

Klein ball model $\{P(x, y, z) \mid x^2 + y^2 + z^2 < 1\}$ is interior of unit sphere in \mathbb{R}^3 . The unit sphere is absolute of this model. This means that points of unit sphere represent points in infinity of the hyperbolic space. Hyperbolic lines and planes are parts of Euclidean lines and planes. The distance between points P and Q is given by the formula

$$d(P, Q) = \frac{1}{2} \log \frac{|QA||PB|}{|PA||QB|}, \quad (1)$$

where A and B are endpoints of the chord containing P and Q and $|\cdot|$ denotes the Euclidean distance.

Klein ball model is closely related to Klein projective model or pseudosphere model. Namely for point $P(x, y, z)$ from the Klein model, one can consider homogenous coordinates $P(x : y : z : 1)$. There are unique coordinates $\bar{P}(x_1, x_2, x_3, x_4)$ representing the same point and satisfying the relation

$$-x_1^2 - x_2^2 - x_3^2 + x_4^2 = 1, x_4 > 0. \quad (2)$$

This means that one can regard hyperbolic space as pseudosphere (2) in Minkowski vector space $\mathbb{R}^{(3,1)}$, with inner product \cdot given by matrix $J = \text{diag}(-1, -1, -1, 1)$.

It is interesting that formula (1) for distance translates into

$$d(P, Q) = \cosh^{-1}(\bar{P} \cdot \bar{Q}),$$

where $\bar{P}(x_1, x_2, x_3, x_4)$ and $\bar{Q}(y_1, y_2, y_3, y_4)$ are coordinates of these points. Therefore, the isometries of Klein projective model are those projective transformations that preserve the pseudosphere, i.e. the inner product given by matrix J . The importance of this model is its linear nature: the lines and planes are linear and isometries are represented as multiplication of vectors by 4×4 matrices that belong to classical linear group $SO(3, 1)$.

1.2 Half-space model

Half-space model consists of all points $P(x, y, z)$ from \mathbb{R}^3 satisfying the relation $z > 0$. The plane $z = 0$ is absolute of this model. Hyperbolic lines are half-circles orthogonal to the absolute (i.e. with center on the absolute and lying in a plane orthogonal to the absolute) and Euclidean rays orthogonal to the absolute. Planes of this model are half-spheres and half-planes orthogonal to the absolute. To be mathematically correct, we add single infinite point P_∞ to \mathbb{R}^3 . This point compactifies \mathbb{R}^3 to sphere S^3 whereas the absolute $z = 0$ becomes two-dimensional sphere, like absolute in the other two models. Each plane and each line in \mathbb{R}^3 can be regarded as sphere and circle, containing P_∞ . The distance in this model is best described using the metric tensor

$$ds^2 = \frac{dx^2 + dy^2 + dz^2}{z^2}. \quad (3)$$

The isometries are compositions of reflections and inversions with respect to hyperbolic planes. We are interested in half-space model since the horosphere has its simplest representation in this model, as we show in the sequel.

1.3 Poincaré ball model

Poincaré ball model $\{P(x, y, z) | x^2 + y^2 + z^2 < 1\}$ is the interior of unit sphere in \mathbb{R}^3 . The unit sphere is absolute of this model. Hyperbolic Lines are parts of circles orthogonal to the absolute and parts of lines orthogonal to the absolute (i.e. passing through the origin). Hyperbolic planes are parts of spheres orthogonal to the absolute and parts of planes orthogonal to absolute. The metric tensor reads

$$ds^2 = \frac{dx^2 + dy^2 + dz^2}{(1 - x^2 - y^2 - z^2)^2}.$$

The isometries are compositions of reflections and inversions with respect to hyperbolic planes. The mapping

$$f(x, y, z) = (2(x, y, z)) / (1 + x^2 + y^2 + z^2) \quad (4)$$

is isometry that maps point $P(x, y, z)$ from Poincaré to Klein model. Isometry between Poincaré ball model and half-space model is simple composition of translations and spherical inversion

$$g(x, y, z) = \frac{1}{x^2 + y^2 + (z - 1)^2} (4x, 4y, 2(1 - x^2 - y^2 - z^2)). \quad (5)$$

2 Horosphere and the related central projection

2.1 Horosphere

Fix a point O on absolute and point $M \in H^3$. Horosphere (with center O containing point M) is set of images of point M in reflections with respect to all planes containing O . Note that, if O is finite point and M' is image of M then OM is congruent to OM' and the horosphere is hyperbolic sphere with center O . Therefore, one may think of horosphere as of sphere with center in infinity.

From construction of horosphere it follows that all horospheres are mutually congruent in H^3 . We want to find the the simplest one in some model. Consider point P_∞ of the half-plane model as center of horosphere, and any finite point $M(x_0, y_0, z_0)$ of that model. All hyperbolic planes through P_∞ are exactly all Euclidean half-planes orthogonal to the absolute $z = 0$. Therefore, all images M' of M have the same z -coordinate and the horosphere is the plane $z = z_0$. Note that this special horosphere touches the absolute in point P_∞ . Since isometries are compositions of inversions and reflections, all horospheres of half-space model are Euclidean spheres that touch the absolute in its center, or planes parallel to the absolute.

From the isometries (4) and (5) between models we conclude that horospheres in Poincaré ball model are spheres touching the absolute in their center and in Klein ball model ellipsoids touching the absolute. The most simple horosphere $z = z_0$ in half-space model has restriction of the metric tensor (3) equal to

$$ds^2 = \frac{dx^2 + dy^2}{z_0^2},$$

showing that the horosphere is isometric to the Euclidean plane, up to a scale. Since in this case the isometry between horosphere and Euclidean plane is given by the imbedding itself, the setup consisting of half-space model and horosphere $z = z_0$ is the one we use for implementation of the central projection.

Reparameterizing, one can write metric (3) in the form:

$$ds^2 = e^{\frac{2t}{k}} (dx^2 + dy^2) + dt^2,$$

where $k > 0$ is the the curvature constant (in our case $k = 1$). This is so called horospherical coordinate system of the hyperbolic space H^3 , which consists of “concentric” horospheres, parameterized by the real line, $t \in \mathbb{R}$. In each horosphere we have Euclidean plane coordinates (x, y) . This was the basic idea for distance measure of János Bolyai in his absolute geometry (in nowadays formulation, Lobachevskii distinguished the non-Euclidean case from the beginning), introduced without any model.

2.2 Central projection onto horosphere

We have shown that we can embed flat computer screen into hyperbolic space as a part of horosphere. What would an observer see on the screen if he is in the hyperbolic space? In any geometry, light ray carrying visual information from the object to the eye of the observer travels along geodesic, i.e. along straight line of that geometry. Klein model is the the easiest to sketch, since the hyperbolic lines of that model are parts of Euclidean lines. If we place the observer O outside the horosphere (see Figure 2, left) some points cannot be projected (point N) while others have two possible projections along the light ray (point M). On the other hand, if we place the observer inside the horosphere (see Figure 2, right) the projection is well defined for all points of hyperbolic space. For this reason, we chose that observer is inside the horosphere, although it is possible to consider the other case.

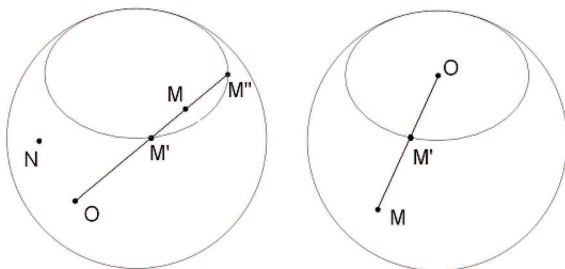


Figure 2: Observer outside and inside the horosphere.

In the half-space model, the point $O(x, y, z)$ is inside the horosphere $z = z_0$ if the condition $z > z_0$ holds. Particular coordinates of point O doesn't matter, so we can chose $O(0, 0, \omega)$, $\omega > z_0$.

Note that all points of half-space model are projected into finite points of horosphere $z = z_0$, except points M of the form $M(0, 0, z)$, $z > \omega$ that are projected into point P_∞ , the infinite point of the horosphere (i.e. screen).

2.3 Central projection of hyperbolic line segment

In order to visualize polyhedra in hyperbolic space we have to understand the projection of hyperbolic line segment.

The line segment is projected into intersection of horosphere and the hyperbolic plane determined by the center of projection O and the segment. For our purposes it is sufficient to consider half-space model, horosphere $z = z_0$ and the point $O(0, 0, \omega)$, $\omega > z_0$.

Recall that hyperbolic line segment is either Euclidean segment orthogonal to the absolute or circular arc orthogonal to the absolute. The following cases are possible:

1. If hyperbolic segment is circular arc AB then hyperbolic plane containing A, B and O is half-sphere. The projection of the segment belongs to intersection of the half-sphere and the horosphere. It is circular arc $A'B'$ in horosphere $z = z_0$ (see Figure 3).
2. If hyperbolic segment is an Euclidean segment AB orthogonal to absolute then the hyperbolic plane OAB is Euclidean half-plane. The central projection of segment AB is
 - a) Euclidean segment $A'B'$ if segment AB have no common point with z -axis above O (that is point $(0, 0, z)$, $z > \omega$;
 - b) Euclidean ray starting from A' if B is of form $B(0, 0, z)$, $z > \omega$. This ray belongs to the intersection line of α and horosphere;
 - c) Two disjoint Euclidean rays starting from A' and B' and belonging to the intersection line of α and horosphere. This happens if segment AB has one inner point of the form $(0, 0, z)$, $z > \omega$;
 - d) Empty set if segment AB is contained in the set $\{(0, 0, z) | z > \omega\}$.

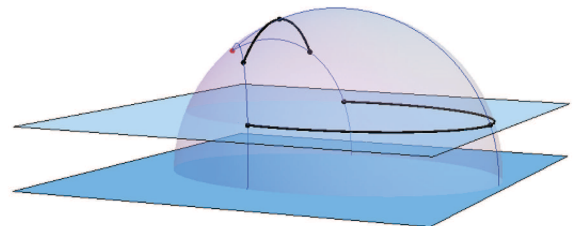


Figure 3: Central projection of hyperbolic line segment.

Note that that there are two circular arcs in horosphere $z = z_0$ with endpoints A' and B' . Therefore, to determine uniquely the projection of segment AB we need the projection C' of some point C belonging to the segment AB . Similar holds for Euclidean ray with the endpoint A' .

2.4 Visibility

We assume that polyhedra consists of faces that are not transparent. Therefore, not all vertices and edges are visible from the center of projection O . We may draw nonvisible edges as dashed lines to improve spacial understanding of the polyhedra (see for example Figure 5). Visibility in Klein ball model coincides with Euclidean visibility. Therefore, to determine the visibility one can use the standard algorithms from \mathbb{R}^3 .

In reality, the human eye sees the part of scene that is inside the cone of vision. In computer graphics this translates into clipping. In this work we don't mind about cone of vision and consider mathematical central projection of polyhedra, i.e. we project points at the front, as well as, at the back of the observer. However, in our package [4] there is an option to draw the circle that separates front and back of the observer. That circle is the intersection of half-sphere with center $(0, 0, 0)$ and radius ω and horosphere $z = h < \omega$.

In Figure 11 (a) this is the black circle. The observer is inside the blue cube, so all its edges are visible, but some part of the cube is in the front and some part is in the back of the observer.

3 Isometries of hyperbolic space

The isometries of hyperbolic space are mathematically well known. The fastest and most elegant way to implement the isometries is to represent them as 4×4 matrices that are applied to column vectors of homogenous coordinates of points in Klein projective model. The homogenous coordinates are of the form $P(x_1 : x_2 : x_3 : x_4)$, where homogeneity means that coordinates $(\lambda x_1 : \lambda x_2 : \lambda x_3 : \lambda x_4)$, for each $\lambda \neq 0$ represent the same point.

As explained in Subsection 1.1 the isometries of Klein model belong to linear group $SO(3, 1)$. They are linear mappings preserving Minkowski inner product, i.e. matrix A of an isometry satisfies $A^T J A = J$, where $J = \text{diag}(-1, -1, -1, 1)$ is diagonal matrix of the inner product. Any hyperbolic isometry is composition of less than four reflections with respect to hyperbolic planes. In this review we don't want to exhaust all isometries, so we present only isometries that are composition at most two plane reflections.

3.1 Reflection with respect to plane (or point)

Plane in the Klein projective model is represented by hyperplane in the vector space $\mathbb{R}^{(3,1)}$, so it has an equation

$$\alpha_1 x_1 + \alpha_2 x_2 + \alpha_3 x_3 + \alpha_4 x_4 = 0$$

The plane has normal vector $n_\alpha = (\alpha_1 : \alpha_2 : \alpha_3 : -\alpha_4)^T$ with respect to the inner product given by matrix J . One can show that the reflection S_α with respect to the plane α has the matrix

$$S[n_\alpha] = I_4 - 2 \frac{n_\alpha n_\alpha^T J}{n_\alpha^T J n_\alpha}, \quad (6)$$

where I_4 is 4×4 identity matrix. It is important to note that the numerator $n_\alpha n_\alpha^T J$ is 4×4 matrix, while denominator $n_\alpha^T J n_\alpha$ is squared norm of vector n_α and therefore a number. Vector n_α is considered column, while n_α^T is row vector.

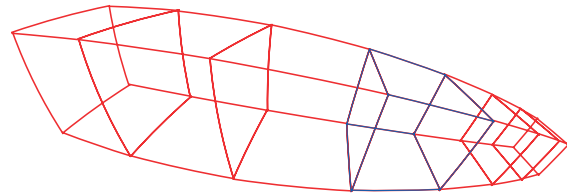


Figure 4: *Cube and its reflections.*

The Figure 4 shows consecutive reflections of the blue cube with respect to the opposite faces (red cubes).

One can show that reflection $S[P]$ with respect to point $P(x_1 : x_2 : x_3 : x_4)$ is given by the matrix

$$S[P] = I_4 - 2 \frac{P P^T J}{P^T J P},$$

which uses the same formula as the reflection 6 with respect to hyperplane. The reflection $S[P]$ is formally defined as composition of three reflections $S[\gamma] \circ S[\beta] \circ S[\alpha]$ with respect to three mutually orthogonal planes α, β and γ that intersect in the point P .

The Figure 5 represents the red cube and its reflection with respect to its vertex.

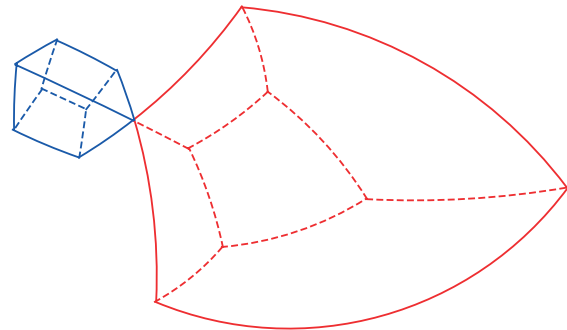


Figure 5: *Central reflection of a cube.*

3.2 Translation

Translation is composition $S[\beta] \circ S[\alpha]$ of two reflections with respect to hyperparallel planes α and β , i.e. the planes that have common normal line n , (that line is unique in hyperbolic geometry). If A and B are intersections of n with α and β , respectively, we call it translation from A to B and denote it $T[AB]$. Note that $T[AB]$ is not translation by vector AB , since the notion of vector is not possible to define in hyperbolic geometry. Many other properties of hyperbolic translation are quite different from its Euclidean counterpart. For example, we have $T[AB](A) = B$ and if $T[AB](M) = N$ then hyperbolic segments AB and MN are not congruent if $M \neq A$. Moreover, the relation $AB \leq MN$ always hold.

This is illustrated in Figure 6 which represent translation of a cube along its edge, i.e. from one vertex to another. Unlike in the Euclidean case, the face of the cube is not translated to the opposite face.

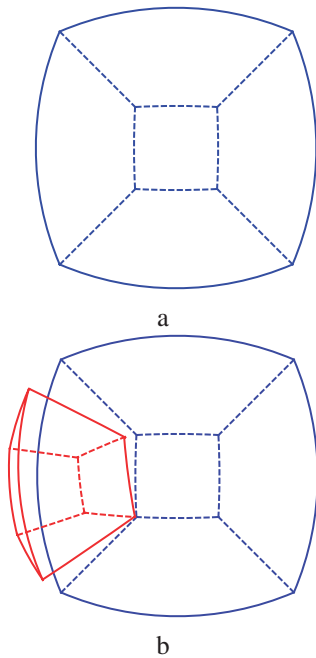


Figure 6: Central projection of cube and its translate.

To find matrix of translation denote by C the hyperbolic midpoint of segment AB . One can show that $T[AB] = S[\beta] \circ S[\alpha] = S[C] \circ S[A]$ and hence its matrix is product of known matrices:

$$T[AB] = S[C]S[A].$$

To find the hyperbolic midpoint C of the segment AB one can use the formula:

$$C = A \sqrt{\frac{(B^T J B)(A^T J B)}{(A^T J A)(A^T J B)}} + B \sqrt{\frac{(A^T J A)(A^T J B)}{(B^T J B)(A^T J B)}}.$$

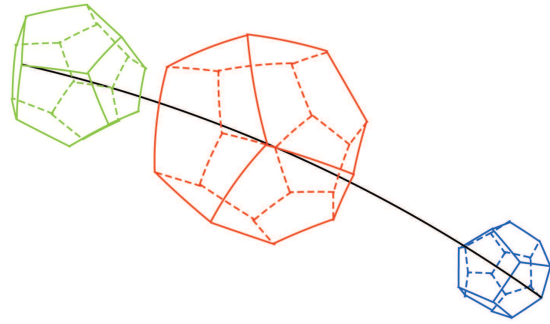


Figure 7: Dodecahedron and its translates.

3.3 Rotation

Hyperbolic rotation is composition $R[p, \phi] = S[\beta] \circ S[\alpha]$ of two reflections with respect to planes that intersect along axes of rotation, the line $p = \alpha \cap \beta$, and the angle between α and β equals $\frac{\phi}{2}$. From formula (1) it follows that Euclidean rotations that fix the origin $O(0,0,0)$ are also hyperbolic rotations. The consequence is that the angles in the origin of the Klein ball model are the same as Euclidean angles (this doesn't hold for any other point of Klein ball model). Therefore, to perform the rotation about line p , translate p to the origin, rotate around translated line p' as Euclidean rotation and finally translate back. To be more specific, if $Q \in p$ is any point of p , then $R[p, \phi]$ is given by the matrix:

$$R[p, \phi] = T[OQ] \circ R[p', \phi] \circ T[QO],$$

where homogenous coordinates of the origin are $O(0 : 0 : 0 : 1)$, $p' = T[QO][p]$, and $R[p', \phi]$ is 4×4 matrix

$$R[p', \phi] = \begin{pmatrix} R^E[p', \phi] & 0 \\ 0 & 1 \end{pmatrix}.$$

Here we denote by $R^E[p', \phi]$ the 3×3 Euclidean matrix of rotation around Euclidean line p' by angle ϕ .

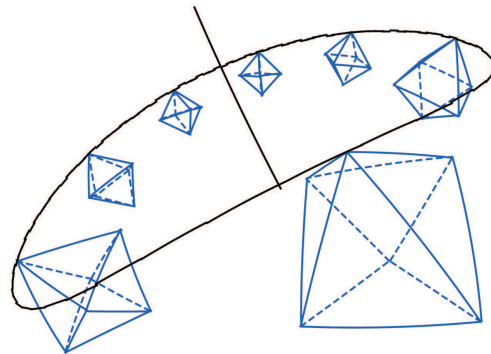


Figure 8: Consecutive rotations of regular octahedron.

3.4 Limit rotation

The limit or horocyclic rotation is probably the most intriguing isometry of hyperbolic plane, since it doesn't exist in Euclidean geometry. Limit rotation is composition $S[\beta] \circ S[\alpha]$ of two reflections with respect to parallel hyperbolic planes α and β , i.e. the planes with single common point on the absolute.

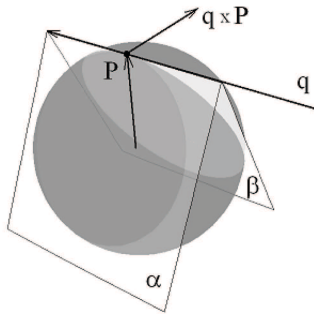


Figure 9: *Horocyclic rotation in the Klein ball model.*

We now describe general formulas of limit rotation. Note that parallel planes α and β in Klein ball model are parts of Euclidean planes α and β that intersect along line q touching the absolute in point $P(x_0, y_0, z_0)$ (see Figure 9).

Planes α and β have normal vector of the form

$$P \cos \phi + (q \times P) \sin \phi = (a_1(\phi), a_2(\phi), a_3(\phi)),$$

for some $\phi = \phi_\alpha, \phi = \phi_\beta$. One can show that the normal vector of these planes in $\mathbb{R}^{(3,1)}$ in homogenous coordinates is given by:

$$n(\phi) = (a_1(\phi) : a_2(\phi) : a_3(\phi) : a_1(\phi)x_0 + a_2(\phi)y_0 + a_3(\phi)z_0).$$

Now, the 4×4 matrix of the limit rotation $S[\beta] \circ S[\alpha]$ is the product $S[n(\phi_\beta)]S[n(\phi_\alpha)]$ of their reflection matrices.

In the Figure 10 a horocyclic rotation is consecutive applied on the blue cube in both directions. One can imagine that “small” cubes converge to a single point in infinity - the center of horocyclic rotation. Of course, all the cubes are congruent but the cubes further from the observer appear smaller.

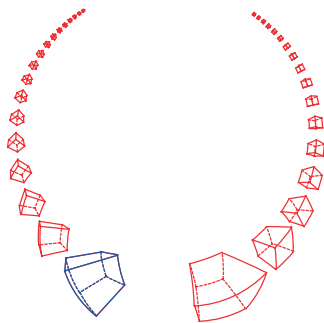


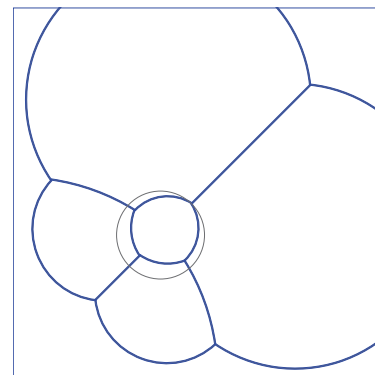
Figure 10: *Horocyclic rotation applied on cube.*

4 Mathematica package L3toHorosphere

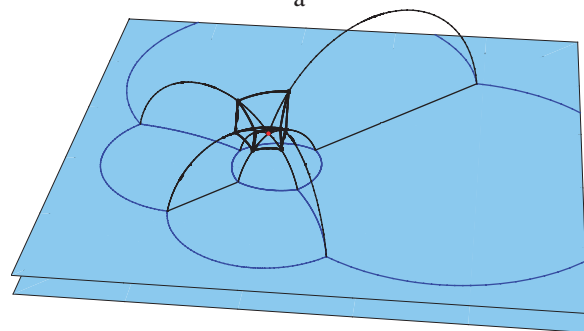
Mathematica package *L3toHorosphere* has three main features:

- central projection of hyperbolic polyhedra onto horosphere;
- visualization of the polyhedra and its projection in half-space model;
- realization of isometries of hyperbolic space.

Within the package, user can define any polyhedral surface in hyperbolic space by defining its faces (as lists of vertices) and vertices (using coordinates in any model). The polyhedral surface than can be repositioned using various hyperbolic isometries. Finally, the projection of the polyhedral surface onto horosphere can be obtained (see figures from Section 3).



a



b

Figure 11: *Observer inside a cube: a) projection, b) situation in the half-space model.*

To achieve better understanding of what's going on, user can also visualize the polyhedral surface and its projection onto horosphere in the half-space model (see Figure 11). Visibility is implemented only for convex polyhedra without boundary.

Now we only list the functions implemented in the package. Some functions have options that can be obtained using command `Options[function]`. More details can be found in the user guide file that is delivered with the package [4].

`drawProjection[ω , k , vert, faces, options]`
- project polyhedra given by `faces` and vertex coordinates `vert` from point $O(0,0,\omega)$ onto horosphere $z = k, \omega > k$.

`modelHS[ω , k , vert, faces, options]` - in half-space model draws polyhedra given by `faces` and vertex coordinates `vert` and its central projection from point $O(0,0,\omega)$ onto horosphere $z = k, \omega > k$.

`reflect[nVector]` - returns 4×4 matrix of reflection with respect to plane α with normal vector `nVector`. The normal vector is given in homogenous coordinates, i.e. is list of 4 numbers.

`reflect[P]` - returns 4×4 matrix of reflection with respect to point P given in homogenous coordinates.

`translate[A,B]` - returns 4×4 matrix of translation from A to B .

`rotate[A,B][ϕ]` - returns 4×4 matrix of rotation by angle ϕ around line AB .

`limitRotate[P][p][ϕ]` - returns 4×4 matrix of limit rotation by “angle” $\phi \neq \frac{\pi}{2} \pmod{\pi}$ around line with direction p and containing unit point P (P orthogonal to p). The points P and p are given in the form (x,y,z) .

`Klein2HS[pt]` - converts Klein projective point `pt` of form $(x_1 : x_2 : x_3 : x_4)$ to half-space model.

`HS2Klein[pt]` - converts half space point `pt` of the form $(x,y,z), z > 0$ to Klein projective point.

`m[A,B]` - returns midpoint of segment AB . All points are in Klein projective coordinates $(x_1 : x_2 : x_3 : x_4)$.

`normalVector[pt1, pt2, pt3]` - returns projective normal vector of plane determined by three points. All points are in Klein projective coordinates $(x_1 : x_2 : x_3 : x_4)$.

`edges[faces]` - returns edges of polyhedra with faces given by `faces`.

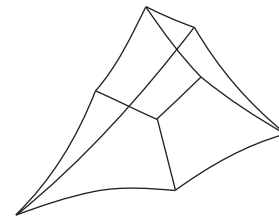
5 Conclusion and future work

The main advantage of our horospherical projection are realistic images of hyperbolic space. As in Euclidean central projection, closer objects appear larger. Furthermore, we don't use models of particular parameterizations - our projections are geometrically invariant and represent what flat Euclidean eye would really see in the hyperbolic space. All other approaches “cheat” in some way. The inevitable drawback of our approach is that projection of hyperbolic segment, in generic case, is circular arc that is complex to render. However, this is not heavy task for modern computers.

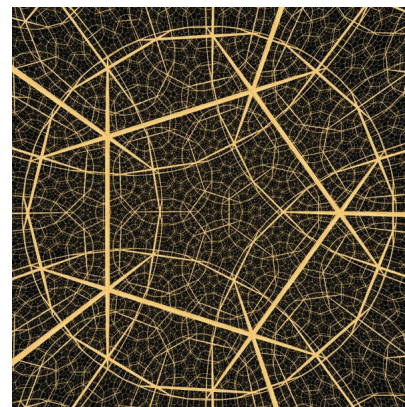
The sum of the angles in a hyperbolic triangle is strictly less than π . Therefore, when sketching hyperbolic objects we usually draw them to be curved concave. However, the projections we get in this paper are curved convex, what may bother someone's intuition. It is interesting that if the observer is placed outside the horosphere (first picture in Figure 2) the projections become curved concave. The Figure 12 (a) shows projection of a cube obtained using this approach.

It would be interesting to implement horospherical central projection of hyperbolic space in some more efficient programming language than Mathematica. This would allow us rendering of more complex hyperbolic scenes and probably lead to very interesting and unusual pictures. However, we are of opinion that even basic Mathematica package [4] we developed, can be very useful for better understanding of hyperbolic geometry and can be used as a good starting point for more complex visualizations.

One of classical scenes is Figure 12 (b), from video *Not knot* ([6]), used to cover many mathematical books. It represents tiling of hyperbolic space with regular dodecahedrons. This classic idea is common inspiration in mathematical art and jewelry making (see [2]). The natural question is to see regular hyperbolic tilings rendered using horospherical central projection.



a



b

Figure 12

Another interesting topic is stereoscopic vision of hyperbolic scenes investigated in the paper [11]. Our preliminary testing, surprisingly, gave positive results, i.e. the human visual system seems to be able to “see” in hyperbolic space.

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