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Steiner Curve in a Pencil of Parabolas

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ABSTRACT

Using the facts from the theory of conics, two theorems that are analogous to the theorems in triangle geometry are proved. If the pencil of parabolas is given by three lines a, b, c, it is proved that, the vertex tangents of all the parabolas in the pencil, envelop the Steiner deltoid curve δ, and the axes of all parabolas in the same pencil envelop further deltoid curve α. Furthermore, the deltoid curves are homeothetic. It is proved that all the vertices in the same pencil of parabolas are located at the 4th degree curve. The above mentioned curves are constructed and treated by synthetic methods.

Key words: Steiner deltoid curve, Wallace-Simson line, pencil of parabolas, vertex tangent

MSC 2010: 51M35, 51M15

Steinerova krivulja u pramenu parabola

SAŽETAK

Koristeći činjenice teorije konika, dokazuju se dva teorema koji su analogoni klasičnih teorema geometrije trokuta. Za pramen parabola zadan trima temeljnim tangentama a, b, c dokazuje se da tjemene tangente svih parabola omataju deltoidu δ, a osi parabola u istom pramenu deltoidu α. Pokazuje se da su deltoide homotetične. Još se dokazuje da sva tjemena parabola u istom pramenu leže na krivulji 4. reda. Spomenute krivulje se konstruiraju i istražuju metodama sintetičke geometrije.

Ključne riječi: Steinerova deltoida, Wallace-Simsonov pravac, pramen parabola, tjemena tangenta

1 Introduction

Some of the (numerous) classical theorems from the geometry of the triangle can be expressed in a different way in order to obtain new theorems in the theory of conics. In this article special attention will be given to the following two well known theorems:

Theorem 1 *If F is any point belonging to the circle k circumscribed to a triangle ABC, then three points W_a, W_b, W_c obtained by orthogonally projecting F, on the three sides of the triangle are collinear. The line thus obtained is called the Wallace-Simson line w of F, [1].*

See Figure 1.

In 1856 Jakob Steiner proved that the envelope of Wallace-Simson lines when F moves around the circumscribed circle to a triangle ABC is a special curve of third class and fourth degree. That curve which has the line at infinity as double ideal tangent, a curve that is tangent to the three sides and to the three altitudes of the triangle, and has three cuspidal points and the three tangent lines on them meet at a point is called the *Steiner deltoid*, [1].

Theorem 2 *The envelope of the Wallace-Simson lines of a triangle ABC is the Steiner deltoid curve, [2].*

See Figure 1.

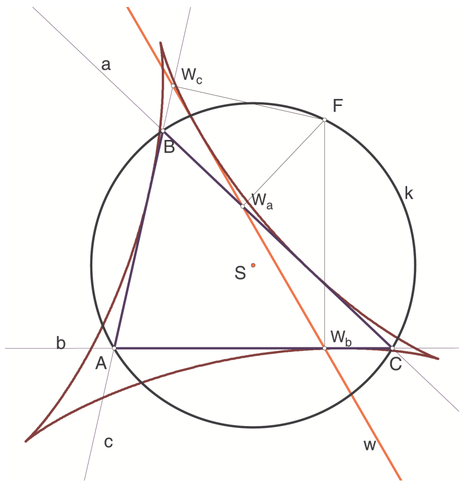

Figure 1

2 Deltoid curves in a pencil of parabolas

It is known for a fact that a pencil of parabolas can be set according to lines a, b, c. Therefore we can prove the following theorems.

Theorem 3 *Let $\{a,b,c\}$ be the pencil of parabolas touching three lines a, b, c. Let F be any point belonging to the circle k circumscribed to the triangle ABC given by the lines a, b, c. The Wallace-Simson line w of the point F is the vertex tangent of one parabola from the pencil.*

Proof. The focus points of all parabolas from the pencil lie on the circumscribed circle k, so the point F is the focus of one (certain) parabola, [3], [4].
It is known for a fact that the pedal curve of a parabola, with respect to pedal point O, is a circular cubic. If the pedal point O is the focus F of the parabola, the pedal curve degenerates into the isotropic lines of the focus F and the vertex tangent. In that case, the vertex tangent is the Wallace-Simson line w, [2]. □

It needs to be highlighted that:

Remark 1 *Each side of the triangle ABC is the Wallace-Simson line of the point antipodal to a vertex of triangle. The side a (b, c) of the triangle is the vertex tangent of the parabola from the pencil of parabolas, that as its focus has the antipodal point to the vertex A (B, C, respectively).*

Remark 2 *In the given pencil of parabolas there are three parabolas degenerated into three pairs of points, the vertex A (B, C) of the triangle and the point at infinite of the line a (b, c, respectively). The altitudes of the triangle ABC are the vertex tangents of three degenerated parabolas, from a pencil $\{a,b,c\}$, respectively they are Wallace-Simson lines of triangle vertices.*

The following theorem is the direct consequence of Theorems 2 and 3.

Theorem 4 *Let the pencil of parabolas touching three lines $\{a,b,c\}$ be given. The envelope of the vertex tangents of the parabolas from the pencil is the Steiner deltoid curve δ, (Figure 2).*

Theorem 5 *If $\{a,b,c\}$ is the pencil of parabolas touching three lines a, b, c, then the envelope of the axes of parabolas from the pencil $\{a,b,c\}$ is a deltoid curve α, that is the dilation image of the Steiner deltoid curve δ, with the center of dilation at the centroid T and a scale factor of -2.*

Proof. Let F be an arbitrary chosen point belonging to the circle k circumscribed to the triangle ABC, and w its associated vertex tangent. When F moves around circle k, the envelope of w is the Steiner deltoid curve denoted as δ. The perpendicular line from F to vertex tangent w is the axes o of the parabola with focus F, from the pencil $\{a,b,c\}$, (Figure 3). Let $A_1B_1C_1$ be the anti complementary triangle of ABC. The sides of the triangle $A_1B_1C_1$ coincide with the axes of the degenerated parabolas from the pencil $\{a,b,c\}$. Furthermore, the deltoid curve α of triangle ABC coincides with Steiner deltoid curve δ (vertex tangent deltoid curve) of anti complementary triangle $A_1B_1C_1$. The triangle ABC is the dilation image of the anti complementary triangle $A_1B_1C_1$ about the centroid T with the factor of $-1/2$. We can conclude that the deltoid curve α is the dilation image of the deltoid curve δ, with the scale factor of -2. □

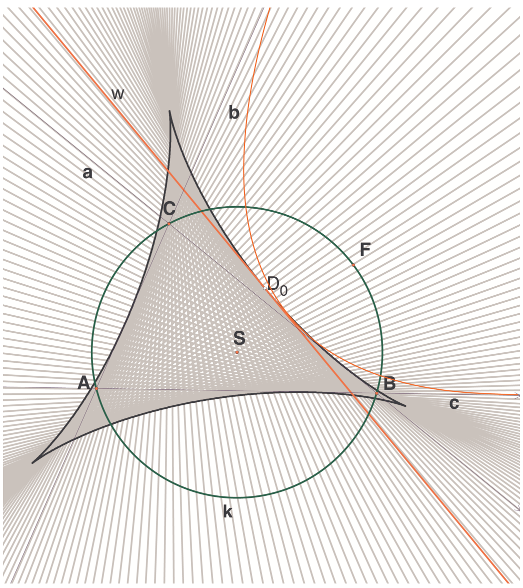

Figure 2

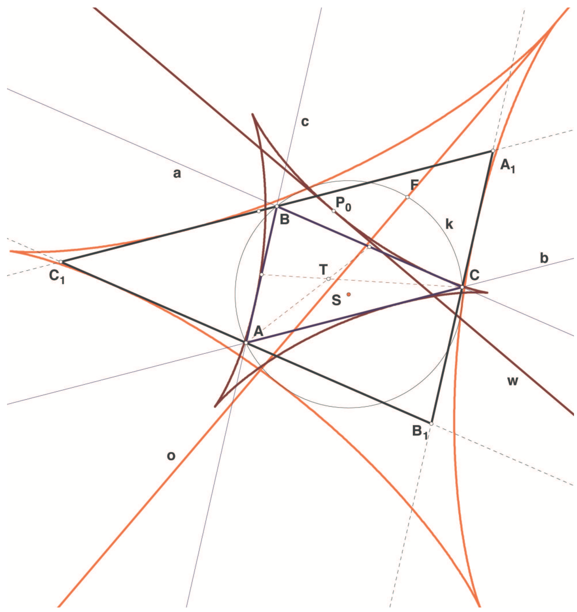

Figure 3

Theorem 6 *Let the pencil of parabolas touching three lines $\{a,b,c\}$ be given. All vertices in the pencil of parabolas $\{a,b,c\}$ lie on the 4^{th} order curve.*

Proof. Let the triangle ABC be determined by lines a, b, c, and let k be its circumscribe circle, $F \in k$. Let the intersection point of the vertex tangent w and the axis o be denoted as T_1. The point T_1 is the vertex of one parabola from the pencil $\{a,b,c\}$.
Let the envelop of the vertex tangent w, and the envelop of the axis o be denoted as δ and α, respectively. We will obtain a bijection between two 3^{rd} class pencils of lines, i.e. two deltoid curves δ and α. Each vertex tangent w of deltoid curve δ is corresponding to the (perpendicular) line o of deltoid curve α. According to Chasles's theorem, this correspondence will result in the 6^{th} order curve, [5]. Since the line at infinity as double tangent of one deltoid curve is corresponding to the line at infinity as double tangent of the other deltoid curve, the 6^{th} order curve degenerates into a quartic β and the line at infinity counted twice. The quartic β touches the line at infinity at the absolute points. Therefore, the line at infinity is an isolated double tangent of three mentioned quartics α, β, δ. □

Remark 3 *The quartic β passes through the vertices A, B, C of the triangle. Vertices A, B, C of the triangle are vertices of three degenerated parabolas from the pencil $\{a,b,c\}$. In Figure 4 focus points of parabolas with vertex tangents a, b, c are denoted as F_a, F_b, F_c. Vertex of the same parabolas are denoted as T_a, T_b, T_c. At these points the quartic β touches the sides of the triangle ABC.*

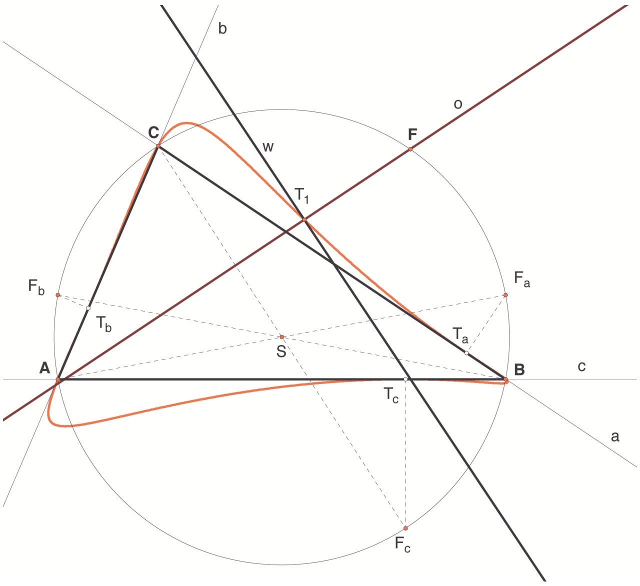

Figure 4

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