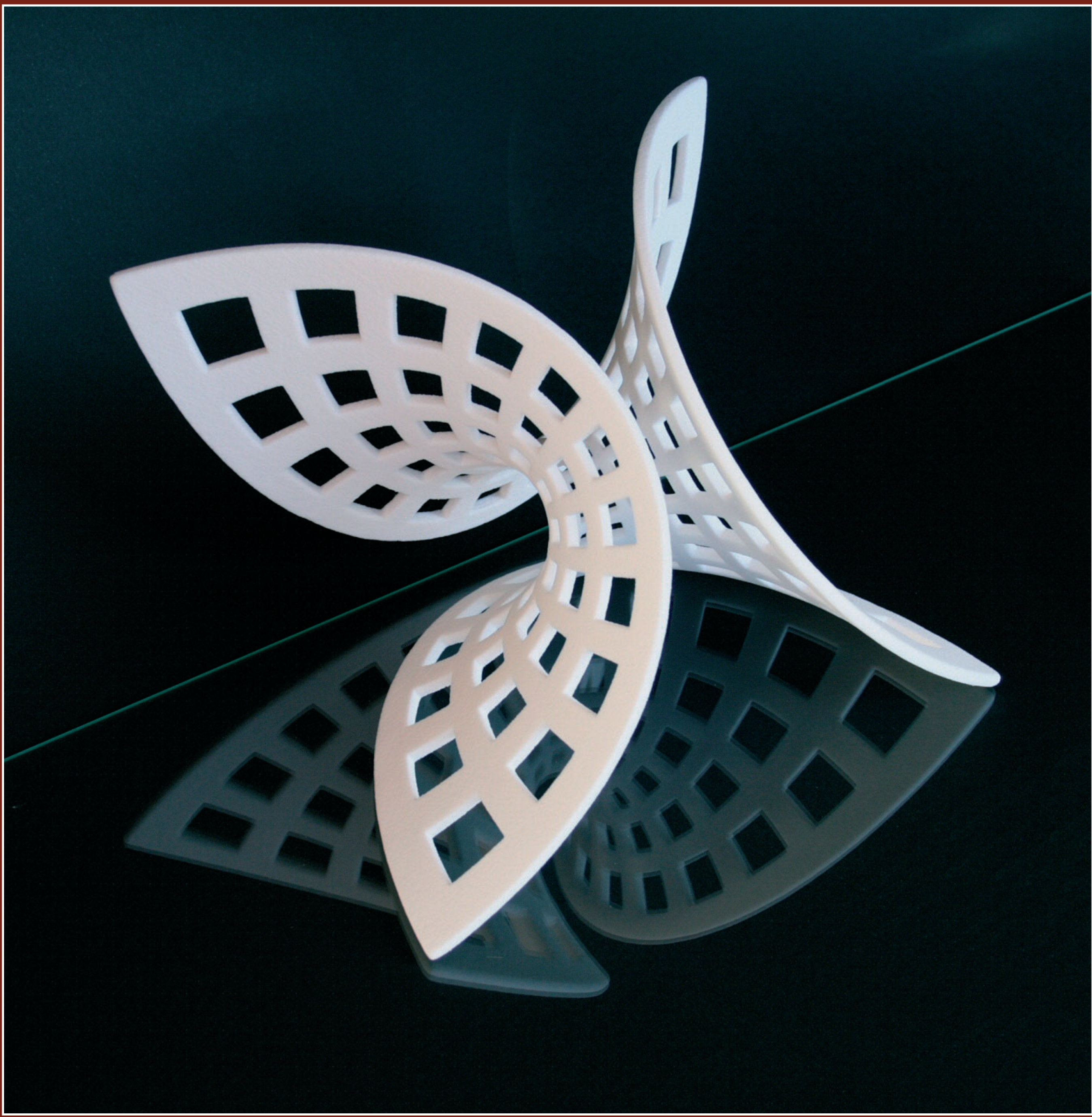
KoG

No. 12. 2008
ISSN 1331-1611

SCIENTIFIC-PROFESSIONAL JOURNAL OF
CROATIAN SOCIETY FOR GEOMETRY AND GRAPHICS



Official publication of the Croatian Society for Geometry and Graphics publishes scientific and professional papers from the fields of geometry, applied geometry and computer graphics.

Founder and Publisher

Croatian Society for Geometry and Graphics

Design

Miroslav Ambruš-Kiš

Layout

Sonja Gorjanc, Ema Jurkin

Cover Illustration

Gregorio Franzoni, *Models of surfaces by 3D printing – photos*

Print

“O-TISAK”, d.o.o., Zagreb

URL address

http://www.hdgg.hr/kog
http://hrcak.srce.hr

Edition

250

Published annually

Guide for authors
Please, see the pages 63, 64

KoG is cited in: Mathematical Reviews, MathSciNet, Zentralblatt für Mathematik

---

This issue has been financially supported by The Ministry of Science, Education and Sport of the Republic of Croatia.

ISSN 1331–1611

KoG

No. 12

Zagreb, 2008

SCIENTIFIC AND PROFESSIONAL JOURNAL OF
CROATIAN SOCIETY FOR GEOMETRY AND GRAPHICS

CONTENTS

IN MEMORIAM

REVIEW

ORIGINAL SCIENTIFIC PAPERS

PROFESSIONAL PAPERS

ISSN 1331–1611

BROJ 12
Zagreb, 2008

ZNANSTVENO-STRUČNI ČASOPIS
HRVATSKOG DRUŠTVA ZA GEOMETRIJU I GRAFIKU

SADRŽAJ

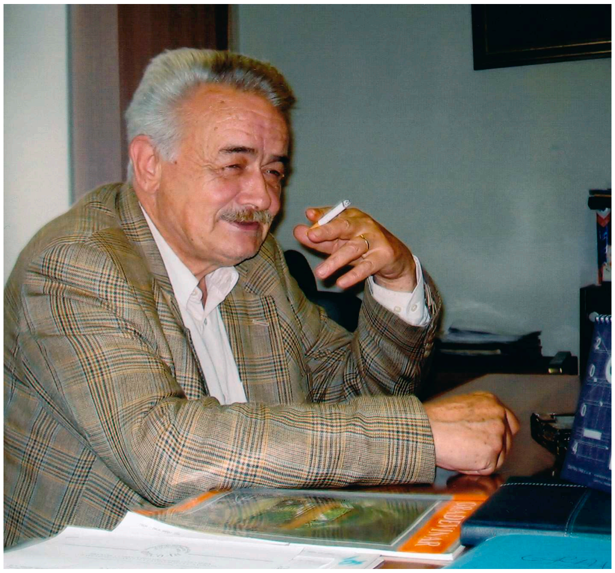
IN MEMORIAM

PREGLEDNI ZNANSTVENI RADOVI

ORIGINALNI ZNANSTVENI RADOVI

STRUČNI RADOVI

SONJA GORJANC, ANA SLIEPČEVIĆ, VLASTA SZIROVICZA



Branko Kučinić

(1936. - 2008.)

Naš profesor BRANKO KUČINIĆ preminuo je 7. listopada 2008. godine u Zagrebu. Bio je umirovljeni redoviti profesor Građevinskog fakulteta Sveučilišta u Zagrebu, dugogodišnji voditelj Katedre za geometriju. Njegova nadahnuta predavanja pamte generacije hrvatskih građevinskih inženjera i geometričara-nacrtnaša.

Rođen je 28. siječnja 1936. godine u Rijeci. Osnovnu školu i gimnaziju polazio je u Bjelovaru, Đurđevcu, Koprivnici i Zagrebu, a 1961. godine diplomirao je matematiku na Prirodoslovno-matematičkom fakultetu Sveučilišta u Zagrebu diplomskim radom iz područja geometrije – *Vitopero kolinearni prostori*.

Još kao apsolvent, od 1959. do 1961., godine predavao je matematiku i fiziku na Osnovnoj školi Orlovac u Zagrebu, a nakon diplome odlazi na jednogodišnje služenje vojnog roka. Godine 1962. zaposlio se kao asistent profesora Vilka Ničea na Katedri za nacrtnu geometriju na tadašnjem Arhitektonsko-građevinsko-geodetskom fakultetu u Zagrebu. Iste se godine AGG fakultet razdvaja na tri fakulteta na kojima profesor Niče vodi nastavu iz Nacrtne geometrije, a Branko Kučinić postaje njegovim asistentom na Građevinskom fakultetu na kojem neprekinuto radi do svog umirovljenja. U zvanje docenta izabran je 1970. godine, u zvanje izvanrednog profesora 1974., a u zvanje redovitog profesora 1986. Godine 2006. odlazi u mirovinu kao redoviti profesor u trajnom zvanju.

Okruženje u kojem je mladi Branko Kučinić, čiji su se talent i sklonost geometriji pokazali već tijekom studija, započeo svoje znanstveno djelovanje bilo je iznimno povoljno. Osobno je kao svoje učitelje uvijek isticao akademike Vilka Ničea i Stanka Bilinskog. Svakodnevno je radio uz profesora Ničea, vrhunskog znanstvenika u području sintetičke i, posebno, sintetičke projektivne geometrije, koji je znao okupiti mlade i sposobne ljude i prenijeti na njih svoju ljubav i oduševljenje za znanstveni rad, te ponuditi velik broj tema za daljnje istraživanje. U svom tekstu [19] Kučinić ga naziva "bardom hrvatske geometrije". Tijekom poslijediplomskog studija Kučinić je došao do nekih originalnih spoznaja iz područja neeuklidske geometrije, predočio ih profesoru Stanku Bilinskom koji je prepoznao njihovu vrijednost te rekao mladom asistentu kako je upravo najavio temu svoje disertacije. Kučiniću je to bio pravi poticaj; već 1966. godine napisao je svoj doktorski rad. Promjena zakona, prema kojem se više nije moglo doktorirati, a da se prethodno ne magistrira, prisilila ga je na izradu još jedne radnje, tako da je 1966. prvo magistrirao [1], a 1967. i doktorirao [2] na Prirodoslovno-matematičkom fakultetu Sveučilišta u Zagrebu te postao jedan od najmlađih doktora matematike u tadašnjoj Jugoslaviji.

Slijedile su godine Kučinićeva vrlo produktivnog znanstvenog rada. U području projektivne i euklidske geometrije izučava posebna izvođenja krivulja i ploha (radovi [3], [4]), a najznačajniji doprinos daje u području hiperboličke geometrije: autor je dvaju novih modela hiperboličke ravnine ($\phi-$modela i $V-$modela), istražuje veze između različitih modela hiperboličke ravnine koji su, naravno, izomorfni, ali se razni pojedinačni problemi u njima različito rješavaju, posebno se bavi Kleinovim i Gyarmathijevim modelom i u njima rješava neke konstruktivne i metričke probleme (radovi [5], [6], [7], [8], [9], [10] i [14]). Od 1970. do 1980. bio je član-suradnik tadašnje JAZU.

Sedamdesete godine prošloga stoljeća također su i razdoblje Kučinićeva intenzivnog društvenog i nastavničkog rada u tadašnjoj SR Hrvatskoj. Godine 1970. imenovan je voditeljem novoosnovane *Katedre za teoretske predmete* na Građevinskom fakultetu u Zagrebu koja je tada u svom matematičkom dijelu imala osam članova. Kučinić je imao viziju "velike katedre za matematiku", inzistirao je na upošljavanju novih nastavnika i većeg broja mladih matematičara. Katedra je 1973. prerasla u *Zavod za matematiku*. Na poslijediplomskom studiju matematike na PMF-u u Zagrebu, akademske godine 1972/73 predavao je kolegij *Modeli hiperboličke geometrije* i bio mentor na dva magistarska rada (Vlasta Szirovicza i Jasna Kos-Modor). Mlade je suradnike poticao na publiciranje, na njegovu je inicijativu i u njegovoj redakciji 1975. godine objavljena stručna knjiga [11] u kojoj je svaki član Zavoda napisao po jedan članak. Zavod je tih godina imao 24 člana te bio jedan od najjačih zavoda za matematiku izvan Matematičkog odjela PMF-a. Manji dio Zavoda činila je *Katedra za geometriju* kojoj je profesor Kučinić bio na čelu sve do umirovljenja. Intenzivno je, zajedno sa svojim suradnicima, radio na osuvremenjivanju nastave nacrtne geometrije i obogaćivanju nastavnog materijala. Može se reći da je u tome bio ispred svog vremena. Aktivnu nastavu, koju je još prije više od trideset godina uveo na Građevinskom fakultetu u Zagrebu, nalazimo danas u temeljima Bolonjskog procesa. Godine 1978. bio je voditelj interdisciplinarne pedagoško-matematičke teme *Kibernetičko zasnivanje nastave geometrijskih predmeta*, koja je nastala kao rezultat suradnje s Pedagoškim odjelom Filozofskog fakulteta u Zagrebu. Zajedno sa suradnikom Slavkom Hozjanom, 1979. godine objavio je prijevod knjige [12]. Uveo je kolegij *Primijenjena geometrija* koji je, sve do posljednje visokoškolske reforme, prihvaćen i predavan na prvoj godini studija na svim građevinskim fakultetima u Republici Hrvatskoj. Predavao je i u vojnim institucijama: Visokoj tehničkoj školi u Zagrebu te Inženjerskoj akademiji u Karlovcu. Sudjelovao je u organizaciji studija građevinarstva u Splitu, Rijeci i Osijeku te je, osim na Građevinskom fakultetu u Zagrebu, nekoliko godina predavao na Građevinskom fakultetu u Splitu (gdje ga je kasnije zamijenila njegova asistentica Zdravka Božikov), a godinu dana i na Odjelu zagrebačkog Građevinskog fakulteta u Varaždinu. Kao jedan od rijetkih izvan Prirodoslovno-matematičkog fakulteta, 1976.-78. godine bio je predsjednik tadašnjeg *Društva matematičara i fizičara SR Hrvatske*.

Rad na Građevinskom fakultetu i bliska suradnja s inženjerima graditeljskih struka uvelike određuju daljnji smjer Kučinićevog rada i istraživanja. U razdoblju nakon 1982. godine područje njegova primarnog interesa postaje primijenjena geometrija – obrada ploha primjenjivih u graditeljstvu. Oživljavajući ideje svog učitelja Vilka Ničea, kombinirajući metode sintetičke i konstruktivne geometrije, osim pravčastih ploha projektivnoga prostora (o kojima u okviru kolegija Primijenjene geometrije na vrlo visokom nivou predaje studentima građevinarstva već na prvoj godini studija), obrađuje one algebarske plohe euklidskoga prostora koje sadrže apsolutnu koniku. Pretpostavlja da su one, zbog bogatstva kružnih presjeka, vrlo primjenjive u graditeljstvu. Rezultat tog rada su interdisciplinarni elaborat (matematika, arhitektura, građevinarstvo) *Kugle višeg reda u graditeljstvu* za Građevinski institut (1984.), radovi [15] i [18] te knjiga [17] koju, u izdanju *Građevinara*, objavljuje u koautorstvu sa suradnicima arhitektima Ivanom Salerom i Olgom Kristoforović. Na izdanju leksikona [22] (glavni urednik Veselin Simović) Kučinić je radio kao jedan od urednika i autora.

U razdoblju osamdesetih godina prošloga stoljeća, Kučinić intenzivira svoj rad na obrazovanju mladih nastavnika i znanstvenika na području nacrtne i sintetičke geometrije. Ponovno na tragu profesora Ničea, koji je početkom sedamdesetih bio suosnivačem Postdiplomskog studija za nacrtnu geometriju i perspektivu na Arhitektonskom fakultetu u Beogradu, Kučinić uz velike nastavničke obveze na svom fakultetu, gdje je godinama (prema anketama provedenim među studentima) jedan od najboljih profesora, izvodi nastavu i na tom beogradskom studiju. Kolegij *Hiperbolička geometrija i modeli* predaje u tri ciklusa, 1979.-81., 1984.-86. i 1989.-90. Pod njegovim mentorstvom magistrirali su Ivan Saler, Sonja Gorjanc i Lidija Pletenac, a doktorirali Olga Kristoforović, Ivan Saler i Ivanka Babić. U to je vrijeme rad na doktorskoj disertaciji pod njegovim mentorstvom započela i Ana Sliepčević, koja ga je petnaest godina kasnije naslijedila na mjestu voditelja Katedre za geometriju na Građevinskom fakultetu u Zagrebu. Bio je vrlo cijenjen u krugovima nacrtnaša diljem tadašnje Jugoslavije koji su se, više od trideset godina, redovito, svake druge godine, sastajali na svojim savjetovanjima. Posljednje, *17. jugoslavensko savjetovanje za nacrtnu geometriju* održano je u Zagrebu 1990. godine, Kučinić je bio organizator koji je pokrenuo izdavanje Zbornika savjetovanja (urednica je bila Vlasta Szirovicza) te inicirao i formalnu registraciju strukovne udruge. Tako je 1990. osnovano *Jugoslavensko udruženje za nacrtnu geometriju i inženjersku grafiku*, a na osnivačkoj je skupštini Branko Kučinić jednoglasno izabran za predsjednika. To je udruženje, zbog rata i raspada Jugoslavije, ubrzo prestalo sa svojim izvorno zacrtanim djelovanjem. Međutim, potreba za stvaranjem strukovnog udruženja nacrtnaša (specifične grupacije geometričara na tehničkim fakultetima), koju je Branko Kučinić već davno uočio, ostala je izražena i u samostalnoj Republici Hrvatskoj. Godine 1994. osnovano je *Hrvatsko društvo za konstruktivnu geometriju i kompjutorsku grafiku* (današnje Hrvatsko društvo za geometriju i grafiku) čija je prva predsjednica bila Vlasta Ščurić-Čudovan, a prvi dopredsjednik Branko Kučinić koji je tu dužnost obnašao do 1998. godine. Ta je udruga 1996. izdala prvi broj znanstveno-stručnog časopisa KoG u kojem je Kučinić objavio članak [20].

Na Kučinićevo poznavanje hiperboličkog prostora mogla se osloniti i Ivanka Babić koja je u svojoj doktorskoj disertaciji uvela nacrtnu geometriju u hiperbolički prostor. Izgradila je novi *M-model* koji je omogućio konstruktivnu obradu hiperboličkog prostora Mongeovom metodom projiciranja i perspektivom. U tom su koautorstvu objavljeni posljednji Kučinićevi znanstveni radovi [19] i [21].

Branko je bio lucidan, pronicljiv i duhovit čovjek, s velikim razumijevanjem za ljude, naročito mlade. Volio je društvo, bio ljubitelj dobre hrane i pića te izvrsno kuhao. Puno nas je puta ugostio odlično pripremljenim specijalitetima. Bio je naš učitelj, kolega i prijatelj. Sjećat ćemo ga se s poštovanjem i ljubavlju.

Popis radova

[1] Kučinić, Branko: *Apsolutna konika*, magistarski rad, 1966.

[2] Kučinić, Branko: *Teorija mreže konika i neki modeli geometrije hiperboličke ravnie*, disertacija, 1967.

[3] Kučinić, Branko: *Über Zusammenhang zweier Herleitungen ebener zirkularer Kurven 3-er Ordnung vom Geschlecht Null*, Glasnik matematički, **3**(23)(1968), 105-116

[4] Kučinić, Branko; Kranjčević Eduard: *Polarno - podernye krivye i poverhnosti*, Glasnik matematički, **4**(24)(1969), 139-156

[5] Kučinić, Branko: *Ein Modell der hyperbolischen Ebene in der Teorie der Kegelschnittnetze*, Glasnik matematički, **5**(25)(1970), 319-333

[6] Kučinić, Branko: *Specialjnyj slučaj odnoj modeli geometrii giberboličeskoj ploskosti*, Rad JAZU, **394**(1971), 159-165

[7] Kučinić, Branko: *Veza ϕ modela i Kleinovog modela geometrije hiperboličke ravnine*, Zbornik na trudovite I-Matematika, V. kongres na matematičarite na Jugoslavija, Skopje (1973), 131-137

[8] Kučinić, Branko: *O nekotoryh novejših modeljah giberpoličeskoj plosksti*, Glasnik matematički, **10**(30)(1975), 101-108

[9] Kučinić, Branko: *Grafičko određenje hiperboličkog kuta u Kleinovom modelu i jedno izvođenje Laguerrove formule*, Matematika I (1975), 20-22

[10] Kučinić, Branko: *O V–modeljah giperboličeskoj ploskosti*, Rad JAZU, **379**(1975), 117-118

[11] Kučinić, Branko; i dr.: *Iz matematičkog mozika*, Školska knjiga, Zagreb, 1975 (članak u zbirci)

[12] A. Savelov: *Ravninske krivulje*, Školska knjiga, Zagreb, 1979 (prijevod s ruskog B. Kučinić i S. Hozjan)

[13] Kučinić, Branko: *Jedan model projektivne ravnine*, Zbornik Građevinskog fakulteta II, (1975), 5-7

[14] Kučinić, Branko: *Sintetička veza Kleinovog i Gyarmathijevog modela h–ravnine*, Zbornik za prirodne nauke Matice srpske, **63**(1982), 183-186

[15] Kučinić, Branko: *Metarmofoze kugle u graditeljstvu*, Građevinar, **34**(1982), 301-308

[16] Kučinić, Branko: *O krivulji 2. reda*, Jezik, **3**(1984)94

[17] Kučinić, Branko; Saler, Ivan; Kristoforović, Olga: *Oble forme u graditeljstvu*, GRAĐEVINAR, Zagreb, 1992 (knjiga)

[18] Kučinić, Branko: *Brza industrijska gradnja sakralnih i drugih velikih objekata*, Građevinar, **45**(1993), 141-145

[19] Babić, Ivanka; Kučinić, Branko: *M–Modell des hyperbolischen* H^3 *– Raums in der Möbius–ebene*, Rad HAZU, (467)11(1994), 67-75

[20] Kučinić, Branko: *200 godina sustavnog grafičkog komuniciranja*, KoG, 1(1996), 35-37

[21] Babić, Ivanka; Kučinić, Branko: *Hypebolische Perspektive*, KoG, 2(1997), 29-34

[22] *Leksikon građevinarstva*, Masmedia, Zagreb, 2002. (Kučinić Branko je jedan od grupe urednika i autora)

Übersichtsartikel
Angenommen am 15.10.2008.

GUNTER WEISS
FRANZ GRUBER

Den Satz von THALES verallgemeinern - aber wie?

Herrn Prof. Dr. Hellmuth Stachel zum 65. Geburtstag gewidmet.

How to generalize Thales' Theorem?

ABSTRACT

The classical theorem of Thales can be generalized – depending on someones interpretation – in several elemental or abstract ways. This paper tries to classify such generalizations without claim of completeness. We structured this work in a way that is suitable for educational purposes by emphasizing aspects of mathematical research. Beside more or less known facts, we present new insights and approaches to one of the most important theorems of geometry.

Key words: Thales' Theorem 3D, constrained motions, spatial kinematics, trihedron

MSC 2000: 51M04

Kako poopćiti Talesov teorem?

SAŽETAK

Klasičan Talesov teorem moguće je poopćiti – ovisno o interpretciji – na nekoliko elementarnih i apstraktnih načina. U ovom se radu pokušava, ne zahtijevajući potpunost, klasificirati te generalizacije. Svojom strukturom rad je prilagođen obrazovnim svrhama s naglasakom na matematičkom istraživanju. Uz više ili manje poznate činjenice, predstavljamo nove uvide i pristupe jednom od najznačajnijih teorema geometrije.

Ključne riječi: Talesov 3D teorem, ograničeno gibanje, prostorna kinematika, trobrid

1 Der klassische Satz von Thales

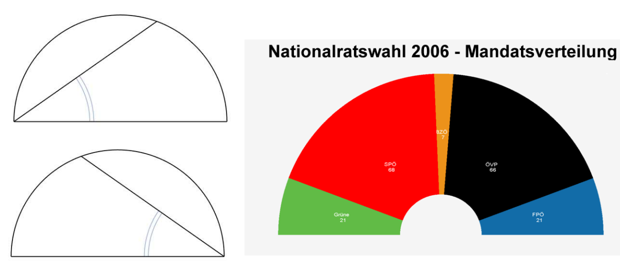
Schlampig formuliert lautet er so:

"Jeder Winkel im Halbkreis ist ein rechter."

Formulierungen dieser Art überleben, ebenso wie "der Pythagoras", die Vergessensjahre nach dem Ende der schulischen Ausbildung bis ins hohe Alter. Man sollte sich als Lehrer die Zeit nehmen, wissensunbelastete Schüler im Elternhaus nach dem Satz von Thales fragen zu lassen und die gesammelten Antworten inhaltlich statistisch auswerten! Anschließlich könnte man auf die Diskrepanz zwischen Gesagtem und Gemeintem hinweisen (vgl.Abb 1(a)) und so zur Schärfung des Sprachvermögens der Schüler beitragen. Übrigens kann man sich auch aus Spaß gleich an die Umkehrung des unscharf formulierten Thales-Satzes machen, die dann zu folgendem Statement Anlass gäbe (vgl.Abb. 1(b), [15]):

"Jeder Rechte ist ein Winkel im Halbkreis! "

- was offensichtlich so nicht stimmt.



(a) (b)

Abbildung 1: *Beispiel eines linken bzw. rechten Winkels im Halbkreis (a) und "Satz-Umkehrung" (b)*

Wie ist also der Satz zu formulieren, dass er dem (üblicherweise richtig) Gemeinten, Abb.2, entspricht?

(T1) *"Sind A und B Durchmesser-Endpunkte eines Kreises k und ist S ein (von A und B verschiedener) Kreispunkt, so ist der Winkel $\angle ASB$ ein rechter, sein Winkelmaß $\sphericalangle ASB$ also $\pi/2$ oder 90°."*

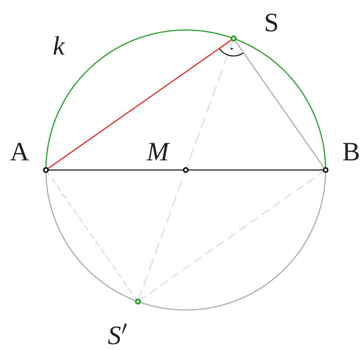


Abbildung 2: *Der Satz von Thales mit Ergänzungen für einen Beweis*

Beweis etwa durch Punktspiegelung von S am Mittelpunkt M von k, (vgl. auch [19]): Das entstehende Viereck $\{ASBS'\}$ besitzt gleich lange Diagonalen (Kreisdurchmesser) mit gleichem Halbierungspunkt. Es ist also ein Rechteck.

Menschen denen "*genau dann*" - Formulierungen - noch oder schon wieder - fremd sind, werden aus dem obigen Beweis intuitiv bereits die Umkehrung des Satzes (T1) mitnehmen und keine Beweisnotwendigkeit mehr verspüren: S nicht auf $k \Rightarrow$ Diagonalen von $\{ASBS'\}$ ungleich lang, aber gemeinsame Mitte M, $\Rightarrow$ kein Rechteck, sondern schiefes Paralellogramm. Eh klar! Und trotzdem muss man beim expliziten Formulieren der Umkehrung fast immer helfen:

(**T2**) "*Ist ΔASB ein rechtwinkeliges Dreieck mit Hypothenuse $[A,B]$, so geht der Kreis mit Durchmesserstrecke $[A,B]$ durch S.*"

Über die Limesfigur in der Grenzlage $S \to A$ oder $S \to B$ wird man bis zur 9.Schulstufe vielleicht noch nicht reden wollen und können. Ein Grund mehr, den Satz in der (gymnasialen) Oberstufe wieder hervor zu holen und an entsprechende Lehrplaninhalte (Vektorrechnung, Differentialrechnung) zu koppeln.

Erarbeitet man mit Schülern den Satz von Thales, so werden bestimmt auch kinematische Formulierungen kommen:

(**T3**) "*Gleiten die Schenkel eines Rechtwinkelhakens durch zwei feste Punkte A und B, so durchläuft der Scheitel S einen (Halb-) Kreis k über dem Durchmesser $[A,B]$*"

Der so erklärte "Thales-Zwangslauf" ist die Umkehrung einer Ellipsenbewegung, wie sie bekanntlich durch einen klassischen Ellipsenzirkel repräsentiert wird.

(**T4**) "*Gleitet der Scheitel S eines Rechtwinkelhakens entlang eines Kreises k und ein Winkelschenkel durch einen festen Punkt A von k, so umhüllt der zweite Winkelschenkel den A gegenüberliegenden Punkt B (Gegenpunkt) von k.*"

Diese Formulierungen werden zum Ausgangspunkt für sehr unterschiedliche Verallgemeinerungen, von denen einige den Schulstoff weit hinter sich lassen. Dies gilt im gleichen Maße auch für die folgende Version des Thales - Satzes:

(**T5**) "*Fällt man auf die Geraden a eines Büschels mit Scheitel A aus einem Punkt $B \neq A$ die Normalen b, so erfüllen die Schnittpunkte S zugeordneter Geraden a und b einen Kreis k mit Durchmesser $[A,B]$.*"

Bei dieser Auffassung sind die Sonderlagen $S = A$ und $S = B$ in natürlicher Weise miterfasst. Inhaltlich mit (T5) ident, aber in eine andere Verallgemeinerungsrichtung führend, ist folgende Formulierung:

(**T6**) "*Die Fußpunktkurve eines Geradenbüschels $\{a \mid A \in a \subset \pi;\ A, \pi$ fest$\}$ für einen festen Punkt $B \in \pi, (B \neq A)$, als Pol ist ein Kreis k, der Thaleskreis über $[A,B]$.*"

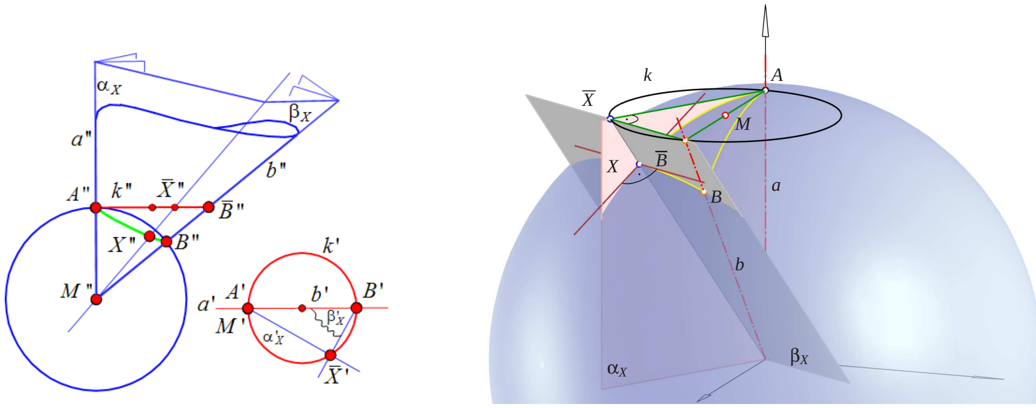
2 Grundidee der Verallgemeinerung des Satzes von Thales

Die *klassische*, *ebene* Figur besteht aus Punkten A, B, einem Kreis oder Halbkreis k mit den Gegenpunkten A, B und dem Winkelscheitel S mit den Schenkeln SA und SB, die zu einander normal sind. Wir werden daher (T1) in verschiedene, zum Teil völlig unabhängige Richtungen verallgemeinern können.

Verzichtet man auf Rechtwinkeligkeit der Winkelschenkel unter Beibehaltung der übrigen Elemente, gelangt man zur bekannten Ausssage des *"Peripheriewinkelsatzes"*. Wir werden diesen Verallgemeinerungsstrang hier nicht weiter verfolgen.

3 Sphärische Version des Satzes von Thales

Es ist nahe liegend, zunächst das Wort *klassisch*, das i.w. *euklidisch* meint, durch *"nicht-euklidisch"* zu ersetzen. Auch diesen Verallgemeinerungsstrang werden wir hier nicht im Detail verfolgen, sondern nur den der elementaren Anschauung zugänglichen sphärischen Fall untersuchen, der ja die elliptische ebene Geometrie repräsentiert. Es ist dabei zunächst zweckmäßig, von (T5) auszugehen und sinngemäß die geradlinigen Winkelschenkel zu Großkreisbogen abzuändern.



(a) Auf- Grundriss (b) Axonometrie

Abbildung 3: *Konstruktion der Scheitel-Ortslinie für den sphärischen Satz von Thales*

Zur sphärischen Konstruktion der Ortslinie des Scheitels S werden die Angabeelemente A und B aus dem Kugelmittelpunkt M auf die Tangentialebene π in A projiziert. Das Orthogonalstehen der Winkelschenkelbogen a und b durch A bzw. B reproduziert sich dabei in orthogonalen Ebenen um MA und MB, deren Spuren in π gleichfalls rechtwinkelig sind und demnach dort einen gewöhnlichen Thaleskreis k' erzeugen. M verbunden mit k' ist also ein *"orthogonaler (Kreis-) Kegel"* Γ, denn seine Kreisschnittebene π ist normal zu einer seiner Erzeugenden, nämlich zu MA. Die gesuchte Ortslinie k ist demnach (ein Ast der) Schnittkurve dieses orthogonalen Kegels Γ mit der Kugel und somit eine Kurve 4. Ordnung, ein *"sphärischer Kegelschnitt"* (Abb.3).

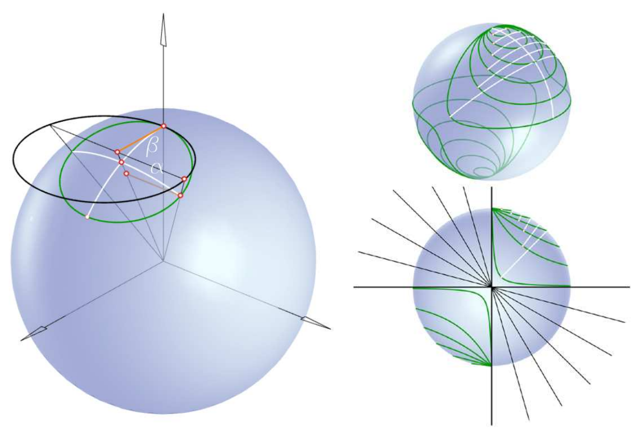
Der Kegel $\Gamma := M \vee k'$ erfüllt die Gleichung

$$x^2 + y^2 - \tan(2\alpha) yz = 0 \qquad (1)$$

Wir verwenden dabei M als Ursprung, MA als z-Achse und MAB als yz-Ebene eines kartesischen Koordinatensystems mit $\overline{MA} = 1$ und α messe den halben Winkel $\sphericalangle AMB$. Hieraus folgt, dass der Aufriss k'' der gesuchten Thales-Kurve k auf einer Hyperbel h mit Mitte M liegt, deren Asymptoten durch die Normalen zu MA bzw. MB repräsentiert sind. Die halbe Hauptachsenlänge von h ist dabei $\frac{1}{2}\sin(2\alpha)$, sodass sich für den halben Nebenachsenbogen β von k die Länge

$$\sin\beta = \tan\alpha \quad \text{bzw.} \quad \beta = \arcsin(\tan\alpha) \qquad (2)$$

ergibt (Abb.4(a)). Man beachte, dass stets $\alpha < \beta$ gilt und dass ein klassisch "elliptisches" Erscheinungsbild von k nur für $2\alpha < 90°$auftritt. Für $\alpha = 90°$ zerfällt k in zwei Kreisbögen in Ebenen normal zur Symmetrieebene MAB. Für $2\alpha > 90°$ ist einer der Punkte A oder B durch seinen Gegenpunkt zu ersetzen (Abb.5).



(a) $\sin\beta = \tan\alpha$ (b) Sphärische Kegelschnitte

Abbildung 4: *Illustrationen zu (ST1)*

(ST1) *Sphärischer Thales-Satz: Der Ort der Scheitel sphärischer rechter Winkel, deren Schenkelbogen durch zwei Kugelpunkte A und B (B nicht Gegenpunkt von A) gehen, ist ein sphärischer Kegelschnitt mit dem Achsbogen* $[A,B]$. *Hat* $[A,B]$ *die sphärische Länge* 2α, *so ist die zweite Achse von der Länge* $2\beta = 2\arcsin(\tan\alpha)$. Würde man von der kinematischen Auffasssung (T4) ausgehen, so ist auch folgende Idee ganz natürlich und auf die Kugel verallgemeinerbar:

(T4') *Gleitet in der euklidischen Ebene ein Rechtwinkelhaken* (S,a,b) *mit seinem Scheitel S längs eines Kreises k, während sein Schenkel a durch einen festen Punkt A gleitet, so umhüllt der Schenkel b i.A. eine Kurve* 2. *Ordnung* k_b *(Kegelschnitt), die "Anti-Fußpunktkurve" von k bezüglich des Pols A. Diese Kurve degeneriert in einen Punkt B genau für A aus k.*

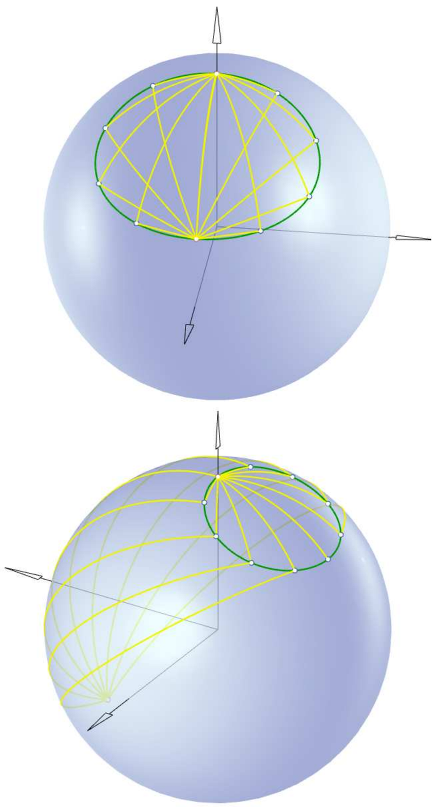
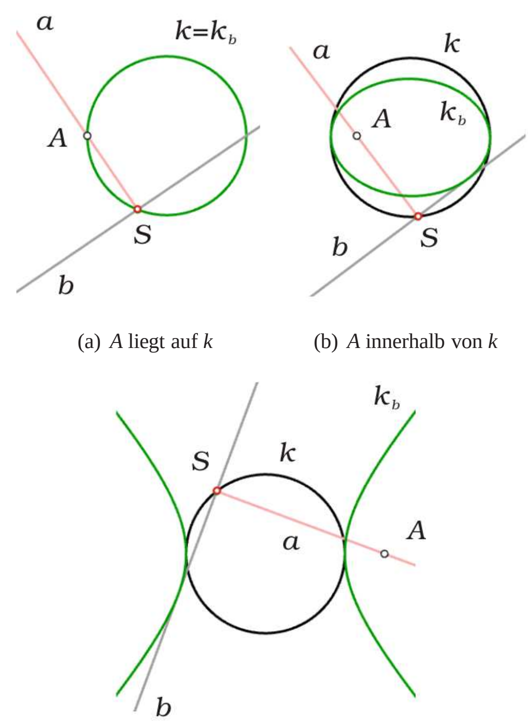


Abbildung 5: *Sphärischer Satz von Thales* $2\alpha \lesseqgtr 90°$



(a) *A* liegt auf *k* (b) *A* innerhalb von *k*

(c) *A* ausserhalb von *k*

Abbildung 6: *Antifusspunktkurven / Illustrationen zu Satz (T4')*

Diese elementare Verallgemeinerung des Thales-Satzes (vgl.Abb.6) ist mit Mitteln der analytischen Geometrie oder der ebenen Kinematik explizit zu erfassen.

Die sphärische Version dieses Satzes ist vom Rechen- und Beweisaufwand schwieriger. Auch hier das Ergebnis natürlich eine Anti-Fusspunktkurve. Das Hüllgebilde von *b* wird wegen (ST1) i.A. sicher nicht kreisförmig oder gar punktförmig ausfallen können! (Im Sonderfall, dass *A* Mittelpunkt von *k* ist, stimmt k_b mit *k* überein, ist also doch kreisförmig.)

4 Räumlich elementare Verallgemeinerungen des Satzes von Thales

Ersetzt man bei den Grundbegriffen das Wort "eben" durch "räumlich" und behält alle übrigen Elemente bei, so ergibt die Drehung um die Durchmessergerade *AB* von *k* als Scheitelort von rechten Winkeln eine Kugel:

(**T1'**) *Sind* A, B *Durchmesserendpunkte einer Kugel* K^2 *und ist S ein von A und B verschiedener Kugelpunkt, so ist* $\angle ASB$ *ein rechter Winkel. Umgekehrt, ist* ΔAPB *ein rechtwinkeliges Dreieck, dann gehört P der Kugel* K^2 *an.*

Es ist naheliegend dass die Aussage auch für die Hyperkugel K^{d-1} im d-dimenionalen euklidischen Raum ($d \geqslant 2$) gelten muss, wobei $d = 2$ verwendete Beweisidee unmittelbar brauchbar bleibt.

Die Erweiterung des Schauplatzes auf (euklidische) Räume E^n höherer Dimension n erlaubt auch die dimensionsmäßige Verallgemeinerung der am Satz von Thales beteiligten Elemente: Es können statt Punkten A, B Unterräume A^k bzw.B^l verwendet werden und statt der Winkelschenkel *a* und *b* orthogonale und gemeinsam ganz E^n aufspannende Unterräume α^{k+i}, β^{l+j} durch A^k bzw.B^l. Der Schnittort zugeordneter "Schenkel-Räume" ist dann i.A. eine orthogonale Hyperquadrik oder ein orthogonaler Kegel, ein Rotationszylinder oder eine Hyperkugel. Für $n = 3$ sind die interessanten unter den möglichen Fällen in den folgenden Figuren (Abb.7) dargestellt. Wir wollen Ergebnisse höherdimensionaler Verallgemeinerungen im Folgenden mit (nD-Ti) kennzeichnen. Somit sind Aussagen, die das Erzeugnis orthogonaler Unterraumbündel mit der Bezeichnung (nD-T5) zusammenzufassen.

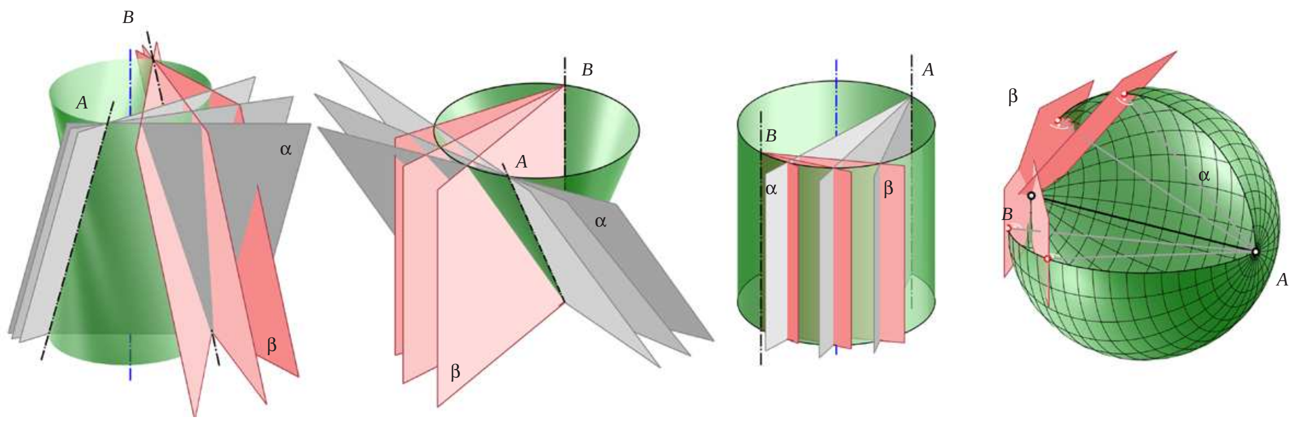


Abbildung 7: *Die Fälle (3D-T5): Durch orthogonale Unterraumbüschel oder Unterraumbündel erzeugten Thales-Quadriken des* E^3

Bemerkung: Der orthogonale Kegel als Erzeugnis orthogonal gekoppelter Ebenenbüschel mit scheidenden Achsen kommt bereits in 3 und Abb. 3 vor. Orthogonale Hyperboloide und Kegel, sowie Drehzylinder treten als "gefährliche Flächen der Fotogrammetrie" auf: Stammen die in zwei Fotos eines Objektes erkennbaren mindestens 7 Bildpaare von Raumpunkten, die einer gefährlichen Fläche angehören, so ist die Rekonstruktion des (euklidisch ausgemessenen) Objektes nicht möglich, vgl. [8] und [16].

5 Der rechte Winkel als zerfallende Kurve 2. Ordnung

Ein Paar normaler Geraden a,b in der euklidischen Ebene p kann als "ausgeartete " gleichseitige Hyperbel aufgefasst werden und ist Asymptotenpaar eines Büschels homothetisch liegender gleichseitiger Hyperbeln. Dabei ergeben sich zwei Fragen:

(1) Was hüllt jede der homothetisch gelegenen Hyperbeln h bei der Thales-Bewegung (T3) ein?

(2) Welchen Zwanglauf bestimmt eine durch zwei feste Punkte A,B gleitende gleichseitige Hyperbel? Die beiden Fragestellungen werden hier nur durch zwei Figuren visualisiert (Abb. 9(a) und 9(b)), eine analytische Behandlung, die Zusammenhänge mit der Ellipsenbewegung offenbart, unterbleibt hier jedoch.

6 Die Punkte A und B als singuläre Kurve 2. Klasse; orthoptische Linien

Aus den Punkten des Thales-Kreises k sieht man die Strecke $[A,B]$ unter festem (rechten) Winkel, k ist also eine spezielle *isoptische Linie* für diese Strecke. Fasst man nun das Punktepaar (A,B) als singuläre Kurve 2.Klasse auf, die z.B. in eine konfokale Schar von Ellipsen eingebettet gedacht werden kann, so ist die Frage nahe liegend, welche isoptischen Linien bei Kegelschnitten auftreten, wenn wir als Sichtwinkel stets einen rechten Winkel fordern. (Solche $\alpha = 90°$- isoptische Linien werden üblicherweise "orthopische Linien" genannt. Für die Begriffsbildung "isoptisch" und "orthoptisch" vgl. [17] und [18].) Es zeigt sich – und dies ist ein altbekanntes Ergebnis –, dass die orthoptischen Linien von Mittelpunktskegelschnitten Kreise sind; für Parabeln ergibt sich deren Leitgerade als orthopische Linie, vgl. Abb. 10.

Natürlich kann man die Punkte A,B für sich und unabhängig voneinander zu Kurven verallgemeinern und auf diesen Kurven a und b einen Rechtwinkelhaken "reiten" lassen. Der Scheitel S dieses Hakens durchläuft dabei die orthoptische Linie des Kurvenpaares (a,b). Bleibt A ein Punkt, so entsteht die Fußpunktskurve von b bezüglich A als Pol. Abb. 8 zeigt Spezialfälle, wenn a und b kreis- oder punktförmig angenommen sind.

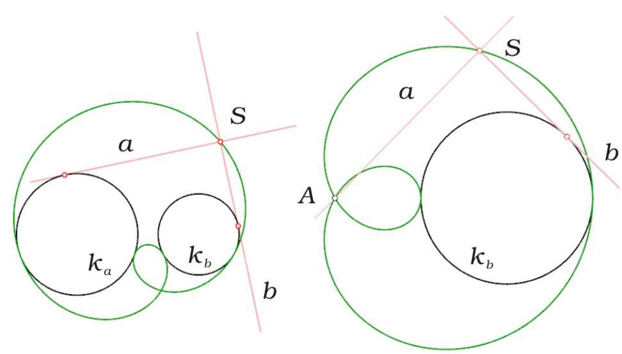
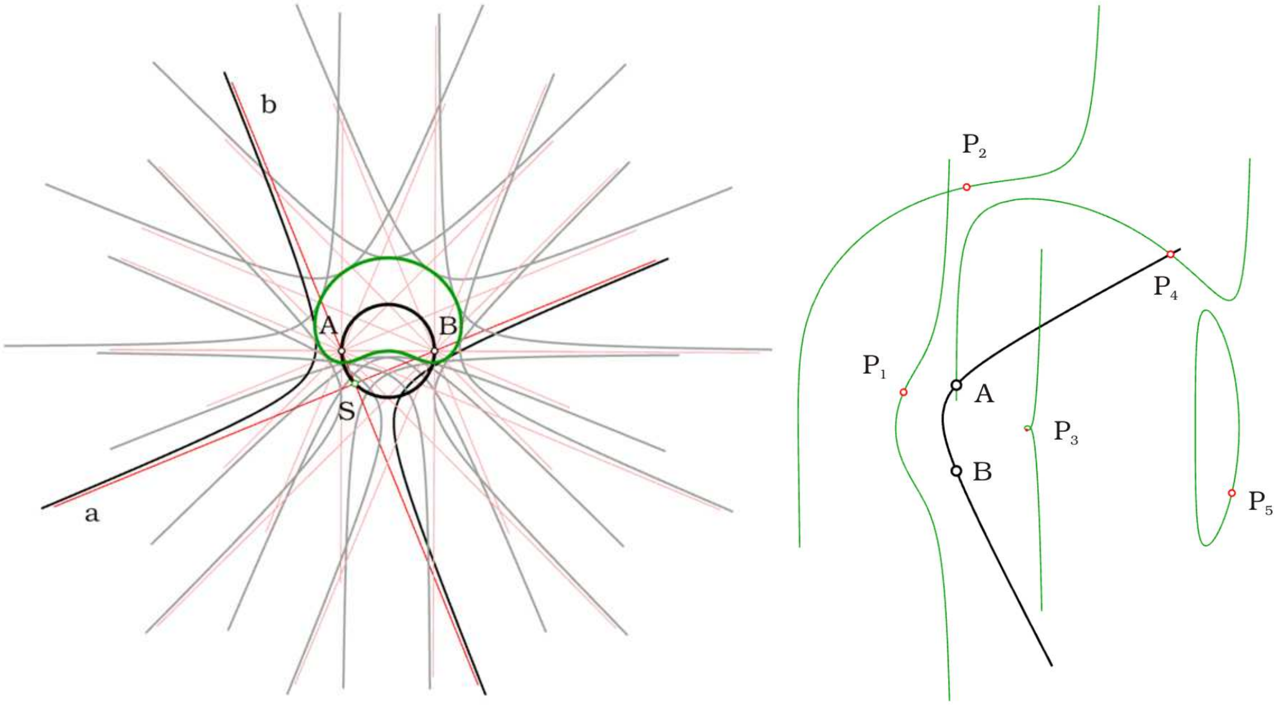


Abbildung 8: *Orthoptische Linie eines Kreispaares und allgem. Fusspunktkurve eines Kreises.*



(a) Hüllgebilde einer mit (S, a, b) homothetischen Hyperbel *h* beim Thales-Zwanglauf (T3)

(b) Bahn(en) des Mittelpunktes und verschiedener allgemeiner Punkte P_i beim Zwanglauf einer durch zwei Punkte gleitenden gleichseitigen Hyperbel

Abbildung 9: *Der rechte Winkel aufgefasst als Asymptoten gleichseitiger Hyperbeln*

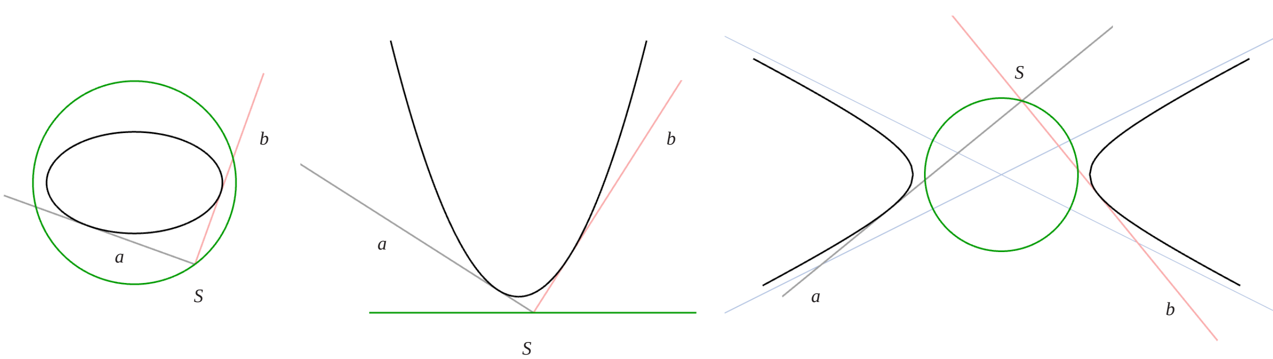


Abbildung 10: *Orthoptische Linie einer Ellipse, einer Parabel und einer Hyperbel*

7 Höherdimensionale Analoga der Rechtwinkeligkeit

Wir suchen nach "vernünftigen" Verallgemeinerungen eines rechten Winkels bzw. eines Winkels vom Maß $\pi/2$, (also einem Viertel des Vollkreisumfanges). Einem rechten Winkel höherdimensional analog ist ein n-Bein aus paarweise orthogonalen (Halb-)-Geraden. Dagegen könnte man (im Fall der Dimension 3), jedes Dreikant mit (sphärischem) Eckenwinkelmaß $\pi/2$ als Analogon eines ebenen Winkels vom Maß $\pi/2$ auffassen. Es gibt also im E^3 bis auf Bewegungen eine zweiparametrige Menge solcher $\pi/2$ - Dreikante.

Einen anderen Verallgemeinerungsweg betreten wir, wenn wir im Sinne von **4.** $\{a,b\}$ als niedrigstdimensionalen Fall eines quadratischen Kegels Γ auffassen. Um ihn mit der Orthogonalität zu verbinden, möge man sich der "Spur-0-Kegel" erinnern. Das sind elliptische Kegel mit einer Gleichung in projektiven Koordinaten derart, dass die zugehörige symmetrische Bilinearform auf eine (singuläre) symmetrische $(4x4)$-Matrix mit der Hauptdiagonalglieder-Summe 0, also der Spur 0 führt. Die kennzeichnende geometrische Eigenschaft dieser Kegel ist die, dass auf ihnen eine stetige Schar orthogonaler Dreibeine aus Kegelerzeugenden existiert (vgl. [12], [6]). Ein orthogonales Dreibein aus Erzeugenden lässt sich also längs Γ herum bewegen. Ein triviales Beispiel eines solchen Kegels ist der die Kanten einer Würfelecke enthaltende Drehkegel. Hat der Basiskreis eines solchen Drehkegels den Radius 1, so ist seine Höhe $1/\sqrt{2}$.

Bemerkung: Für höhere Dimension d ergibt sich eine Höhe von $1/\sqrt{d-1}$ über der Mitte der Leitsphäre K^{d-2} vom Radius 1, wie man unter Zuhilfenahme der Maßverhältnisse beim Hyperwürfel unschwer ableitet. Sinngemäß müssen wir nun auch das Punktepaar A,B als niedrigstdimensionalen Kreis auffassen, der nun "Leitkreis" für einen Spur-0-Kegel zu sein hat. Es ergibt sich folgender verallgemeinernder Sachverhalt:

(**3D-T1**) *Die durch einen Kreis k legbaren Spur-0-Kegel haben Spitzen, die einem abgeplatteten Drehellipsoid mit dem Achsenverhältnis $\sqrt{2}:1$ angehören, welches k zum Äquatorkreis hat.*

Denkt man sich Γ durch ein konkret gegebenes orthogonales Dreibein erzeugt, so ergibt sich folgende Formulierung, vgl. Abb. 11, sowie [2] und [10]:

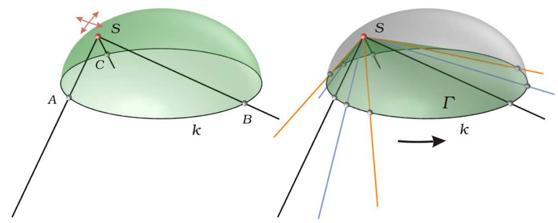


Abbildung 11: *Dreiparam. Beweglichkeit eines orthogonalen Dreibeins längs eines festen Kreises k: Drehellipsoid als zweidimensionaler Scheitelort S plus einparam. Bewegung längs orthogonalem Kegel $\Gamma = S \vee k$.*

(**3D-T3**) *Treffen die Schenkel eines orthogonalen Dreibeins stets einen Kreis k, so erfüllen die Scheitel S der so erklärten* 3*-parametrigen (!) Menge von Dreibeinen eine Fläche, nämlich ein abgeplattetes Drehellipsoid mit dem Achsenverhältnis* $\sqrt{2}:1$, *welches k zum Äquatorkreis hat.*

Diese Aussage ist zunächst unerwartet, bestimmt der beschriebene Zwanglauf doch eine dreiparametrige Dreibeinmenge. Erstaunlicherweise lässt sich der Beweis mit Mitteln der Elementargeometrie führen, wobei Eigenschaften der Euler-Geraden und der Satz vom Normalriss eines Rechten Winkels zur Anwendung kommen.

Der folgende, von [2] in Details abweichende elementargeometrische Beweis gliedert sich in drei Schritte:

a) die Festlegung eines Treffpunktedreiecks des Dreibeins $\{S;a,b,c\}$ mit k zu (in der Kreisebene) gewähltem Grundriss S' des Scheitels S und gewähltem Punkt $C \in k$ des Dreibeinschenkels c,

b) die Bestimmung der einparametrigen Menge von Treffpunktedreiecken zu festem Scheitelgrundriss S', die wegen (3D-T1) erwartungsgemäß den selben Scheitel S haben; wobei also zu zeigen ist, dass bei verschiedener Wahl von C die zugehörigen Scheitel S die selbe Höhe über der Kreisebene haben, und

c) gezeigt werden muss, dass der sicher drehsymmetrische Ort der Scheitel S eine Meridianellipse der in (3D-T3) behaupteten Gestalt hat.

Zu a): Bezeichnen A,B,C die Treffpunkte der Schenkel des Ortho-3-Beins, so ergibt sich nach dem "Satz vom Rechtwinkelbild", nach dem sich jede Ebenennormale im Normalriss orthogonal zur Ebenenspur abbildet, dass der Grundriss S' des Scheitels S stets in den Höhenschnittpunkt von ΔABC fällt. Vom Dreieck ΔABC kennt man also einerseits den Umkreis samt Mittelpunkt M und den Höhenschnittpunkt S', somit die Euler-Gerade e und den Schwerpunkt $G \in e$ mit $\overline{MG} = 2\overline{GS'}$. Zu beliebig, aber nicht auf e gewählter Ecke $C \in k$ ist somit $CS' = h_c$ eine Dreieckshöhe, zu der die durch M gehende Mittelsenkrechte m_c der Seite $c := [A,B]$ parallel sein muss; $CG =: s_c$ ist eine Schwerlinie. Sie trifft m_c im Seitenmittelpunkt C^* von c und c ist damit als Normale zu m_c durch C^* festgelegt. Die Seitengerade c trifft k in den gesuchten Treffpunkten A und B, vgl. Abb. 12 (links).

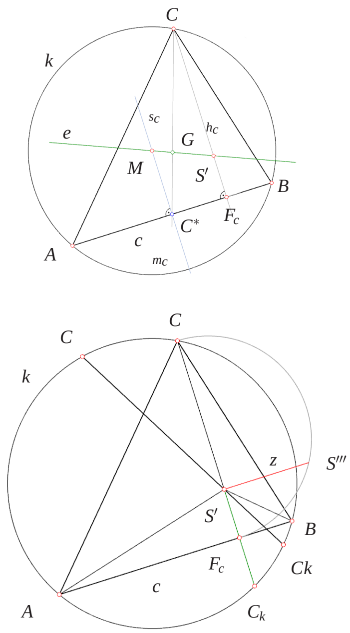


Abbildung 12: *Skizzen zum Beweis a) und b)*

Zu b): Um die Höhe von S über der Ebene π von k zu bestimmen, verwenden wir $CS'S$ als Seitenriss-Ebene. (Den in dieser Ebene liegenden Höhenfußpunkt $c \cap h_c$ bezeichnen wir mit F_c.) In diesem Seitenriss erscheint das sicher rechtwinkelige Dreieck ΔCSF_c unverzerrt und wir lesen nach dem Höhensatz für rechtwinkelige Dreiecke ab:

$$\overline{S'S}^2 = \overline{S'C} \cdot \overline{S'F_c}. \tag{3}$$

Nun schneidet h_c den Umkreis k von ΔABC noch in einem Punkt C_k, für den bekanntlich gilt:

$$\overline{F_cS'} = \overline{F_cC_k}, \tag{4}$$

sodass damit folgt:

$$\overline{S'S}^2 = \frac{1}{2}\overline{S'C} \cdot \overline{S'C_k}. \tag{5}$$

Der Sehnensatz für (k, S') besagt nun aber, dass

$$\overline{S'C} \cdot \overline{S'C_k} = const. \tag{6}$$

für jede Wahl von C auf k ist. Demnach muss auch $\overline{S'S}^2$ einen von C unabhängigen festen Wert haben, vgl. Abb. 12 (rechts).

Zu c): Für den Nachweis, dass der wegen b) nur zweidimensionale Scheitelort ein abgeplattetes Drehellipsoid Φ ist, verfolgen wir einen Meridian m von Φ, indem wir von fest gewähltem C auf k ausgehen und S' auf der Verbindungslinie CM wählen. Damit muss ΔABC ein gleichschenkliges Dreieck sein. Zu jeder Wahl der Basisseite c normal $h_c = e = CM$ gehört ein rechtwinklig-gleichschenkliges Dreieck ΔASB, sodass $SF_c = \frac{1}{2}AB$ sein muss, (F_c bezeichnet wieder den Höhenfußpunkt von h_c auf c). Andererseits liegt S auf dem Thaleskreis über $[CF_c]$, da auch das Dreieck ΔCSF_c rechtwinklig ist, vgl. Abb.13.

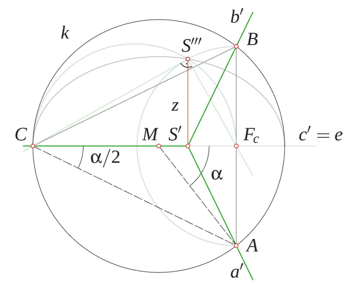


Abbildung 13: *Skizze zum Beweis c)*

Mit $\alpha := \sphericalangle AMF_c$ besitzt es die Kathetenlängen

$$\overline{SF_c} = \sin\alpha \text{ und } \overline{SC} = \cos\frac{\alpha}{2}(1+\cos\alpha), \tag{7}$$

wobei wir für den Radius von k o.B.d.A. den Wert 1 verwenden. Für den Normalriss $S' \in CM$ ergibt sich demnach in Abhängigkeit von α

$$r(\alpha) = \overline{MS'} = 1 - \cos^2\frac{\alpha}{2}(1+\cos\alpha), \tag{8}$$

und die Höhe z von S berechnet sich zu

$$z(\alpha) = \overline{S'S} = \sin\frac{\alpha}{2}\cos\frac{\alpha}{2}(1+\cos\alpha), \tag{9}$$

sodass $(r(\alpha), z(\alpha))$ die Ellipsengleichung

$$\frac{r^2}{1} + \frac{z^2}{\frac{1}{2}} = 1 \tag{10}$$

erfüllt.

□

Bemerkung: Obschon der Sachverhalt (3D-T3) durch die Aussagen zu (3D-T1) mit erledigt ist, scheint der elementargeometrische Beweis von Interesse zu sein, sind doch die benützten Werkzeuge im Schulstoff angesiedelt, also ‘Folklore’. Er kann daher von jedem an Mathematik interessierten Laien leicht nachvollzogen werden. Der angegebene Beweis “rechtfertigt” auch die Tradierung elementargeometrischer Sätze als ein seltenes Beispiel für die Verwendung der Euler-Geraden als Beweishilfsmittel!

8 Umstülpung des Würfels nach P. Schatz

In [9] findet sich auf Seite 37 im Zusammenhang mit der Umstülpung des Würfels (siehe Abb. 14) die Formulierung "Es erfolgt übrigens die Bezeichnung des umstülpbaren Würfels als "rechtwinkliges Sechsflach" in Anbetracht seines Verhältnisses zur Umkugel, dem im Zweidimensionalen das Verhältnis des rechtwinkligen Dreiecks zum Kreis entspricht."

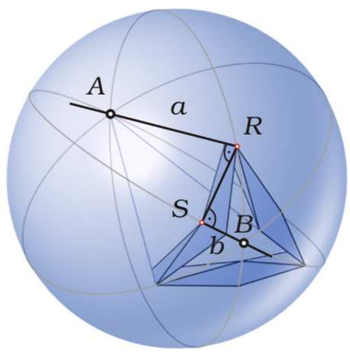


Abbildung 14: *Vom Würfel abgeleitete umstülpbare Tetraederkette von P. Schatz (vgl. auch [20]).*

Man könnte also meinen, auch hier eine räumliche Verallgemeinerung des Satzes von Thales vorzufinden. Der Umstülpvorgang ändert aber den Radius der Umkugel des Sechsflachs. Dennoch lässt sich eine Thales-Bewegung extrahieren, wenn man den Bewegungsvorgang eines der Tetraeder der sechsgliedrigen Schatzschen Kette von Abb. 14 für sich studiert. Dabei wird das Tetraeder auf die drei aufeinander folgenden Rechtwinkelhaken reduziert, die mit den Schenkeln a,b durch feste Punkte A,B gleiten sollen (Abb. 15). Wir fragen nach der Menge von Geraden, die die Gemeinnormale *c* der zueinander normalen Geraden *a* und *b* durchläuft. Diese Geradenmenge besteht aus Drehreguli (also einer Erzeugendenschar von Drehhyperboloiden) mit der gemeinsamen Achse *AB*. Andererseits ist auch ein Bewegungsvorgang denkbar, der *c* nur parallel verschiebt und einen Thales-Zylinder erzeugt. Damit berühren alle möglichen Geraden *c* eine mit der Thales-Kugel über *AB* konzentrische Kugel und schließen mit *AB* konstanten Winkel ein. Die Menge c ist also eine Geradenkongruenz, die als Schnitt eines Kugeltangentenkomplexes und des Treffgeradenkomplexes eines Fernkreises entsteht. (Für diese der Liniengeometrie entstammende Begriffswelt siehe z.B [7].)

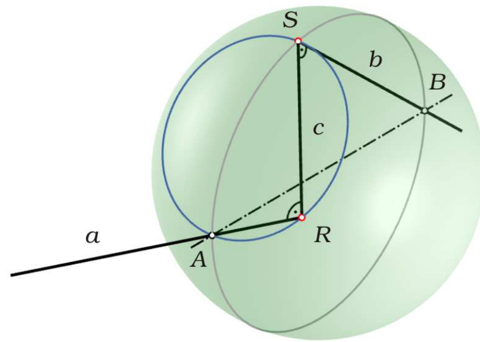


Abbildung 15: *Räumlicher Rechtwinkelhaken aRcSb dessen Schenkel a und b durch die festen Punkte A und B gleiten.*

9 Weitere Analoga zum Satz von Thales

Von den bisherigen Verallgemeinerungen ausgehend lassen sich weitere Verallgemeinerungslinien eröffnen. Zum Beispiel erlaubt der Begriff der Fußpunktskurve nahe liegende räumliche Verallgemeinerungen im Sinne eines Thales-Satzes, wenn man nur die Schenkel a,b des ursprünglichen Rechtwinkelhakens durch zwei totalorthogonale Unterräume ersetzt, die ihrerseits zwei Steuerflächen oder -kurven berühren oder durch Punkte A,B hindurchgehen. Der Fall der durch orthogonale Geraden- und Ebenenbündel erzeugten Thales-Kugel von Abb. 7 gehört beispielsweise hierher.

Es ist unmittelbar einsichtig, dass das ebene Problem orthoptischer Kurven, wie es in Kapitel 6 behandelt wurde, auf zwei Arten in den Raum zu übertragen ist:

(**3D-T3a**) Gesucht ist der Ort der Scheitel eines orthogonalen Dreibeins, dessen Schenkel drei Steuerflächen berühren, wobei die Steuerflächen zusammenfallen oder auch zu Kurven oder Punkten ausarten können.

(**3D-T3b**) Gesucht ist der Scheitel eines orthogonalen Dreiflachs, dessen Ebenen drei Steuerflächen berühren, wobei die Steuerflächen zusammenfallen oder auch zu Kurven oder Punkten ausarten können.

(3D-T3a) soll durch das folgende Beispiel illustriert werden: Gesucht ist der Ort der Scheitel *S* orthogonaler Dreibeine (S,a,b,c), deren Schenkel drei nicht notwendig windschiefe Steuergeraden p,q,r treffen. Jeder nicht durch eine Steuergerade gehende ebene Schnitt der drei Steuergeraden gibt Anlass zu einem Treffpunktedreieck ΔABC. Damit dieses das Spurpunktedreieck eines orthogonalen Dreibeins sein kann, muss es notwendig spitzwinklig ausfallen. Der Grenzfall rechtwinkliger Spurendreiecke sei zugelassen. Die Schnittebenen-Normale durch den Höhenschnittpunkt von ΔABC trägt dann zwei zur Schnittebene symmetrisch liegende Punkte S. Dieses Punktepaar ergibt sich auch als Schnittpunktepaar der Thales-Kugeln über den Dreiecksseiten $[A,B],[B,C],[C,A]$, vgl. Abb. 16.

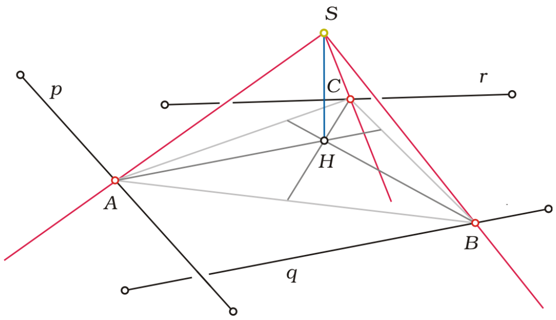


Abbildung 16: *Scheitel S eines orthogonalen Dreibeins, dessen Schenkel drei Leitgeraden p,q,r treffen.*

Da die Menge der Schnittebenen dreidimensional ist, haben wir von einer dreiparametrigen Menge von Dreibeinen auszugehen. Wir überlegen, dass zu jeder Lösung S eine dreidimensionale Umgebung von Lösungen S existiert. Dazu beschreiten wir einen anderen, etwas aufwändigeren Lösungsweg: Wir gehen von einem zunächst beliebig (nicht auf den Leitgeraden) gewählten Raumpunkt S als Auge einer Zentralprojektion auf eine (beliebig gewählte) Bildebene π aus. Dann ist den Zentralrissen p^c, q^c, r^c ein Dreieck ΔABC so einzuschreiben, dass der Hauptpunkt der Perspektive dessen Höhenschnittpunkt ist. Unter der (i.A.) einparametrigen Menge dieser Dreiecke finden sich zwei (im algebraischen Sinn) so, dass sich die Thales- Kugeln über den Seiten im Augpunkt schneiden. Die Beweglichkeit eines (nicht orthogonalen) Dreikants entlang dreier Geraden findet sich auch bei der von H. Stachel entdeckten Bewegungen, die die beiden Tetraeder einer Stella Octangula gegeneinander trotz Übergeschlossenheit des kinematischen Systems ausführen können, vgl. [13] und [14].

Mit diesem in Abb. 16 visualisierten Beispiel hat man in gewisser Weise auch einen "Satz von Thales im Linienraum" behandelt. Auf die Untersuchung weiterer, liniengeometrisch motivierter Verallgemeinerungen des Satzes von Thales wird hier verzichtet.

Zur Illustration von (3D-T3b) gehen wir vom Spezialfall eines einzigen Kreises k als der (dreifachen) Steuerkurve aus und bestimmen die orthogonalen Dreiflache, deren Ebenen k berühren. In gewisser Weise dual zur Aufgabe (3D-T3) gehen wir also von einem spitzwinkligen Tangentendreieck von k aus, bestimmen dessen Höhenschnittpunkt H und schneiden die Thales-Kugeln über den Dreiecksseiten, um das zum Tangentendreieck gehörige Paar von Scheiteln S möglicher Dreiflache zu gewinnen, vgl. Abb. 17.

In Abb. 17 wurde von einem (spitzwinkligen) gleichschenkligen Tangentendreieck ΔABC ausgegangen, dessen Symmetrieachse die Seite $[B,C]$ im Berührpunkt D mit k trifft. Ist I der Mittelpunkt von k und α bzw. γ der Innenwinkel von ΔABC in A bzw. C und bezeichnet H den Höhenschnittpunkt von ΔABC, so gilt

$$\overline{DC} =: p = \cot\frac{\gamma}{2} > 1 \quad \text{(im Grenzfall p=1)},$$

$$\overline{AI} =: q = \frac{1}{cos\gamma} = \frac{p^2+1}{p^2-1}$$

$$\overline{DH} =: r = p\cdot\cot\gamma = \frac{1}{2}(p^2-1), \overline{IH} = 1-r = \frac{1}{2}(3-p^2) =: y$$

Hieraus folgt für die z-Kote des Scheitels S des Dreiflaches nach dem Höhensatz die Beziehung

$$z^2 = (q+y)r = (q+y)(1-y) = q+y+qy-y^2,$$

sodass schließlich z und y die Gleichung

$$y^2+z^2 = q+y-qy = 2\frac{p^2-1}{p^2-1} = 2 \quad (p>1)$$

erfüllen.

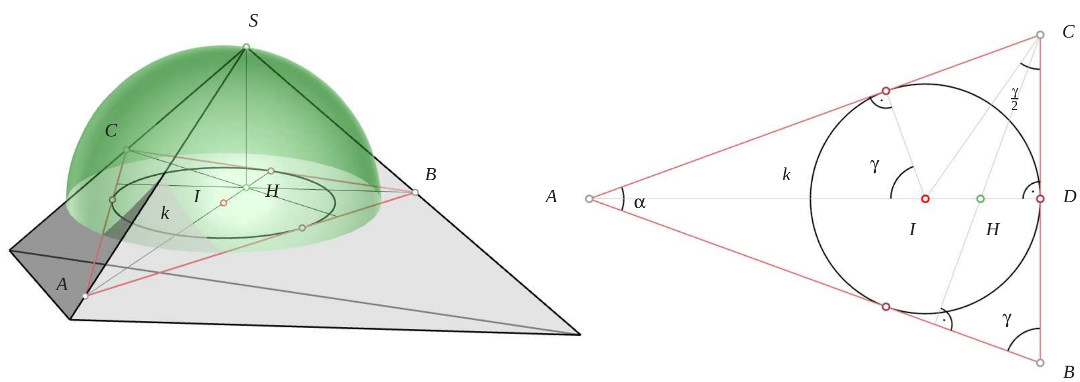


Abbildung 17: *Der Scheitelort der einen Kreis k berührenden orthogonalen Dreiflache ist eine mit k konzentrische Kugel.*

Wir haben noch zu zeigen, dass alle Tangentendreiecke {A,B,C} die den selben Höhenschnittpunkt *H* haben, zum selben *z*-Wert von *S* führen. Die Menge dieser Tangentedreiecke sind sämtlich einer festen Ellipse *e* ein- und dem Kreis *k* umbeschrieben. (*e* und *k* bilden eine Angabe eines von *J-V. Poncelet* stammenden poristischen Problems der ebenen projektiven Geometrie, bei dem entweder keine oder unendlich viele Lösungen auftreten, vgl. [3] und Abb.18) Die Ellipse *e* ist dabei durch zwei solche Dreiecke bestimmt. Wir übergehen die diesbezüglichen Rechnungen.

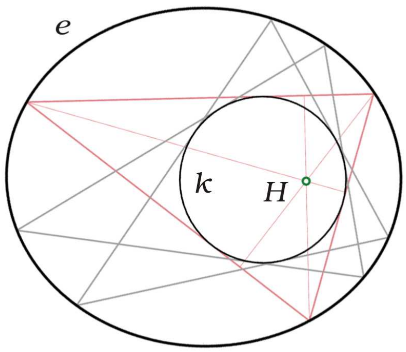


Abbildung 18: *Dem Kreis k umschriebene Dreiecke mit gemeinsamem Höhenschnittpunkt H sind einer festen Ellipse e einbeschrieben.*

Somit gilt der Satz

(**3D-T3b**) *Berühren die Ebenen eines orthogonalen Dreiflachs einen Kreis k vom Radius* 1*, so gehört der Scheitel des Dreiflachs einer mit k konzentrischen Kugel vom Radius* $\sqrt{2}$ *an.*

Für diese Aussage haben die Autoren keinen Hinweis in der Literatur gefunden. Sie lässt eine (nD-T3) analoge Dimensionsverallgemeinerung zu, die auf eine Thales-Hyperkugel vom Radius $\sqrt{n-1}$ führt.

Mit (3D-T3b) ist abschließend ein sehr natürlicher Verallgemeinerungsstrang des Satzes von Thales gefunden, der aus den Formulierungen des ursprünglichen Thales-Satzes im Sinne von Kapitel 4 folgt:

(**T7**) *"Fasst man das Punktepaar A,B als ausgearteten Klassenkegelschnitt (also als Tangentenmenge) auf, so liegen die Schnittpunkte S orthogonaler Tangenten auf einem Kreis, dem Thales-Kreis über* $[A,B]$*."*

(**3D-T7**) *Fasst man einen Kreis k als ausgeartete Klassenquadrik (also als Tangentialebenenmenge) auf, so liegen die Schnittpunkte S orthogonaler Tangentialebenentripel auf einer mit k konzentrischen Kugel vom* $\sqrt{2}$*-fachen des Kreisradius.*

Herrn M. Hamann verdanken die Autoren den Hinweis, dass dieser Verallgemeinerungsstrang auch die Verbindung zu "dualen Holditch-Sätzen im Dreidimensionalen" im Sinne von [1] knüpft.

10 Schlussbemerkung

Wir haben einige Wege aufgezeigt, wie der Satz von Thales in allgemeinere Fragestellungen als Spezialfall eingebettet werden könnte. Obwohl dabei Vieles nur grob angerissen wurde, so wird hoffentlich dennoch zumindest die Reichhaltigkeit dieser möglichen Verallgemeinerungen sichtbar und sollte zu eigenständigem forschenden Tun anregen. Darüber hinaus sollten Prinzipien mathematischer Überlegung exemplarisch vorgestellt werden: Das Einordnen eines Sachverhaltes in ein allgemeines Schema einerseits und das Bilden von Analogien andererseits. Beide Prinzipien erfordern zwar eine gewisse Breite an mathematisch-geometrischer Grundbildung, bei in der Schule nicht gehemmter natürlicher Neugier der Schüler wird der Erwerb dieser Bildung aber zu keiner Zeit als Last empfunden. Für diese Bildungsaufgabe mag der vorliegende Text in Zusammenhang mit der Einübung einer dynamischen Graphik-Software nutzbar sein. Jedenfalls soll Schülern und Studenten mit den vorliegenden Materialien, die durch einschlägige Literatur und Internet-Recherche noch zu ergänzen sind, auch verdeutlicht werden, dass Geometrie und Mathematik entgegen landläufiger Meinung keinesfalls abgeschlossene Wissensgebiete sind!

Es sollte nicht unerwähnt bleiben, dass für den überwiegenden Teil der Figuren die programmierbare Graphics Software 'Open Geometry GL' (vgl. [4] und [5]) benutzt wurde.

Literatur

[1] BROMANN, A., *Holditch's Theorem*, Math. Magazine 54 No.3 (1981).

[2] BUBECK, H., *Auf der Suche nach einer „einfachen" räumlichen Entsprechung zum Satz von Thales*, MNU 47/5 (1994), 264-268.

[3] FRIEDELMEYER, J-P., *Le théorème de cloture de Poncelet comme source d'inspiration et lieu de rencontre des nouvelles methods en géométrie*, (Lumigny, 2005). `www.univ-nancy2.fr/poincare/colloques/hgmc2005/Friedel-meyer_Jean_Pierre.pdf`

[4] GLAESER, G.- STACHEL, H., *Open Geometry: Open GL + Advanced Geometry*, Springer 1999 (ISBN 0387 985999).

[5] GLAESER, G. - SCHRÖCKER, H-P., *Handbook of Programming using Open Geometry GL*, Springer Professional Computing 2002 (ISBN 0387 95272-1).

[6] HAVLICEK, H. - WEISS, G., *Altitudes of a Tetrahedron and Traceless Quadratic Forms*, The American Mathematical Monthly, Vol. 110, No. 8 (Oct., 2003), pp. 679-693.

[7] HOSCHEK, J., *Liniengeometrie*, HTB, B.I. Mannheim 1971.

[8] KRAMES J., *Über die bei der Hauptaufgabe der Luftphotogrammetrie auftretenden 'gefährlichen' Flächen*, Bildmessung und Luftbildwesen (1942), 1-18.

[9] SCHATZ, P., *Rythmusforschung und Technik*, Verlag freies Geistesleben, 1998. 37-38.

[10] SCHUMANN, H., *Schulgeometrie im virtuellen Raum*, Franzbecker 2007, ISBN 978-3-88120-463-7.

[11] SEEBACH, K., *Verallgemeinerung des Satzes von Thales auf 3 Dimensionen*, PM 34(1992), 209-212.

[12] STACHEL, H., *A remarkable overconstrained spherical motion*, In J. Lenarcic and M. Husty (eds), Advances in robot kinematics: analysis and control (Salzburg, 1998), Kluwer Acad. Publ., Dordrecht, 1998 (ISBN 0-7923-5169-X): pp. 287-296. `http://www.geometrie.tuwien.ac.at/stachel/iftom.pdf`

[13] STACHEL, H., *Ein bewegliches Tetraederpaar*, Elem. D. Math. 43 (1988), 65-75.

[14] STACHEL, H., *Zwei bemerkenswerte bewegliche Strukturen*, J. Geom. 43 (1992), 14-21.

[15] STEINER TH., `http://commons.wikimedia.org/wiki/Image:Mandate2006.png`

[16] WUNDERLICH, W., *Zur Eindeutigkeitsfrage der Hauptaufgabe der Photogrammetrie*, Monatshefte Math.Phys. 50 (1941), 151-184.

[17] WUNDERLICH, W., *Kurven mit isoptischem Kreis*, Aequationes math. 6 (1971), 71-81.

[18] WUNDERLICH, W., *Die isoptischen Kurven von Zykliden*, Z.Angew.Math.Mech. 17 (1937), 56.

[19] `http://de.wikipedia.org/wiki/Satz_des_Thales`

[20] ZAHAUREK, F., *Der umstülpbare Würfel nach Paul Schatz*, 1999, `http://www.fzk.at/wuerfel/index.html`

Gunter Weiß
Technical University of Dresden
e-mail: Gunter.Weiss@tu-dresden.de
Franz Gruber
University of Applied Arts Vienna
e-mail: Franz.Gruber@uni-ak.ac.at

Original scientific paper
Accepted 20. 11. 2008.

EMA JURKIN

Automorphic Inversion and Circular Quartics in Isotropic Plane

Automorphic Inversion and Circular Quartics in Isotropic Plane

ABSTRACT

In this paper circular quartics are constructed by automorphic inversion (inversion that keeps the absolute figure fixed) as the images of conics. They are classified depending on their position with respect to the absolute figure. It is shown that only 1, 2 and 4-circular quartics can be obtained by automorphic inversion.

Key words: isotropic plane, circular quartic, automorphic inversion

MSC 2000: 51M15, 51N25

Automorfna inverzija i cirkularne kvartike u izotropnoj ravnini

SAŽETAK

U ovom su radu cirkularne kvartike konstruirate pomoću automorfne inverzije (inverzija koja apsolutnu figuru ostavlja fiksnom) kao slike konika. Klasificirane su s obzirom na položaj prema apsolutnoj figuri. Pokazano je da se automorfnom inverzijom konike mogu dobiti samo 1, 2 i 4 cirkularne kvartike.

Ključne riječi: izotropna ravnina, cirkularna kvartika, automorfna inverzija

1 Introduction

An *isotropic plane* I_2 is a real projective plane where the metric is induced by a real line f and a real point F incidental with it, [4]. The ordered pair (f,F) is called the *absolute figure* of the isotropic plane.

In the affine model of the isotropic plane where the coordinates of the points are defined by

$$x=\frac{x_1}{x_0}, \quad y=\frac{x_2}{x_0},$$

the absolute line f is determined by the equation $x_0=0$ and the absolute point F by the coordinates $(0,0,1)$.

The projective transformations that map the absolute figure onto itself form a 5-parametric group $\mathcal{G}_5$. They have equations of the form

$$\overline{x}=a+dx, \quad \overline{y}=b+cx+ey.$$

$\mathcal{G}_5$ is called the *group of similarities* of the isotropic plane, [1], [4]. Its subgroup $\mathcal{G}_3$ $\mathcal{G}_5$, consisting of the transformations of the form

$$\overline{x}=a+x, \quad \overline{y}=b+cx+y,$$

is called the *group of motions* of the isotropic plane. It preserves the quantities such as the distance between two points, snap between two parallel points or the angle between two lines. Thus, it has been selected for the fundamental group of transformations.
The ordered pair $(I_2,\mathcal{G}_3)$ is called the *isotropic geometry*.

All straight lines through the absolute point F are called *isotropic lines* and all points incidental with f are called *isotropic points*.
Two points $A(a_1,a_2)$ and $B(b_1,b_2)$ are called *parallel* if they lie on the same isotropic line. In that case, the *span* is defined by $s(A,B)=b_2-a_2$. For two non-parallel points the *distance* is defined by $d(A,B)=b_1-a_1$.

There are seven types of regular conics classified according to their position with respect to the absolute figure, [1], [4]. An *ellipse* (*imaginary* or *real*) is a conic that intersects the absolute line in a pair of conjugate imaginary points. If a conic intersects the absolute line in two different real points, it is called a *hyperbola* (of *1st* or *2nd type*, depending on whether the absolute point is outside or inside the conic). A conic passing through the absolute point is called a special *hyperbola* and a conic touching the absolute line is called *a parabola*. If a conic touches the absolute line at the absolute point, it is said to be a *circle*.

A curve in the isotropic plane is *circular* if it passes through the absolute point, [5]. Its *degree of circularity* is defined as the number of its intersection points with the

absolute line f falling into the absolute point. If it does not share any common point with the absolute line except the absolute point, it is *entirely circular*, [3].
The circular curve of order four can be 1, 2, 3 or 4-circular. The absolute line can intersect it, touch it, osculate it or hyperosculate it at the absolute point. The absolute point can be simple, double or triple point of the curve.

2 Automorphic inversion in isotropic plane

Definition 1 *An inversion with respect to the pole P and the fundamental conic q is a mapping where any point and its image are conjugate with respect to the conic q and their connecting line passes through the point P.*

The inversion is an involution, [2], [6]. Any point of the fundamental conic is mapped into itself. The lines joining a point to its image are called the *rays*. They are fixed lines as entities, but their points are not fixed. Let p be the polar line of the point P with respect to the fundamental conic q. The intersection points of the line p with the conic q are denoted by P_1 and P_2 and their polar lines by p_1 and p_2. These three points and three lines are said to be the *fundamental elements* of the given mapping. The fundamental points are the singular points of the inversion ($P \mapsto p, P_1 \mapsto p_1, P_2 \mapsto p_2$), and any point of the fundamental line is mapped into the corresponding fundamental point.

The inversion maps the curve k of order n into the curve $\overline{k}$ of order $2n$. Since k intersects any fundamental line in n points, $\overline{k}$ has three multiple points of order n in the fundamental points. If k passes through some of the fundamental points, $\overline{k}$ splits into the corresponding polar line and a curve of order $n-1$. The curve $\overline{k}$ passes through common points of the curve k and the fundamental conic.

Since we are interested in the property of circularity, we will restrict our interest on the inversions that keep the absolute figure fixed.

Definition 2 *An inversion which maps absolute figure into itself is called the automorphic inversion.*

According to [5] the following theorem is valid:

Theorem 1 *There are five types of the automorfic inversion:*

(1) The fundamental conic q is a special hyperbola. The pole P is a point of the absolute line, different from its intersections with the fundamental conic.

(2) The fundamental conic q is a special hyperbola. The pole P is an intersection point of the absolute line and the fundamental conic different from the absolute point.

(3) The fundamental conic q is a special hyperbola. The pole P is the absolute point.

(4) The fundamental conic q is a circle. The pole P is a point of the absolute line different from the absolute one.

(5) The fundamental conic q is a circle. The pole P is the absolute point.

Proof. The absolute point F has to be mapped into itself. Therefore, F has to be a point of the fundamental conic. Accordingly, the fundamental conic is either a special hyperbole or a circle. Since the absolute line f has to be mapped into itself, it has to be a ray of inversion. Consequently, the pole P is a point of the absolute line. □

A curve of order four can be obtained by inversion of a conic, but also by inversion of a curve of higher order passing through the fundamental points. For example, the inverse image of a cubic passing through two fundamental points is a curve of order six that splits into two lines and a curve of order four. We will study the quartics $\overline{k}$ obtained as an inverse images of a conic k that does not pass through the fundamental points. The conditions that the conic k has to fulfill in order to obtain a circular quartic of certain type will be determined for each of the types of the inversion.

2.1 Equation of fundamental conic

Every curve of second order is given by the equation of the form

$$a_{00}x_0^2 + a_{11}x_1^2 + a_{22}x_2^2 + 2a_{01}x_0x_1 + 2a_{02}x_0x_2 + 2a_{12}x_1x_2 = 0, \quad (1)$$

or in the affine coordinates

$$a_{00} + a_{11}x^2 + a_{22}y^2 + 2a_{01}x + 2a_{02}y + 2a_{12}xy = 0.$$

Considering the isotropic motion

$$\overline{x} = x - \frac{a_{01}a_{22} - a_{02}a_{12}}{a_{12}^2 - a_{11}a_{22}}, \quad \overline{y} = y - \frac{a_{02}a_{11} - a_{01}a_{12}}{a_{12}^2 - a_{11}a_{22}}$$

we get a simpler form of the equation

$$a_{11}x^2 + a_{22}y^2 + 2a_{12}xy + a = 0.$$

If $a_{12}^2 - a_{11}a_{22} = 0$, the conic is either a parabola or a circle.

The equation of parabola touching the absolute line $[1,0,0]$ at the point $(0,1,0)$ due to

$$\begin{bmatrix} a_{00} & a_{01} & a_{02} \\ a_{01} & a_{11} & a_{12} \\ a_{02} & a_{12} & a_{22} \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \mu \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix},$$

is of the form

$$a_{00} + y^2 + 2a_{01}x + 2a_{02}y = 0.$$

Now, by the isotropic motion

$$\overline{x} = x + \frac{a_{00} - a_{02}^2}{2a_{01}}, \quad \overline{y} = y + a_{02}$$

it can be transformed into

$$y^2 + 2a_{01}x = 0, \quad a_{01} \neq 0.$$

1-circular conic (a special hyperbola) due to $a_{22} = 0$ has the equation of the form

$$a_{11}x^2 + 2a_{12}xy + a = 0, \quad a_{12} \neq 0.$$

The assumption that the other intersection point of the special hyperbola and the absolute line is the point $(0,1,0)$ leads to even simpler form

$$xy + a = 0. \tag{2}$$

If the conic given by (1) is 2-circular, the line $f[1,0,0]$ is the tangent of the conic at the point $F(0,0,1)$, consequently $a_{12} = a_{22} = 0, a_{02} \neq 0$. Therefore, the equation

$$a_{11}x^2 + 2a_{01}x + 2a_{02}y + a_{00} = 0,$$

is obtained. Since a_{11} should not equal zero, we can chose $a_{11} = 1$. After applying the isotropic motion

$$\overline{x} = x, \quad \overline{y} = \frac{a_{00}}{2a_{02}} + \frac{a_{01}}{a_{02}}x + y$$

the equation becomes

$$x^2 + 2a_{02}y = 0. \tag{3}$$

2.2 Equations of inversion

Let the fundamental conic q be a special hyperbola and let the pole of the inversion $P(0, p_1, p_2)$ be a point of the absolute line. Without loss of generality we can assume that q has the equation $xy - 1 = 0$. Let us determine the coordinates of the image $\overline{T}(1, \overline{\alpha}, \overline{\beta})$ of a given point $T(1, \alpha, \beta)$. The polar line t of the point T is determined by

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & \frac{1}{2} \\ 0 & \frac{1}{2} & 0 \end{bmatrix} \begin{bmatrix} 1 \\ \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} -1 \\ \frac{1}{2}\beta \\ \frac{1}{2}\alpha \end{bmatrix},$$

Or, in other words, the line with the equation

$$-2 + \beta x + \alpha y = 0. \tag{4}$$

Connecting line TP is given by

$$\begin{vmatrix} x_0 & x_1 & x_2 \\ 0 & p_1 & p_2 \\ 1 & \alpha & \beta \end{vmatrix} = 0,$$

which is equivalent to

$$p_1\beta - p_2\alpha + p_2x - p_1y = 0. \tag{5}$$

The point $\overline{T}$ is the intersection of the lines t and TP so its coordinates can be determined by solving the system of the linear equations (4) and (5) and equal

$$\overline{\alpha} = \frac{p_2\alpha^2 - p_1\alpha\beta + 2p_1}{p_2\alpha + p_1\beta}, \quad \overline{\beta} = \frac{-p_2\alpha\beta + p_1\beta^2 + 2p_2}{p_2\alpha + p_1\beta}.$$

Thus, the inversion is given by

$$\overline{x} = \frac{p_2x^2 - p_1xy + 2p_1}{p_2x + p_1y}, \quad \overline{y} = \frac{-p_2xy + p_1y^2 + 2p_2}{p_2x + p_1y}. \tag{6}$$

If the pole is the absolute point $F(0,0,1)$ the previous expressions are turned into

$$\overline{x} = x, \quad \overline{y} = -y + \frac{2}{x}. \tag{7}$$

If the pole is the other intersection of the fundamental conic with the absolute line, i.e. the point $P(0,1,0)$, (6) becomes

$$\overline{x} = -x + \frac{2}{y}, \quad \overline{y} = y. \tag{8}$$

In the general case when the pole $P(0,1,p), p \neq 0$, is the point of the absolute line not belonging to the fundamental conic, the inversion is determined by

$$\overline{x} = \frac{px^2 - xy + 2}{px + y}, \quad \overline{y} = \frac{-pxy + y^2 + 2p}{px + y}. \tag{9}$$

Therefore, we conclude that inversion, fundamental conic of which is a circle $x^2 - y = 0$ (without loss of generality we can assume $a_{02} = -\frac{1}{2}$) and pole $P(0, p_1, p_2)$ is a point of the absolute line, has the equations

$$\overline{x} = \frac{-p_2x + 2p_1y}{2p_1x - p_2}, \quad \overline{y} = \frac{-2p_2x^2 + 2p_1xy + p_2y}{2p_1x - p_2}. \tag{10}$$

In the case when the pole is the absolute point $F(0,0,1)$ equalities (10) are turned into

$$\overline{x} = x, \quad \overline{y} = x^2 - y. \tag{11}$$

The general case, when the pole is different from the absolute point, may be simplified by choosing $(0,1,0)$ for its coordinates. Then, the inversion is given by

$$\overline{x} = \frac{y}{x}, \quad \overline{y} = y. \tag{12}$$

2.3 Circular quartics obtained by automorphic inversion of conic

Theorem 2 *An automorphic inversion in the isotropic plane maps the* 2^{nd}*-order curve* k *not passing through the fundamental points of the inversion into the* 4^{nd}*-order curve* $\overline{k}$. *The degree of circularity of the curve* $\overline{k}$ *depends on the type of the inversion and on the position of the conic* k *with respect to the fundamental and absolute elements as follows:*

- *If the fundamental conic* q *is a special hyperbola and the pole* P *is an isotropic point,* $P \neq F$, $\overline{k}$ *is 1-circular (if* k *is a special hyperbola) or 2-circular (if* k *is a circle) quartic.*
- *If the fundamental conic* q *is a special hyperbola and the pole* P *is the absolute point,* $\overline{k}$ *is 2-circular quartic.*
- *If the fundamental conic* q *is a circle and the pole* P *is an isotropic point,* $P \neq F$, $\overline{k}$ *is 2-circular quartic*
- *If the fundamental conic* q *is a circle and the pole* P *is the absolute point,* $\overline{k}$ *is 4-circular quartic.*

Proof. Detailed proofs of all facts will be given only in the case of the inversion of type (1). Since the approach is similar in all the other cases for the inversions of types (2)-(5), only the facts will be stated in those cases.

Type (1)

Let us consider the inversion of type (1) with the fundamental conic q given by the equation $xy-1=0$ and the pole $P(0,1,p)$.

The other two fundamental points are $P_1(p,\sqrt{-p},-p\sqrt{-p})$ and $P_2(p,-\sqrt{-p},p\sqrt{-p})$. Three fundamental lines p, p_1, p_2 are given by equations $y=-px$, $y=px-2\sqrt{-p}$, $y=px+2\sqrt{-p}$, respectively.

The inversion

$$\overline{x}=\frac{px^2-xy+2}{px+y}, \quad \overline{y}=\frac{-pxy+y^2+2p}{px+y}$$

maps the conic k

$$a_{00}+a_{11}x^2+a_{22}y^2+2a_{12}xy+2a_{01}x+2a_{02}y=0 \quad (13)$$

into the quatric $\overline{k}$

$$a_{11}p^2x^4-2p(a_{12}p+a_{11})x^3y+(a_{22}p^2+4a_{12}p+a_{11})x^2y^2-$$
$$-2(a_{22}p+a_{12})xy^3+a_{22}y^4+2a_{01}p^2x^3-2a_{02}p^2x^2y-$$
$$-2a_{01}xy^2+2a_{02}y^3+((a_{00}+4a_{12})p^2+4a_{11}p)x^2-$$
$$-2(2a_{22}p^2+(4a_{12}-a_{00})p+2a_{11})xy+(4a_{22}p+a_{00}+4a_{12})y^2+$$
$$+4p(a_{02}p+a_{01})x+4(a_{02}p+a_{01})y+4(a_{22}p^2+2a_{12}p+a_{11})=0.$$

The coefficient a_{11} must not equal $-a_{22}p^2-2a_{12}p$, since otherwise conic k would pass through the pole P and constructed quartic $\overline{k}$ would split into the line $px+y=0$ and a cubic.

The intersections of the absolute line $x_0=0$ with the quartic $\overline{k}$ are the points coordinates of which satisfy the equation

$$a_{11}p^2x_1^4-2p(a_{12}p+a_{11})x_1^3x_2+(a_{22}p^2+4a_{12}p+a_{11})x_1^2x_2^2-$$
$$-2(a_{22}p+a_{12})x_1x_2^3+a_{22}x_2^4=0.$$

Some short calculation turns it into

$$(x_2-px_1)^2(a_{11}x_1^2-2a_{12}x_1x_2+a_{22}x_2^2)=0.$$

It is obvious from here that the pole $P(0,1,p)$ is two times counted common point of the absolute line and the quartic. The absolute point is one of the intersections if and only if $a_{22}=0$, and it is two times counted common point if and only if $a_{12}=0$, too.

If $a_{22}=0$, the conic k is a special hyperbola with the equation

$$k \quad \dots \quad a_{00}+a_{11}x^2+2a_{12}xy+2a_{01}x+2a_{02}y=0, \quad (14)$$

and the quartic $\overline{k}$ is 1-circular, Figure 1.
If $a_{12}=a_{22}=0$, the conic k is a circle

$$k \quad \dots \quad a_{00}+a_{11}x^2+2a_{01}x+2a_{02}y=0, \quad (15)$$

and the quartic $\overline{k}$ is 2-circular.
Therefore, the degree of circularity of the constructed quartic $\overline{k}$ equals the degree of circularity of the conic k.

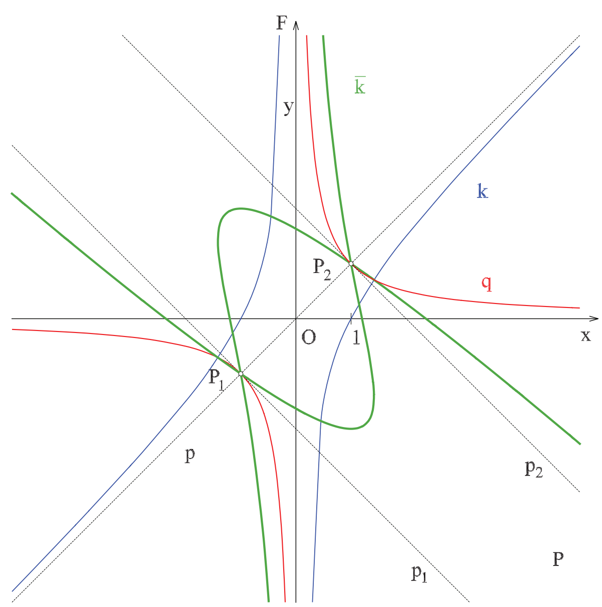


Figure 1

We should determine the tangent of the quartic $\overline{k}$ at the absolute point. Any line through the point $F(0,0,1)$, except the absolute one, has the equation of the form $x = m$, i.e. $x_1 = mx_0$. Its intersections with the quatric $\overline{k}$ are the points coordinates of which satisfy the equation

$$(4a_{01}mp+4a_{11}m^2p+4a_{02}mp^2+(a_{00}+4a_{12})m^2p^2+$$
$$+2a_{01}m^3p^2+a_{11}m^4p^2+4a_{11}+8a_{12}p)x_0^4+$$
$$+2(2a_{01}-2a_{11}m+2a_{02}p+(a_{00}-4a_{12})mp-a_{11}m^3p-$$
$$-a_{02}m^2p^2-a_{12}m^3p^2)x_0^3x_2+$$
$$+(a_{00}+4a_{12}-2a_{01}m+a_{11}m^2+4a_{12}m^2p)x_0^2x_2^2+$$
$$+2(a_{02}-a_{12}m)x_0x_2^3=0.$$

The line is a tangent at the point F if $x_0 = 0$ is double root of the equation above, and that is if and only if $a_{02} - a_{12}m = 0$.
Hence in the case of k being the special hyperbola given by (14), the line $x = \frac{a_{02}}{a_{12}}$ is a tangent of the quartic.
If k is the circle ($a_{12} = 0$) given by the equation (15), there is no line different from the absolute one that touches the quartic at the absolute point.

Any line passing through the pole $P(0,1,p)$ and different from the absolute one has the equation of the form $mx_0 - px_1 + x_2 = 0$, i.e. $y = px + m$. We need to determine m corresponding with a tangent. Its intersections with the quartic satisfy the equation

$$(2a_{02}m^3+(a_{00}-4a_{12})m^2+4(a_{01}+a_{02}p)m+4(a_{11}+2a_{12}p))x_0^4+$$
$$+(-2a_{12}m^3+2(3a_{02}p-a_{01})m^2+4(-a_{11}+a_{00}p)m+$$
$$+8p(a_{01}+a_{02}p))x_0^3x_1+$$
$$+((a_{11}-2a_{12}p)m^2+4p(-a_{01}+a_{02}p)m+4a_{00}p^2)x_0^2x_1^2=0.$$

Obviously, $x_0 = 0$ is a double solution for each m. Therefore, P is a double point of the curve $\overline{k}$. $x_0 = 0$ is a triple solution if $(a_{11}-2a_{12}p)m^2+4p(-a_{01}+a_{02}p)m+4a_{00}p^2$ equals zero.
It follows that the lines

$$y = px+2p\frac{a_{01}-a_{02}p\pm\sqrt{(a_{01}-a_{02}p)^2-a_{00}(a_{11}-2a_{12}p)}}{a_{11}-2a_{12}p}$$

are the tangents of the quartic at the pole P. The reality of these lines depends on the sign of the expression under the root, and P is a cusp if the expression equals zero. The reality of the points $T_{1,2}(1,t_{1,2},-pt_{1,2})$, at which the conic k intersects the fundamental line p, depends on the same expression, where

$$t_{1,2} = \frac{a_{02}p-a_{01}\pm\sqrt{(a_{02}p-a_{01})^2-a_{00}(a_{11}-2a_{12}p)}}{a_{11}-2a_{12}p}.$$

In the special case when $a_{11} = 2a_{12}p$ and the special hyperbola k intersects the absolute line at the absolute point and the point $A(0,1,-p)$ at which the line is intersected with the polar p of the pole P, one of the tangents of the quartic at the pole P coincides with the absolute line, while the other tangent has the equation $y = px + \frac{a_{00}p}{a_{01}-a_{02}p}$.
If $a_{01} = a_{02}p$, too, the conic k touches the fundamental line p at the point A. Therefore, the point P is a cusp at which both tangents coincide with the absolute line.

Figure 1 displays 1-circular quatric $\overline{k}...x^4+3x^3y+3x^2y^2+xy^3-7x^2-6xy-3y^2-8=0$ obtained as the inverse image of the special hyperbola $k...1-x^2+xy=0$ by the inversion $\overline{x} = \frac{-x^2-xy+2}{-x+y}$, $\overline{y} = \frac{xy+y^2-2}{-x+y}$ with the fundamental conic $q...xy-1=0$ and the pole $P(0,1,-1)$. The tangent of the quartic at the absolute point is the line $x = 0$, while the absolute line is the tangent at the cusp P. Each of the fundamental lines $p_1...y=-x-2$, $p_2...y=-x+2$ intersects k at two different real points. Therefore, fundamental points $P_1(-1,-1)$, $P_2(1,1)$ are the nodes at which tangents have the equations $y = \frac{-9\pm4\sqrt{3}}{3}x+\frac{12\pm4\sqrt{3}}{3}$, $y = \frac{-9\pm4\sqrt{3}}{3}x+\frac{12\mp4\sqrt{3}}{3}$, respectively.

Type (2)

An inversion with the fundamental conic $q...xy-1=0$ and the pole $P(0,1,0)$ is given by

$$\overline{x} = -x+\frac{2}{y}, \quad \overline{y} = y.$$

The image of the point $T(x,y)$ is the point $\overline{T}(-x+\frac{2}{y},y)$. Obviously, $d(T,Q) = d(Q,\overline{T})$, where Q is the intersection of the ray PT with the fundamental conic q, Figure 2.
The image of the conic k with the equation (13) is the quartic $\overline{k}$:

$$a_{11}x^2y^2-2a_{12}xy^3+a_{22}y^4-2a_{01}xy^2+2a_{02}y^3-$$
$$-4a_{11}xy+(a_{00}+4a_{12})y^2-4a_{01}y+4a_{11}=0.$$

The quartic $\overline{k}$ is circular if and only if $a_{22} = 0$, i.e. if and only if k is circular.
The conic k intersects the absolute line $x_0 = 0$ at the points $(0,-2a_{12},a_{11})$ and $(0,0,1)$, while the quartic $\overline{k}$ intersects it at the points coordinates of which satisfy the equation

$$x_1x_2^2(a_{11}x_1-2a_{12}x_2)=0.$$

It is obvious that $x_1 = 0$ is a solution, hence $F(0,0,1)$ is an intersection. Since $x_2 = 0$ is a double solution, $(0,1,0)$ is double joint point of the quartic with the absolute line. If $a_{12} \neq 0$, the fourth intersection is the point $(0,2a_{12},a_{11})$. If $a_{12} = 0$, then $a_{11} \neq 0$ (since otherwise the conic k is a line) and the point $(0,0,1)$ is the fourth intersection. In that case

k is a circle and $\overline{k}$ touches the absolute line at the absolute point.

It is easy to prove that $P(0,1,0)$ is a double point of the quartic at which both tangents are identical with the fundamental line $p\,[0,0,1]$.

In order to make the quartic $\overline{k}$ circular, it is necessary for conic k to be circular. More precisely, $\overline{k}$ is 1-circular if k is 1- circular, and $\overline{k}$ is 2-circular if k is 2- circular. The tangent of the conic k at the point F is the line $a_{02}x_0 + a_{12}x_1 = 0$, while the tangent of the quatric $\overline{k}$ at that point is the line $a_{02}x_0 - a_{12}x_1 = 0$.

The inversion $\overline{x} = -x + \dfrac{2}{y}, \overline{y} = y,$ maps the circle k with the equation $x^2 - y = 0$ into the 2-circular quartic $\overline{k}...x^2y^2 - y^3 + -4xy + 4 = 0$ touching the absolute line at the absolute point. The quartic has a cusp with the tangent $y = 0$ at the pole of the inversion, Figure 2.

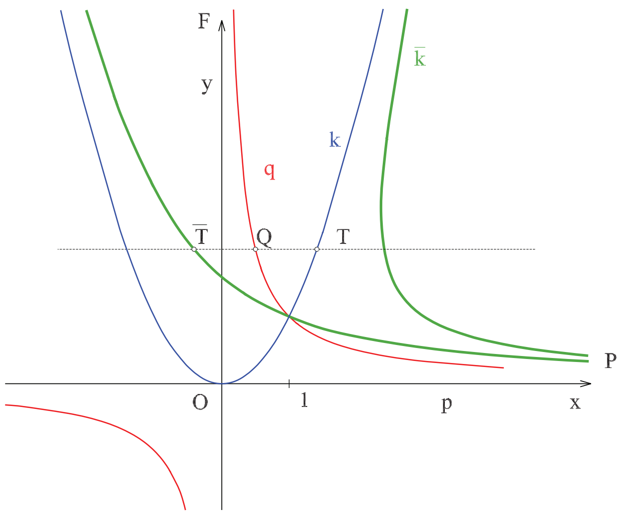


Figure 2

Type (3)

Let us now consider the inversion

$$\overline{x} = x, \quad \overline{y} = -y + \frac{2}{x}$$

with the fundamental conic $q...xy - 1 = 0$. The absolute point $F(0,0,1)$ is the pole and its polar line $x = 0$ is the fundamental line of the inversion. The point $(0,1,0)$ is another intersection of the fundamental conic with the absolute line.

The point $\overline{T}(x, -y - \dfrac{2a}{x})$ is the inverse image of the point $T(x,y)$. Clearly, $s(T,Q) = s(Q,\overline{T})$, where Q is the intersection of the ray FT with the fundamental conic q, Figure 3.

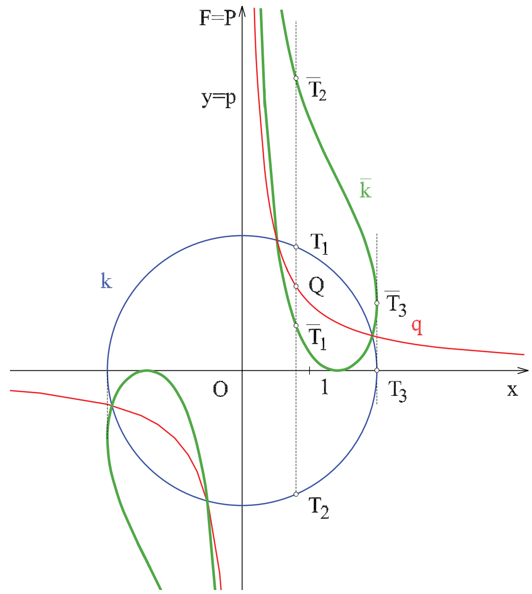


Figure 3

Let the conic k be given with the equation (13). The equation of the constructed quartic $\overline{k}$ is then

$$a_{11}x^4 - 2a_{12}x^3y + a_{22}x^2y^2 + 2a_{01}x^3 - 2a_{02}x^2y + \\ + (a_{00} + 4a_{12})x^2 - 4a_{22}xy + 4a_{02}x + 4a_{22} = 0.$$

The coefficient a_{22} should not equal zero since in that case the conic k would pass through the absolute point (the pole) and the quartic $\overline{k}$ would split into the line with the equation $x = 0$ and a cubic.

The intersections of the conic k with the absolute line are points $\left(0, -a_{12} \pm \sqrt{a_{12}^2 - a_{11}a_{22}}, a_{11}\right)$ if $a_{11} \neq 0$ and $(0,1,0)$, $(0, a_{22}, -2a_{12})$ if $a_{11} = 0$. If $a_{12}^2 = a_{11}a_{22}$, the conic k touches the absolute line.

Let us now determine the intersections of the quartic $\overline{k}$ and the absolute line $x_0 = 0$. Their projective coordinates should satisfy the equation

$$a_{11}x_1^4 - 2a_{12}x_1^3x_2 + a_{22}x_1^2x_2^2 = 0.$$

It is evident that $x_1 = 0$ is a double solution of the equation. Therefore, the point $F\,(0,0,1)$ is their two times counted common point. Since $a_{22} \neq 0$, $x_1 = 0$ cannot be a triple solution.

If $a_{11} \neq 0$, the other two intersections are points $\left(0, a_{12} \pm \sqrt{a_{12}^2 - a_{11}a_{22}}, a_{11}\right)$.

If $a_{11} = 0$, the intersections of the quartic with the absolute line are point $(0,0,1)$ counted twice and points $(0,1,0)$, $(0, a_{22}, 2a_{12})$. If a_{12} equals zero, $\overline{k}$ touches the absolute line f.

Our next goal is to determine the equation of the tangent of the quartic at the absolute point.
Any line t through F (except the absolute one) is given by equation $x_1 = mx_0$, i.e. $x = m$. The coordinates of its intersections with the quartic satisfy the equation

$$x_0^2[(a_{11}m^4+2a_{01}m^3+(a_{11}+4a_{12})m^2+4a_{02}m+4a_{22}^2)x_0^2+ \\ +(-2a_{12}m^3-2a_{02}m^2-4a_{22}m)x_0x_2+a_{22}m^2x_2^2]=0.$$

Due to the fact that $x_0 = 0$ is a double solution for each m we conclude that $(0,0,1)$ is a double point of the curve.
$x_0 = 0$ is a triple solution if m equals zero. In that case the solutions are given by

$$x_0^2\left[4a^2a_{22}^2x_0^2\right]=0.$$

Thus $x_0 = 0$ is a quadruple solution.
Consequently, the line p $(x_1 = 0)$ is two times counted tangent of the quartic $\overline{k}$ at the absolute point F.

Figure 3 displays an inverse image $\overline{k}$ of the conic $k...x^2+y^2-4=0$ with respect to the fundamental conic $q...xy-1=0$. The inversion is given by the equations $\overline{x}=x$, $\overline{y}=-y+\frac{2}{x}$, and the quartic $\overline{k}$ by the equation $x^4+x^2y^2-4x^2-4xy+4=0$. The quartic intersects the absolute line at the double point F at which both tangents coincide with the y-axis and the pair of conjugate imaginary points $(0,1,i)$ and $(0,1,-i)$.

Type (4)

If the circle $q...x^2-y=0$ is the fundamental conic and the point $P(0,1,0)$ is the pole of inversion

$$\overline{x}=\frac{y}{x},\quad \overline{y}=y,$$

the conic k given by the equation (13) is mapped by this transformation into the quartic

$$a_{22}x^2y^2+2a_{02}x^2y+2a_{12}xy^2+a_{00}x^2+2a_{01}xy+a_{11}y^2=0. \tag{16}$$

Coefficients a_{00}, a_{11}, a_{22} must not equal zero, since otherwise k would pass through some of the fundamental points.
The intersection points of the conic k with the fundamental lines $x_0 = 0$, $x_1 = 0$, $x_2 = 0$ are the points $(0,a_{22},-a_{12}\pm\sqrt{a_{12}^2-a_{11}a_{22}})$, $(a_{22},0,-a_{02}\pm\sqrt{a_{02}^2-a_{00}a_{22}})$, $(a_{11},-a_{01}\pm\sqrt{a_{01}^2-a_{00}a_{11}},0)$, respectively.
Since $F(0,0,1)$, $P(0,1,0)$ are the intersection points of the quartic (16) with the absolute line and each of them two times counted, an inversion of the type (4) produces 2-circular quartics, Figure 4.

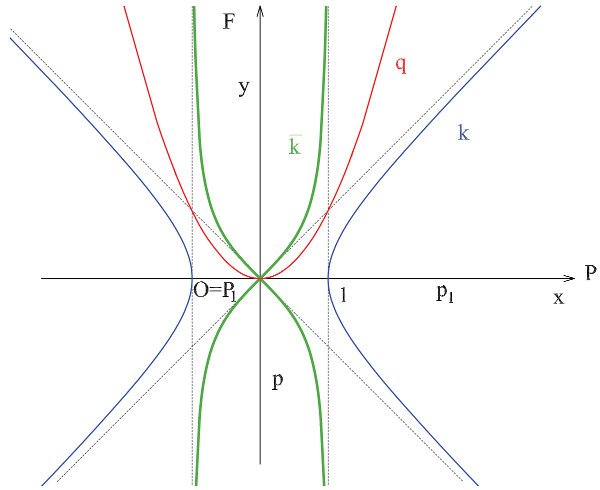


Figure 4

Its tangents at the double point $F(0,0,1)$ are given by

$$x=\frac{-2a_{12}\pm\sqrt{a_{12}^2-a_{11}a_{22}}}{2a_{22}},$$

at the pole $P(0,1,0)$ by

$$y=\frac{-a_{02}\pm\sqrt{a_{02}^2-a_{00}a_{22}}}{a_{22}},$$

and at the third fundamental point $P_1(1,0,0)$ by

$$y=\frac{-a_{01}\pm\sqrt{a_{01}^2-a_{00}a_{11}}}{a_{11}}x.$$

Obviously, the reality of the tangents and the type of the double point depend on the reality of the intersections of the conic k with the corresponding fundamental line.

The conic $k...1-x^2+y^2=0$ and its inverse image $\overline{k}...x^2y^2+x^2-y^2=0$ obtained by the inversion $\overline{x}=\frac{y}{x}$, $\overline{y}=y$ are presented in Figure 4. The quartic possesses a node at the absolute point and an isolated double point at $(0,1,0)$. Its tangents at the point F are lines $x=\pm1$, at the pole P the lines $y=\pm i$ and at the fundamental point P_1 the lines $y=\pm x$.

Type (5)

Let us now suppose that the circle $q...x^2-y=0$ is the fundamental conic and the absolute point $F(0,0,1)$ is the pole of an inversion. The inversion is given by

$$\overline{x}=x,\quad \overline{y}=2x^2-y.$$

This type of inversion is analogous with the ordinary inversion in the Euclidean plane.

The inverse image of the point $T(x,y)$ is the point $\overline{T}(x,2x^2-y)$. So, $s(T,Q)=s(Q,\overline{T})$, where Q is the intersection of the ray FT with the fundamental conic, Figure 5.

The image of the conic (13) is the quartic $\overline{k}$

$$4a_{22}x^4+4a_{12}x^3-4a_{22}x^2y+(a_{11}+4a_{02})x^2-2a_{12}xy+$$

$$+a_{22}y^2+2a_{01}x+4a_{02}y+a_{00}=0.$$

Obviously, $a_{22}=0$ is not allowed. In that case the conic (13) would pass through the pole and the constructed quartic $\overline{k}$ would split into the absolute line and a cubic.

An easy computation shows that the absolute point is a double point of the curve at which both tangents coincide with the absolute line. Therefore, $\overline{k}$ is an entirely circular quartic.

The quartics $4x^4-4x^2y-x^2+y^2+1=0$, $4x^4-4x^2y+x^2+y^2-4=0$, $4x^4-4x^2y+y^2-x=0$ in Figure 5 possess a node, an isolated double point, or a cusp at the absolute point, respectively. They are obtained as images of the conics $x^2-y^2=1$, $x^2+y^2=4$, $y^2=x$ by inversion $\overline{x}=x,\quad \overline{y}=2x^2-y.$ □

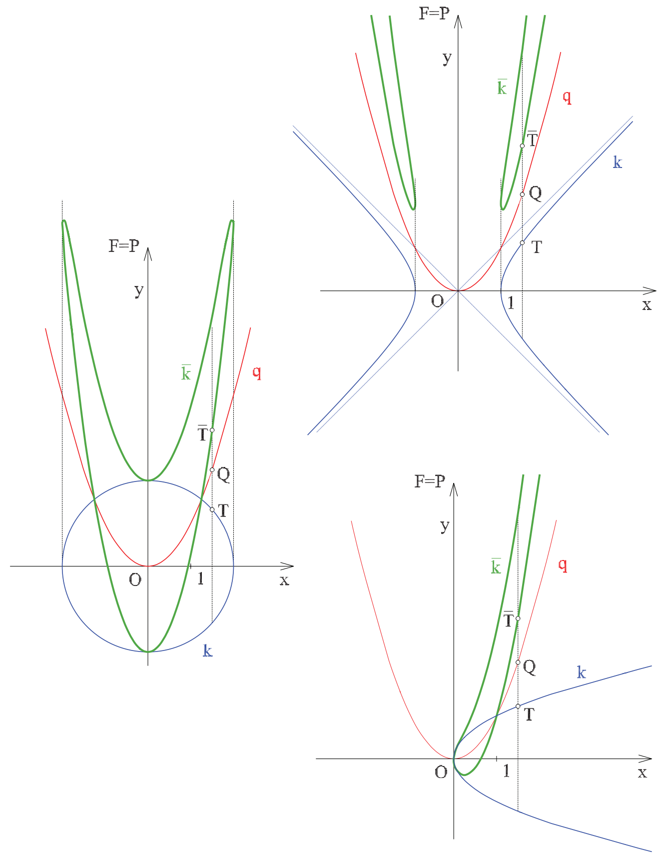


Figure 5

References

[1] J. BEBAN-BRKIĆ, Isometric Invariants of Conics in the Isotropic Plane - Classification of Conics. *Journal for Geometry and Graphics* **6** (2002), No.1, 17–26.

[2] V. NIČE, Krivulje i plohe 3. i 4. reda nastale pomoću kvadratne inverzije, *Rad HAZU* **274** (1945), 153–195.

[3] D. PALMAN, Vollständig zirkuläre Kurven 4. Ordnung der isotropen Ebene. *Rad JAZU* **10** (1991), 15–32.

[4] H. SACHS, *Ebene Isotrope Geometrie*, Wieweg, Braunschweig-Wiesbaden, 1987

[5] V. SZIROVICZA, *Circular Cubics in Isotropic and Hyperbolic Plane.* Dissertation, Department of Mathematics, University of Zagreb, 2001. (in Croatian)

[6] V. SZIROVICZA, A. SLIEPČEVIĆ, Die allgemeine Inversion in der isotropen Ebene. *Rad HAZU* **491** (2005), 153–168.

Ema Jurkin

Faculty of Mining, Geology and
Petroleum Engineering

Pierottijeva 6, 10000 Zagreb, Croatia

e-mail: ejurkin@rgn.hr

Original scientific paper
Accepted 1.12.2008.

H. Bariş Çolakoğlu
Rüstem Kaya

Regular Polygons in the Taxicab Plane

Regular Polygons in the Taxicab Plane

ABSTRACT

In this paper, we define taxicab regular polygons and determine which Euclidean regular polygons are also taxicab regular, and which are not. Finally, we investigate the existence or nonexistence of taxicab regular polygons.

Key words: taxicab distance, Euclidean distance, protractor geometry, regular polygon

MSC 2000: 51K05, 51K99.

Pravilni poligoni u taxicab ravnini

SAŽETAK

U radu definiramo pravilne taxicab poligone i određujemo koji su pravilni euklidski poligoni ujedno i pravilni taxicab poligoni, a koji nisu. Naposljetku, ispitujemo postojanje ili nepostojanje pravilnih taxicab poligona.

Ključne riječi: taxicab udaljenost, euklidska udaljenost, protractor geometrija, pravilni poligon

1 Introduction

A *metric geometry* consists of a set $\mathcal{P}$, whose elements are called *points*, together with a collection $\mathcal{L}$ of non-empty subsets of $\mathcal{P}$, called *lines*, and a distance function d, such that
1) every two distinct points in $\mathcal{P}$ lie on a unique line,
2) there exist three points in $\mathcal{P}$, which do not lie all on one line,
3) there exists a bijective function $f : l \to \mathbb{R}$ for all lines in $\mathcal{L}$ such that $|f(P) - f(Q)| = d(P,Q)$ for each pair of points P and Q on l.

A metric geometry defined above is denoted by $\{\mathcal{P},\mathcal{L},d\}$. However, if a metric geometry satisfies the plane separation axiom below, and it has an angle measure function m, then it is called *protractor geometry* and denoted by $\{\mathcal{P},\mathcal{L},d,m\}$.

4) For every l in $\mathcal{L}$, there are two subsets H_1 and H_2 of $\mathcal{P}$ (called *half planes* determined by l) such that
(i) $H_1 \cup H_2 = \mathcal{P} - l$ ($\mathcal{P}$ with l removed),
(ii) H_1 and H_2 are disjoint and each is convex,
(iii) If $A \in H_1$ and $B \in H_2$, then $[AB] \cap l \neq \varnothing$.

The taxicab metric was given by Minkowski [8] at the beginning of the last century. Later, taxicab plane geometry was introduced by Menger [6], and developed by Krause [5], using the taxicab metric $d_T(P,Q)=|x_1 - x_2|+|y_1 - y_2|$ instead of the well-known Euclidean metric $d_E(P,Q)=[(x_1-x_2)^2+(y_1-y_2)^2]^{1/2}$ for the distance between any two points $P = (x_1,y_1)$ and $Q = (x_2,y_2)$ in the Cartesian coordinate plane ($\mathbb{R}^2$). If L_E is the set of all lines, and m_E is the standard angle measure function of the Euclidean plane, then $\{\mathbb{R}^2,L_E,d_T,m_E\}$ is a model of protractor geometry, and it is called *taxicab plane* (see [3], [7]). The taxicab plane is one of the simple non-Euclidean geometries since it fails to satisfy the side-angle-side axiom, but it satisfies all the remaining twelve axioms of the Euclidean plane (see [5]). It is almost the same as the Euclidean plane $\{\mathbb{R}^2,L_E,d_E,m_E\}$ since the points are the same, the lines are the same, and the angles are measured in the same way. However, the distance functions are different. Since taxicab plane have distance function different from that in the Euclidean plane, it is interesting to study the taxicab analogues of topics that include the distance concept in the Euclidean plane. During the recent years, many such topics have been studied in the taxicab plane (see [10]). In this work, we study regular polygons in the taxicab plane.

2 Taxicab Regular Polygons

As in the Euclidean plane, a *polygon* in the taxicab plane consists of three or more coplanar line segments; the line segments (*side*s) intersect only at endpoints; each endpoint (*vertex*) belongs to exactly two line segments; no two line segments with a common endpoint are collinear. If the number of sides of a polygon is n for $n \geqslant 3$ and $n \in \mathbb{N}$, then the polygon is called an *n-gon*. The following definitions for polygons in the taxicab plane are given by means of the taxicab lengths instead of the Euclidean lengths:

Definition 1 *A polygon in the plane is said to be taxicab equilateral if the taxicab lengths of its sides are equal.*

Definition 2 *A polygon in the plane is said to be taxicab equiangular if the measures of its interior angles are equal.*

Definition 3 *A polygon in the plane is said to be taxicab regular if it is both taxicab equilateral and equiangular.*

Definition 2 does not give a new equiangular concept because the taxicab and the Euclidean measure of an angle are the same. That is, every Euclidean equiangular polygon is also the taxicab equiangular, and vice versa. However, since the taxicab plane has a different distance function, Definition 1 and therefore Definition 3 are new concepts. In this study, we answer the following question: Which Euclidean regular polygons are also the taxicab regular, and which are not? Also we investigate the existence and nonexistence of taxicab regular polygons.

Proposition 1 *Let A, B, C and D be four points in the Cartesian plane such that $A \neq B$ and $d_E(A,B) = d_E(C,D)$, and let m_1 and m_2 denote the slopes of the lines AB and CD, respectively.*

(i) *If $m_1 \neq 0 \neq m_2$, then $d_T(A,B) = d_T(C,D)$ iff $|m_1| = |m_2|$ or $|m_1 m_2| = 1$.*

(ii) *If $m_i = 0$ or $m_i \to \infty$, then $d_T(A,B) = d_T(C,D)$ iff $m_j = 0$ or $m_j \to \infty$, where $i, j \in \{1,2\}$ and $i \neq j$.*

Proof. We know from [4] that for any two points P and Q in the Cartesian plane that do not lie on a vertical line, if m is the slope of the line PQ, then

$$d_E(P,Q) = \rho(m) d_T(P,Q) \tag{1}$$

where $\rho(m) = (1+m^2)^{1/2} \diagup (1+|m|)$. If P and Q lie on a vertical line, that is $m \to \infty$, then $d_E(P,Q) = d_T(P,Q)$.

(i) Let $m_1 \neq 0 \neq m_2$ and $d_E(A,B) = d_E(C,D)$; then by Equation (1), $\rho(m_1) d_T(A,B) = \rho(m_2) d_T(C,D)$. If $d_T(A,B) = d_T(C,D)$, then $\rho(m_1) = \rho(m_2)$, that is $(1+m_1^2)^{1/2} \diagup (1+|m_1|) = (1+m_2^2)^{1/2} \diagup (1+|m_2|)$. Simplifying the last equation, we get $(|m_1|-|m_2|)(|m_1 m_2|-1) = 0$. Therefore $|m_1| = |m_2|$ or $|m_1 m_2| = 1$. If $|m_1| = |m_2|$ or $|m_1 m_2| = 1$, then $m_2 = m_1$, $m_2 = -m_1$, $m_2 = 1/m_1$ or $m_2 = -1/m_1$, and one can easily see by calculations that $\rho(m_1) = \rho(m_2)$. Therefore $d_T(A,B) = d_T(C,D)$.

(ii) Let $m_i = 0$ or $m_i \to \infty$. Then $\rho(m_i) = 1$. If $d_T(A,B) = d_T(C,D)$, then $\rho(m_j) = 1$. Thus, $m_j = 0$ or $m_j \to \infty$. If $m_j = 0$ or $m_j \to \infty$, then $\rho(m_j) = 1$, and therefore $d_T(A,B) = d_T(C,D)$. □

The following corollary follows directly from Proposition 1, and plays an important role in our arguments.

Corollary 2 *Let A, B and C be three non-collinear points in the Cartesian plane such that $d_E(A,B) = d_E(B,C)$. Then, $d_T(A,B) = d_T(B,C)$ iff the measure of the angle ABC is $\pi/2$ or A and C are symmetric about the line passing through B, and parallel to anyone of the lines $x = 0$, $y = 0$, $y = x$ and $y = -x$.*

Note that Proposition 1 and Corollary 2 indicate also Euclidean isometries of the plane that do not change the taxicab distance between any two points: The Euclidean isometries of the plane that do not change the taxicab distance between any two points are all translations, rotations of $\pi/2$ and $3\pi/2$ radians around a point, reflections about lines parallel to anyone of the lines $x = 0$, $y = 0$, $y = x$ and $y = -x$, and their compositions; there is no other bijections of $\mathbb{R}^2$ onto $\mathbb{R}^2$ which preserve the taxicab distance (see [9]).

3 Euclidean Regular Polygons in Taxicab Plane

Since every Euclidean regular polygon is already taxicab equiangular, it is obvious that a Euclidean regular polygon is taxicab regular if and only if it is taxicab equilateral. So, to investigate the taxicab regularity of a Euclidean regular polygon, it is sufficient to determine whether it is taxicab equilateral or not. In doing so, we use following concepts:

Any Euclidean regular polygon can be inscribed in a circle and a circle can be circumscribed about any Euclidean regular polygon. A point is called the *center* of a Euclidean regular polygon if it is the center of the circle circumscribed about the polygon. A line l is called *axis of symmetry* (AOS) of a polygon if the polygon is symmetric about l, and in addition, if l passes through two distinct vertices of the polygon then l is called the *diagonal axis of symmetry* (DAOS) of the polygon. Clearly, every AOS of a Euclidean regular polygon passes through the center of the polygon.

Now, we are ready to investigate the taxicab regularity of Euclidean regular polygons.

Proposition 3 *No Euclidean regular triangle is taxicab regular.*

Proof. Since the Euclidean lengths of two consecutive sides are the same, and the angle between two consecutive sides is not a right angle, by Corollary 2, any two consecutive sides must be symmetric about a line parallel to anyone of the lines $x = 0$, $y = 0$, $y = x$ and $y = -x$, in order to have the same taxicab length. Suppose two consecutive sides are symmetric about a line parallel to anyone of the lines $x = 0$, $y = 0$, $y = x$ and $y = -x$. Figure 1 and Figure 2 show such Euclidean regular triangles. A simple calculation shows that none of the other two AOS's is parallel to anyone of the lines $x = 0$, $y = 0$, $y = x$ and $y = -x$. So, the triangles in Figure 1 and Figure 2 are not taxicab equilateral. Thus, no Euclidean regular triangle is taxicab regular. □

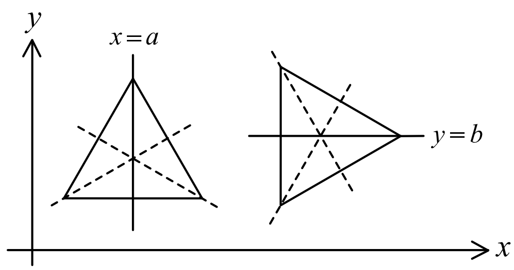


Figure 1

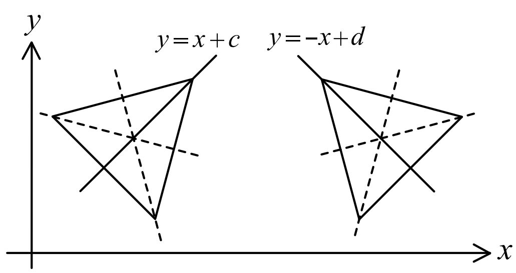


Figure 2

Corollary 4 *No Euclidean regular hexagon is taxicab regular.*

Proof. It is clear that every Euclidean regular hexagon is the union of six Euclidean regular triangles, and by Proposition 1 the taxicab lengths of the sides of one of the Euclidean regular triangles are the same as the taxicab lengths of corresponding parallel sides of the Euclidean regular hexagon as shown in Figure 3. Since no Euclidean regular triangle is taxicab equilateral, no Euclidean regular hexagon is taxicab equilateral, either. Thus, no Euclidean regular hexagon is taxicab regular. □

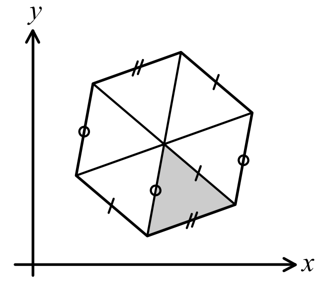


Figure 3

Proposition 5 *Every Euclidean regular quadrilateral (Euclidean square) is taxicab regular.*

Proof. Since every side of the Euclidean square has the same Euclidean length and the angle between every two consecutive sides is a right angle (see Figure 4), by Corollary 2, every side has the same taxicab length. So, every Euclidean square is taxicab equilateral, and therefore is taxicab regular. □

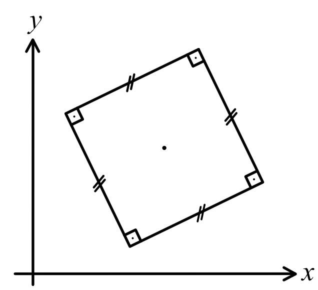


Figure 4

Proposition 6 *Every Euclidean regular octagon, one of whose DAOS's is parallel to anyone of the lines $x = 0$, $y = 0$, $y = x$ and $y = -x$, is taxicab regular.*

Proof. Let us consider the case $x = 0$. Clearly, every Euclidean regular octagon has four DAOS's, and if a DAOS of a Euclidean regular octagon is parallel to the line $x = 0$, then the other DAOS's are parallel to anyone of the lines $y = 0$, $y = x$ and $y = -x$ (see Figure 5). Since every two consecutive sides of such a Euclidean regular octagon are symmetric about a line parallel to anyone of the lines $x = 0$, $y = 0$, $y = x$ and $y = -x$, and every side has the same Euclidean length, by Corollary 2, these sides have the same taxicab length. So, a Euclidean regular octagon, one of whose DAOS's is parallel to the line $x = 0$ is taxicab equilateral, and therefore is taxicab regular. The other cases are similar. □

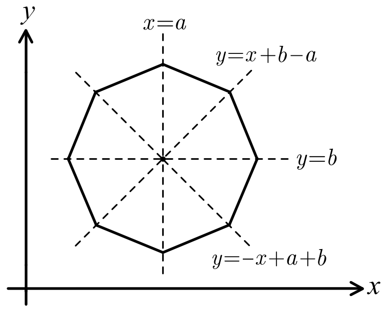


Figure 5

Theorem 7 *No Euclidean regular polygon, except the ones in Proposition 5 and Proposition 6, is taxicab regular.*

Proof. Let us classify Euclidean regular polygons as $(2n-1)$-gons and $2n$-gons for $n \geq 2$ ($n \in \mathbb{N}$), and investigate them separately:

(i) The Euclidean regular $(2n-1)$-gons: The case $n = 2$ is proved in Proposition 3. Let $n > 2$. It is clear that the number of AOS's of a Euclidean regular $(2n-1)$-gon is $2n-1$ (≥ 5), and each AOS of a Euclidean regular $(2n-1)$-gon passes through a vertex and the center of the polygon. Therefore, there exists at least one AOS which is not parallel to the lines $x = 0$, $y = 0$, $y = x$ and $y = -x$. Then, there are at least two consecutive sides symmetric about a line which is not parallel to the lines $x = 0$, $y = 0$, $y = x$ and $y = -x$. Also we know that the angle between two consecutive sides of Euclidean regular $(2n-1)$-gons is not a right angle. By Corollary 2, these consecutive sides do not have the same taxicab length. Thus, if $n > 2$, then Euclidean regular $(2n-1)$-gons are not taxicab equilateral, and therefore are not taxicab regular. That is, no Euclidean regular $(2n-1)$-gon is taxicab regular.

(ii) The Euclidean regular $2n$-gons: The case $n = 2$ is included in Proposition 5. The case $n = 3$ is proved in Corollary 4. In order to exclude the case in Proposition 6, let us consider a Euclidean regular octagon, none of whose DAOS's is parallel to anyone of the lines $x = 0$, $y = 0$, $y = x$ and $y = -x$ for the case $n = 4$. By Corollary 2, no two consecutive sides have the same taxicab length. Thus, such a Euclidean regular octagon is not taxicab equilateral, and therefore is not taxicab regular. Let $n > 4$. Clearly, the number of the DAOS's of a Euclidean regular $2n$-gon is n. Therefore, there exists at least one DAOS which is not parallel to the lines $x = 0$, $y = 0$, $y = x$ and $y = -x$. Then, there are at least two consecutive sides symmetric about a line which is not parallel to the lines $x = 0$, $y = 0$, $y = x$ and $y = -x$. Also we know that the angle between two consecutive sides of Euclidean regular $2n$-gons is not a right angle for $n > 4$. By Corollary 2, these consecutive sides do not have the same taxicab length. Thus, Euclidean regular $2n$-gons for $n > 4$ are not taxicab equilateral, and therefore are not taxicab regular. The proof is completed. □

4 Existence of Taxicab Regular 2*n*-gons

Now, we know which Euclidean regular polygons are taxicab regular, and which are not. Furthermore, we also know the existence of some taxicab regular polygons. However, we do not have general knowledge about the existence of taxicab regular polygons. In this section, we determine some of them. The following theorem shows that there exist taxicab regular $2n$-gons by means of taxicab circles. Recall that a taxicab circle with center A and radius r is the set of all points whose taxicab distance to A is r. This locus of points is a Euclidean square with center A, each side having slope ± 1, and each diagonal having length $2r$ (see [2]).

Theorem 8 *There exist two congruent taxicab regular $2n$-gons ($n \geq 2$), having given any line segment as a side.*

Proof. Clearly, the measure of each interior angle of an equiangular $2n$-gon is $\pi(n-1)/n$ radians. Let us consider now any given line segment A_1A_2 in the taxicab plane. It is obvious that $(n-1)$ line segments A_iA_{i+1} ($2 \leq i \leq n$), having the same taxicab length $d_T(A_1, A_2)$, can be constructed such that the measure of the angle between every two consecutive segments is $\pi(n-1)/n$ radians, by using the taxicab circles with center A_i and radius $d_T(A_1, A_2)$, as in Figure 6. Also it is not difficult to see that $\measuredangle A_2A_1A_{n+1} + \measuredangle A_nA_{n+1}A_1 = \pi(n-1)/n$

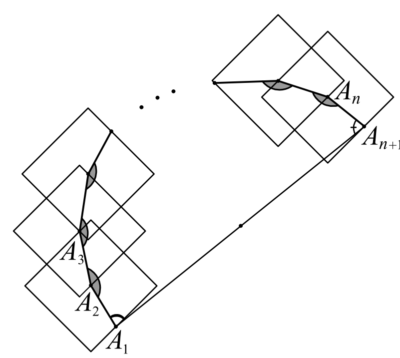


Figure 6

If we continue to construct line segments $A'_iA'_{i+1}$ which are symmetric to A_iA_{i+1} ($1 \leq i \leq n$) about the midpoint of A_1A_{n+1}, respectively, we get a $2n$-gon (see Figure 7). Since symmetry about a point (rotation of π radians around a point) preserves both taxicab lengths and angle measures, we have $d_T(A_i, A_{i+1}) = d_T(A'_i, A'_{i+1}) = d_T(A_1, A_2)$ ($1 \leq i \leq n$) and $\measuredangle A_i = \measuredangle A'_i = \pi(n-1)/n$ ($2 \leq i \leq n$). Also it is not difficult to see that $\measuredangle A_1 = \measuredangle A_{n+1} = \pi(n-1)/n$. Thus, the constructed $2n$-gon is taxicab regular. Furthermore, on the other side of the line A_1A_2, one can construct another taxicab regular $2n$-gon, having the same line segment A_1A_2 as a side, by using the same procedure (see Figure 8).

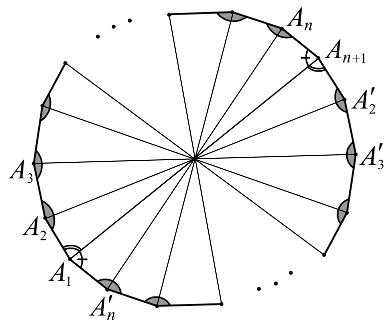


Figure 7

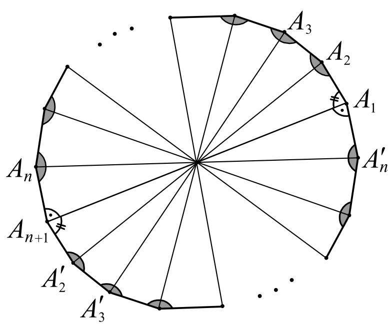


Figure 8

However, it is easy to see that these two taxicab regular $2n$-gons are symmetric about the midpoint of the line segment A_1A_2, and they are congruent. □

In every taxicab regular $2n$-gon, there are n line segments joining the corresponding vertices of the $2n$-gon (A_iA_i', $1 \le i \le n$, for polygons in Figure 7 and Figure 8). We call each of these line segments an *axis* of the polygon. Clearly, axes of every taxicab regular $2n$-gon intersect at one and only one point.

Example. Using the procedure given in the proof of Theorem 8, one can easily construct taxicab regular $2n$-gons, having given any line segment as a side. To give examples, we construct a taxicab regular quadrilateral (taxicab square), a taxicab regular hexagon, and a taxicab regular octagon, having given line segment AB as a side, in Figure 9, 10 and 11:

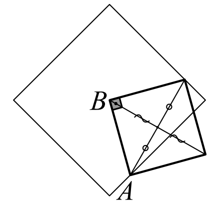


Figure 9

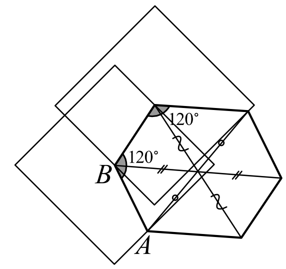


Figure 10

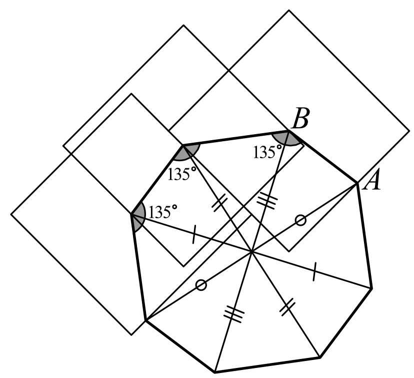


Figure 11

5 More About Taxicab Regular Polygons

By Proposition 5, we know that every Euclidean square is taxicab regular, that is, a taxicab square. By Proposition 11 below, we will see that every taxicab square is also Euclidean regular, that is, a Euclidean square. Thus, the Euclidean and the taxicab squares always have the same shape, and the only regular polygon having this property is square.

Proposition 9 *Let A, B, C and D be four points in the Cartesian plane such that $A \neq B$ and $d_T(A,B) = d_T(C,D)$, and let m_1 and m_2 denote the slopes of the lines AB and CD, respectively.*

(i) If $m_1 \neq 0 \neq m_2$, then $d_E(A,B) = d_E(C,D)$ iff $|m_1| = |m_2|$ or $|m_1 m_2| = 1$.

(ii) If $m_i = 0$ or $m_i \to \infty$, then $d_E(A,B) = d_E(C,D)$ iff $m_j = 0$ or $m_j \to \infty$, where $i, j \in \{1,2\}$ and $i \neq j$.

Proof. The proof is similar to that of Proposition 1. □

The following corollary follows directly from Proposition 9:

Corollary 10 *Let A, B, and C be three non-collinear points in the Cartesian plane such that $d_T(A,B) = d_T(B,C)$. Then, $d_E(A,B) = d_E(B,C)$ iff the measure of the angle ABC is $\pi/2$ or A and C are symmetric about the line passing through B, and parallel to anyone of the lines $x = 0$, $y = 0$, $y = x$ and $y = -x$.*

Proposition 11 *Every taxicab square is Euclidean regular.*

Proof. Since every side of the taxicab square has the same taxicab length and the angle between every two consecutive sides is a right angle, by Corollary 10, every side has the same Euclidean length. So, every taxicab square is Euclidean equilateral, and therefore is Euclidean regular. □

We need a new notion to prove the next proposition: An equiangular polygon with an even number of vertices is called *equiangular semi-regular* if sides have the same Euclidean length alternately. There is always a Euclidean circle passing through all vertices of an equiangular semi-regular polygon (see [11]).

Proposition 12 *Every taxicab regular octagon, one of whose axes is parallel to anyone of the lines $x = 0$, $y = 0$, $y = x$ and $y = -x$, is Euclidean regular.*

Proof. In every taxicab regular octagon, sides have the same Euclidean length alternately since the measure of the angle between any two alternate sides is $\pi/2$ and sides have the same taxicab length, by Proposition 9 and Corollary 10. Therefore, every taxicab regular octagon is equiangular semi-regular. It is obvious that if any two consecutive sides of an equiangular semi-regular polygon have the same Euclidean length, then the polygon is Euclidean regular. Let us consider a taxicab regular octagon, $A_1A_2...A_8$, one of whose axes, let us say A_1A_5, is parallel to the line $y = 0$, for one case (see Figure 12).
Then there exist a Euclidean circle with diameter A_1A_5, passing through points $A_1, A_2, ..., A_8$, and there exist a taxicab circle with center A_1, passing through points A_2 and A_8. Since the Euclidean and the taxicab circles are both symmetric about the line A_1A_5, the intersection points of them, A_2 and A_8, are also symmetric about the same line. Then two consecutive sides A_1A_2 and A_1A_8 have the same Euclidean length by Corollary 10. Therefore, every taxicab regular octagon, one of whose axes is parallel to the line $y = 0$, is Euclidean regular. The other cases are similar. □

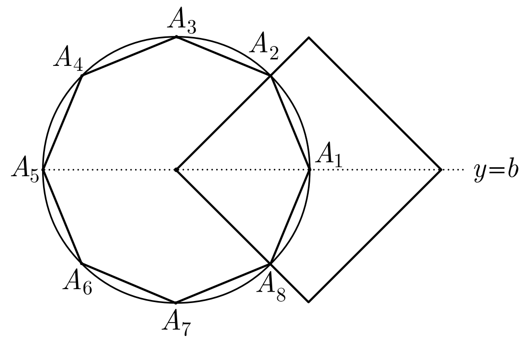


Figure 12

Theorem 13 *No taxicab regular polygon, except the ones in Proposition 11 and Proposition 12, is Euclidean regular.*

Proof. Assume that there exists a taxicab regular polygon, except the ones in Proposition 11 and Proposition 12, that is also Euclidean regular. Then there exists a Euclidean regular polygon, except the ones in Proposition 5 and Proposition 6, that is also taxicab regular. But this is in contradiction with Theorem 7. Therefore, no taxicab regular polygon, except the ones in Proposition 11 and Proposition 12, is Euclidean regular. □

6 On the Nonexistence of Taxicab (2*n*–1) - gons

The following theorem shows that there is no taxicab regular triangle:

Theorem 14 *There is no taxicab regular triangle.*

Proof. Every taxicab equiangular triangle is a Euclidean regular triangle. Since no Euclidean regular triangle is taxicab regular by Proposition 3, no taxicab equiangular triangle is taxicab regular. Therefore, there is no taxicab regular triangle. □

In addition to Theorem 14, we have seen that there is no taxicab regular 5-gon, 9-gon and 15-gon using a computer program called *Compass and Ruler* [12]. However, we could not reach any conclusion by reasoning about the existence or nonexistence of taxicab regular $(2n-1)$-gons for $n = 4$, $n = 6$, $n = 7$ and $n \geq 9$. Our conjecture is that there is no taxicab regular $(2n-1)$-gon since there is no center of symmetry of equiangular $(2n-1)$-gons. It seems interesting to study the open problem: Does there exist any taxicab regular $(2n-1)$-gon?

One can also consider the generalizations and variations of our problem. One of them is determining the regular polygons of the taxicab space. The taxicab distance between points $P = (x_1, y_1, z_1)$ and $Q = (x_2, y_2, z_2)$ in the

Cartesian coordinate space ($\mathbb{R}^3$) is defined by $d_T(P,Q) = |x_1 - x_2| + |y_1 - y_2| + |z_1 - z_2|$ (see [1]). Clearly, the concept of regular polygon can be defined similarly in the taxicab space; and if regular polygons are determined, then one can investigate regular polyhedra in the taxicab space. This is also interesting subject since there are only 5 types of regular polyhedra in the three dimensional Euclidean space. However, the results of this work cannot be generalized directly to the three dimensional taxicab space since the taxicab distance is not uniform in all directions.

References

[1] Z. AKCA, R. KAYA, On the Distance Formulae in Three Dimensional Taxicab Space, Hadronic Journal, Vol 27 (2004), No. 5, 521-532.

[2] H. B. ÇOLAKOĞLU, R. KAYA, A Synthetic Approach to the Taxicab Circles, Applied Science (APPS), Vol. 9 (2007), 67-77.

[3] B. DIVJAK, Notes on Taxicab Geometry, KoG, Vol. 5 (2000/01), 5-9.

[4] R. KAYA, Area Formula for Taxicab Triangles, Pi Mu Epsilon Journal, Vol. 12 (2006), 213-220.

[5] E.F. KRAUSE, *Taxicab Geometry; An Adventure in Non-Euclidean Geometry*, Dover Publications, Inc., New York, 1986.

[6] K. MENGER, *You Will Like Geometry*, Guildbook of the Illinois Institute of Technology Geometry Exhibit, Museum of Science and Industry, Chicago, IL, 1952.

[7] R. S. MILMANN, G. D. PARKER, *Geometry; A Metric Approach with Models*, Springer, 1991.

[8] H. MINKOWSKI, *Gesammelte Abhandlungen*, Chelsa Publishing Co., New York, 1967.

[9] D. J. SCHATTSCHNEIDER, The Taxicab Group, Amer. Math. Monthly, Vol. 91 (1984), No. 7, 423-428.

[10] S. S. SO, Recent Developments in Taxicab Geometry, Cubo Matematica Educacional, Vol. 4 (2002), No. 2, 76-96.

[11] http://eom.springer.de/p/p073580.htm Encyclopaedia of Mathematics, Springer-Verlag, 2002.

[12] http://mathsrv.ku-eichstaett.de/MGF/homes/grothmann/java/zirkel/doc_en/index.html

H. Barış Çolakoğlu
e-mail: bariscolakoglu@hotmail.com

Rüstem Kaya
e-mail: rkaya@ogu.edu.tr

Eskişehir Osmangazi University
Faculty of Arts and Sciences
Department of Mathematics
26480, Eskişehir, Turkey

Original scientific paper
Accepted 10.12.2008.

NILGÜN SÖNMEZ

Trigonometric Proof of Steiner-Lehmus Theorem in Hyperbolic Geometry

Trigonometric Proof of Steiner-Lehmus Theorem in Hyperbolic Geometry

ABSTRACT

In this study, we give a trigonometric proof of the Steiner-Lehmus Theorem in hyperbolic geometry.

Key words: hyperbolic geometry, hyperbolic triangle

MSC 2000: 51K05, 51K99

Trigonometrijski dokaz Steiner-Lehmusovog teorema u hiperboličkoj geometriji

SAŽETAK

U ovom radu dajemo trigonometrijski dokaz Steiner-Lehmusovog teorema u hiperboličkoj geometriji.

Ključne riječi: hiperbolička geometrije, hiperbolički trokut

1 Introduction

Elementary hyperbolic geometry was born in 1903 when Hilbert provided, using the end-calculus to introduce coordinates, a first-order axiomatization for it by adding to the axioms for plane absolute geometry a *hyperbolic parallel axiom* stating that "Through any point P not lying on a line l there are two rays r_1 and r_2, not belonging to the same line, which do not intersect l, and such that every ray through P contained in the angle formed by r_1 and r_2 does intersect l" [2]. The hyperbolic geometry is a non-euclidean geometry. Here in this study, we give hyperbolic version of Steiner-Lehmus theorem. The well-known Steiner-Lehmus theorem states that if the internal angle bisectors of two angles of a triangle are equal, then the triangle is isosceles [1].

Lemma 1 *(Sines Theorem) In the hyperbolic triangle ABC let α,β,γ denote at A,B,C and a,b,c denote the hyperbolic lengths of the sides opposite A,B,C, respectively, then*

$$\frac{\sin\alpha}{\sinh a}=\frac{\sin\beta}{\sinh b}=\frac{\sin\gamma}{\sinh c} \tag{1}$$

[3, p.125].

2 Main results

Theorem 2 *If the internal angle bisectors of two angles of a triangle are equal, then the triangle isn't isosceles.*

Proof. Let BB' and CC' be the respective internal angle bisectors of angles B and C in triangle ABC, and let a,b and c denote the sidelengths in the standard order. As shown in Figure 1, we set

$B=2\beta$, $C=2\gamma$, $u=AB'$, $U=B'C$, $v=AC'$, $V=C'B$.

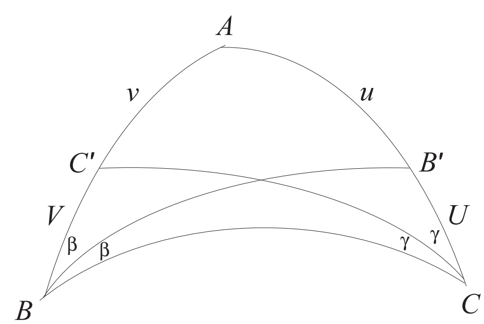


Figure 1

If we use the sines theorem in the triangles $ABC, BB'C, BB'A, BCC', ACC'$
respectively (See Lemma 1), then

$$\frac{\sin A}{\sinh a}=\frac{\sin 2\gamma}{\sinh c}=\frac{\sin 2\beta}{\sinh b} \tag{2}$$

$$\frac{\sinh U}{\sinh BB'} = \frac{\sin\beta}{\sin 2\gamma} \quad (3)$$
$$\frac{\sinh u}{\sinh BB'} = \frac{\sin\beta}{\sin A} \quad (4)$$
$$\frac{\sinh V}{\sinh CC'} = \frac{\sin\gamma}{\sin 2\beta} \quad (5)$$
$$\frac{\sinh v}{\sinh CC'} = \frac{\sin\gamma}{\sin A} \quad (6)$$

If ratios the equations (3) and (4) among themselves, respectively, then

$$\frac{\sinh U}{\sinh u} = \frac{\sin A}{\sin 2\gamma} \quad (7)$$

If ratios the equations (5) and (6) among themselves, respectively, then

$$\frac{\sinh V}{\sinh v} = \frac{\sin A}{\sin 2\beta} \quad (8)$$

Let BB'=CC' and $C > B$ (and $c > b$). Here, there are two cases.

$$\frac{\sinh b}{\sinh u} > \frac{\sinh c}{\sinh v}, \quad \frac{\sinh b}{\sinh u} < \frac{\sinh c}{\sinh v} \quad (9)$$

$$\frac{\sinh b}{\sinh u} : \frac{\sinh c}{\sinh v} = \frac{\sinh b}{\sinh c} \cdot \frac{\sinh v}{\sinh u}$$

If we put the values of $\frac{\sinh b}{\sinh c} = \frac{\sin 2\beta}{\sin 2\gamma}$ in the equation (2) and $\frac{\sinh v}{\sinh u} = \frac{\sin\gamma}{\sin\beta}$ (see (4), (6)) then

$$\begin{aligned}\frac{\sinh b}{\sinh u} : \frac{\sinh c}{\sinh v} &= \frac{\sin 2\beta}{\sin 2\gamma} \cdot \frac{\sin\gamma}{\sin\beta} \\ &= \frac{2\sin\beta\cos\beta}{2\sin\gamma\cos\gamma} \cdot \frac{\sin\gamma}{\sin\beta} \\ &= \frac{\cos\beta}{\cos\gamma} > 1 \quad (10)\end{aligned}$$

Clearly (11) lead to the contradiction $(C > B)$. On the other hand,

$$\begin{aligned}\frac{\sinh b}{\sinh u} - \frac{\sinh c}{\sinh v} &= \frac{\sinh(U+u)}{\sinh u} - \frac{\sinh(V+v)}{\sinh v} \\ &= \frac{\sinh U \cosh u + \cosh U \sinh u}{\sinh u} \\ &\quad - \frac{\sinh V \cosh v + \cosh V \sinh v}{\sinh v} \\ &= \frac{\sinh U}{\sinh u}\cosh u + \cosh U \\ &\quad - \frac{\sinh V}{\sinh v}\cosh v - \cosh V \quad (11)\end{aligned}$$

If we put the values of $\frac{\sinh U}{\sinh u} = \frac{\sin A}{\sin 2\gamma}, \frac{\sinh V}{\sinh v} = \frac{\sin A}{\sin 2\beta}$ in the equations (7) and (8), then

$$\begin{aligned}\frac{\sinh b}{\sinh u} - \frac{\sinh c}{\sinh v} &= \frac{\sin A}{\sin 2\gamma}\cosh u + \cosh U \\ &\quad - \frac{\sin A}{\sin 2\beta}\cosh v - \cosh V\end{aligned}$$

If we put the values of $\frac{\sinh a}{\sinh c} = \frac{\sin A}{\sin 2\gamma}, \frac{\sinh a}{\sinh b} = \frac{\sin A}{\sin 2\beta}$ in the equation (2), then

$$\begin{aligned}\frac{\sinh b}{\sinh u} - \frac{\sinh c}{\sinh v} &= \frac{\sinh a}{\sinh c}\cosh u + \cosh U \\ &\quad - \frac{\sinh a}{\sinh b}\cosh v - \cosh V < 0\end{aligned}$$

$$\frac{\sinh a}{\sinh c}\cosh u + \cosh U < \frac{\sinh a}{\sinh b}\cosh v - \cosh V$$

Because of $C > B$, $V > U$, $v > u$. Hence, $\sinh b < \sinh c$ (and $c > b$). Consequently, the case $C > B$ is satisfied while BB'=CC' . The triangle ABC can't isosceles.

□

References

[1] HAJJA, H., *A short Trigonometric of the Steiner-Lehmus Theorem*, Forum Geometricorum, Vol 8, 39-42, (2008).

[2] PAMBUCCIAN, V., *Axiomatizing geometric constructions*, Journal of Applied Logic, (2007).

[3] STAHL, S., *The Poincare half plane a gateway to modern geometry*, Jones and Barlett Publishers, Boston, 298 p. (1993).

Nilgün Sönmez

e-mail: nceylan@aku.edu.tr

Afyon Kocatepe University

Faculty of Science and Literatures

Department of Mathematics

ANS Campus, 03200 - Afyonkarahisar, Turkey

Original scientific paper
Accepted 18. 12. 2008.

Tatiana Olejníková

Cycloidal Cyclical Surfaces

Cycloidal Cyclical Surfaces

ABSTRACT

The paper presents a family of cycloidal cyclical surfaces, which are created by a movement of a circle alongside the special spatial cycloidal curve, where the circle is located in the curve normal plane and its centre is on this curve. The spatial cycloidal curve can be created by simultaneous revolution of a point about three different axes ${}^3o, {}^2o$ and ${}^1o$ in the space. The form of the cycloidal curve and also of the cycloidal cyclical surface depends on the relative position of the three axes of revolutions, on multiples of angular velocities and orientations of separate revolutions. The analytic representation, the classification of surfaces and some of their geometric properties are derived.

Key words: revolution, translation, angular velocity, cycloidal curve, cyclical surface

MSC 2000: 51A04, 52A05, 14J26

Cikloidne kružne plohe

SAŽETAK

Članak predstavlja porodicu cikloidnih kružnih ploha koje nastaju gibanjem kružnice duž posebne prostorne cikloide, pri čemu kružnica leži u normalnoj ravnini te krivulje, a središte joj je na krivulji. Prostorna cikloida može nastati istodobnom rotacijom točke oko tri različite osi ${}^1o$, ${}^2o$, ${}^3o$, u prostoru. Oblik prostorne cikloide, isto kao i oblik cikloidne kružne plohe, ovisi o međusobnom položaju rotacijskih osi, kratnosti kutnih brzina i orijentacijama pojedinih rotacija. Izvedena je analitička reprezentacija tih ploha, njihova klasifikacija te neka od geometrijskih svojstava.

Ključne riječi: rotacija, translacija, kutna brzina, prostorna cikloida, cikloidna ploha

1 Introduction

A cycloidal cyclical surface can be created by movement of a circle alongside a spatial cycloidal curve. The circle is located in the normal plane of the curve and its centre is on this curve.

The spatial cycloidal curve can be created by simultaneous revolutions of a point about three different lines, axis ${}^3o, {}^2o$ and ${}^1o$. Trajectories of the point P, which revolves about single axes of revolutions are circles ${}^1k$, ${}^2k$, ${}^3k$ located in the planes perpendicular to the axes of revolution ${}^1o$, ${}^2o$, ${}^3o$. With respect to the relative position of axes of revolutions these circles do not necessarily lie in one plane. Form of the spatial cycloidal curve is dependent on the relative position of the axes of revolutions, on the orientations of the single revolutions and on their angular velocities, and also on the position of the revolving point P with respect to the axes of revolutions. Some forms of these curves are studied in [1], [2].

In the next section there is described the creation of one type of the cycloidal cyclical surface for particular relative position of the axes of revolutions (Fig. 1).

Let axis ${}^1o$ be fixed and ${}^1o = z$ in the Cartesian coordinate system (O,x,y,z). Axis ${}^2o$ skew to ${}^1o$, ${}^2o/{}^1o$ creates a 1-sheet hyperboloid of revolutional by its revolutionary movement about axis ${}^1o$ with angular velocity $w_1 = v$ and with orientation determined by parameter q_1 (Fig. 2). Axis ${}^3o$ that is intersect to ${}^2o$, ${}^3o \times {}^2o$, creates a conical surface of revolution by revolution about axis ${}^2o$ with angular velocity $w_2 = m_1 w_1 = m_1 v$ and with orientation determined by parameter q_2 (Fig. 3). Axis ${}^3o$ parallel to ${}^1o$, ${}^3o \parallel {}^1o$, creates a cylindrical surface of revolutional by revolving about axis ${}^1o$ (Fig. 4). In Fig. 5 there are displayed all three surfaces of revolution together. Axis ${}^3o$, which revolves about two axis simultaneously, about axis ${}^2o$ and axis ${}^1o$, creates a two-axial suface of revolution of Euler type - composite ruled, as decribed in [3], (Fig. 6). This surface has six identical branches, because axis ${}^3o$ revolves about axis ${}^2o$ with angular velocity, which is 6-multiple of angular velocity of revolution of the axis ${}^2o$ about axis ${}^1o$.

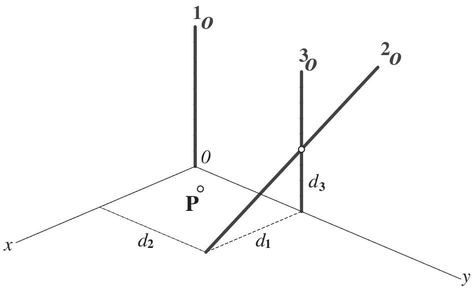


Figure 1

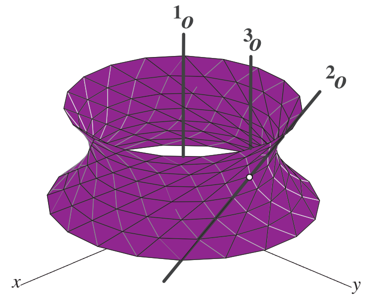


Figure 2

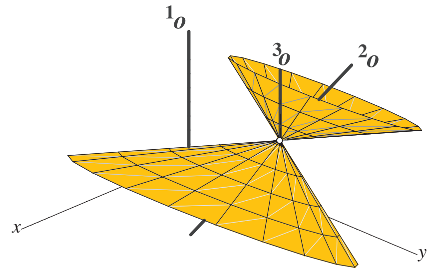


Figure 3

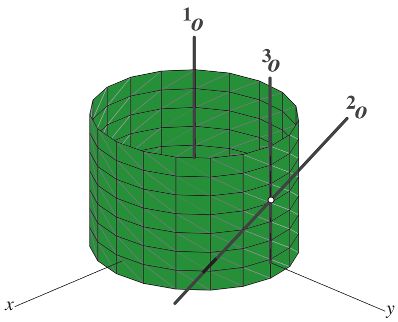


Figure 4

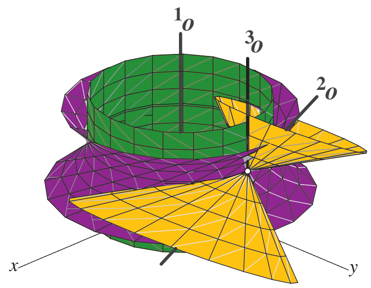


Figure 5

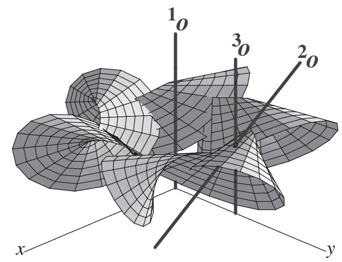


Figure 6

The point P revolves about axis ^3o with angular velocity $w_3 = m_2 w_2 = m_2 m_1 v$ with orientation determined by parameter q_3, where perameters $q_1, q_2, q_3 = \pm 1$ (if $q_i = +1$, $i = 1,2,3$ then revolution is right-handed, if $q_i = -1$ then revolution is left-handed). Trajectory of the point P movement created by its revolution about axis ^1o is circle ^1k (Fig. 7), the circle ^2k is the trajectory of the point P movement about axis ^2o (Fig. 8) and the circle ^3k is the trajectory of the point P movement about axis ^3o (Fig. 9).

The curve k as trajectory of the point P composite revolutionary movement is created by rolling of the circle ^3k on the circle ^2k, which rolls on the circle ^1k simultaneously (Fig. 10). Form of this spatial cycloidal curve is dependent on the relative position of the axes of revolutions ^1o, ^2o, ^3o, on the orientations of the single revolutions and on their angular velocities, and also on the position of the revolving point P with respect to three axes of revolutions.

The cycloidal cyclical surface can be created by moving a circle alongside the curve k, while the circle lies always in the normal plane of the curve k and its centre is on the curve (Fig. 11, Fig. 12-view from above).

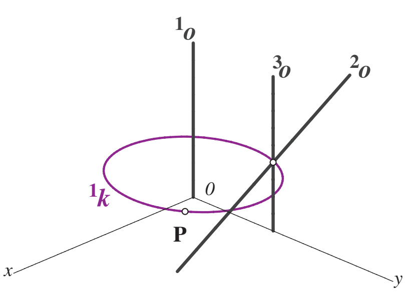


Figure 7

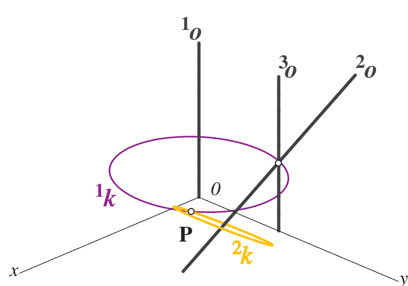


Figure 8

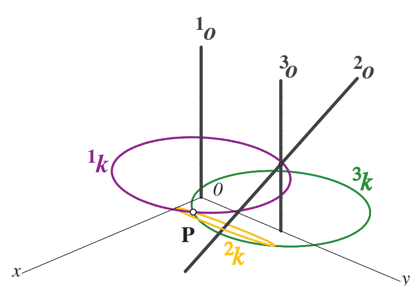


Figure 9

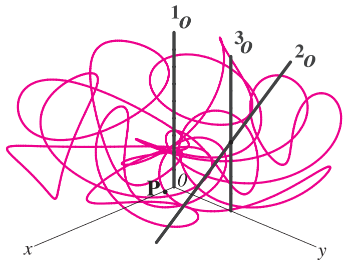


Figure 10

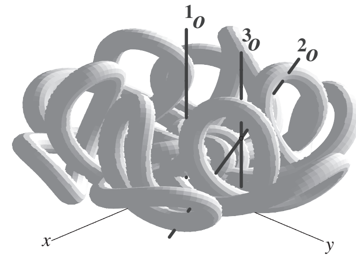


Figure 11

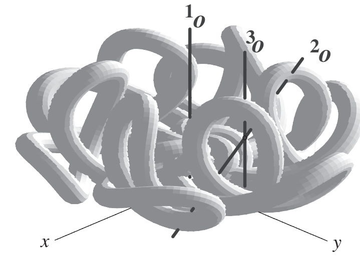


Figure 12

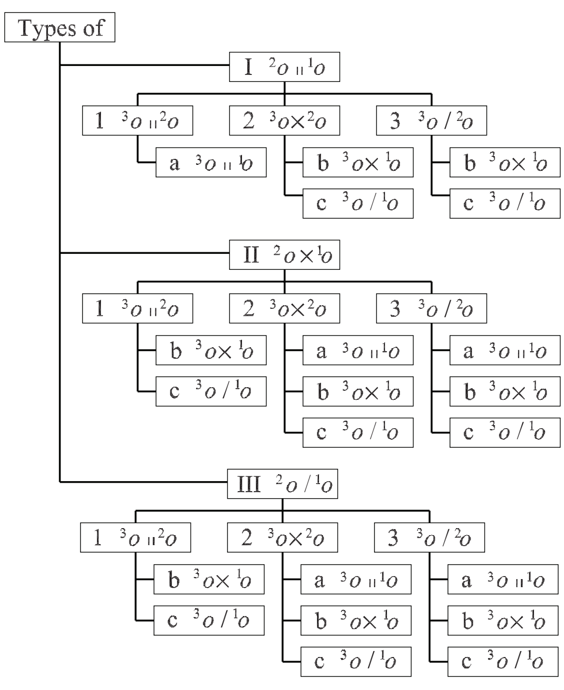
2 Classification of Family of Cycloidal Cyclical Surfaces

The classification of the family of cycloidal cyclical surfaces can be done according to the relative position of axes of revolutions ^3o, ^2o and ^1o, which may be parallel, intersect or skew. The distribution of surfaces within the family is illustrated in the next Graph 1.

Cycloidal cyclical surfaces are distributed in the first level into the three types I, II, III with respect to the relative position of the axes ^2o and ^1o.

Surfaces in all three subclasses I, II, III are distributed in the second level into the three types 1, 2, 3 with respect to the relative position of the axes ^3o and ^2o.

Finally, in the third level, each subgroup of types 1, 2, 3 can be further classified with respect to the relative position of the axes ^3o and ^1o into types a, b or c.



Graph 1

3 Analytical Representation of Cycloidal Cyclical Surface

Let us derive the vector function of the cycloidal cyclical surface for one particular position of the axes of revolutions and for one special position of the point P with respect of these axes, particularly for the surface of type III 2 a. Derivation of the vector function of all other types of surfaces is analogous.

Let the axes of revolution be in the following relative positions: $^1o = z, {}^2o/{}^1o$ (skew), $^3o \times {}^2o$ (intersect), $^3o \parallel {}^1o$ (parallel). The position of axis ^2o in the plane parallel to the coordinate plane (xz), $^2o \subset v'$, $v' \parallel v$, is determined by parameters d_1, d_2, d_3, which determine the position of the intersection points of axis ^2o with the coordinate planes (xy) and (yz) in the Cartesian coordinate system (O,x,y,z). Then $\alpha = \arctan\dfrac{d_3}{d_1}$ is the angle formed by axis ^2o with the coordinate plane (xy) and the position of axis ^3o is determined by parameter d_2, which is the distance between axes ^3o and ^1o, (Fig. 1).

The revolution about axis ^1o with angular velocity $w_1 = v$, in the direction determined by parameter $q_1 = \pm 1$, is represented by matrix

$$\mathbf{T}_1\left(w_1(v), q_1\right) = \mathbf{T}_z\left(w_1, q_1\right), \tag{1}$$

where the matrix $\mathbf{T}_z(w_1,q_1)$ represents revolution about axis z by angle w_1 in the direction determined by parameter q_1 and for $i = 1$ it can be derived from (2)

$$\mathbf{T}_z\left(w_i, q_i\right) = \begin{pmatrix} \cos w_i & q_i \sin w_i & 0 & 0 \\ -q_i \sin w_i & \cos w_i & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{2}$$

The revolution about axis ^2o with angular velocity $w_2 = m_1 w_1$, in the direction determined by parameter $q_2 = \pm 1$, is represented by matrix

$$\begin{aligned} \mathbf{T}_2\left(w_2(v), q_2\right) \quad = \quad & \mathbf{T}\left(-d_1, -d_2, 0\right) \cdot \mathbf{T}_y\left(\alpha, +1\right) \cdot \mathbf{T}_x\left(w_2, q_2\right) \cdot \\ & \cdot \mathbf{T}_y\left(\alpha, -1\right) \cdot \mathbf{T}\left(d_1, d_2, 0\right), \end{aligned} \tag{3}$$

where the matrix $\mathbf{T}_y(\alpha,\pm 1)$ expressed in (4) represents the revolution about axis y by angle α in positive or negative direction, matrix $\mathbf{T}_x(w_2,q_2)$ represents revolution about axis x by angle $w_2 = m_1 v$ in the direction determined by parameter q_2 (5), matrix $\mathbf{T}(\pm d_1,\pm d_2,0)$ represents translation with translation vector $(\pm d_1,\pm d_2,0)$ in (6).

$$\mathbf{T}_y(\alpha,\pm 1) = \begin{pmatrix} \cos\alpha & 0 & \pm\sin\alpha & 0 \\ 0 & 1 & 0 & 0 \\ \mp\sin\alpha & 0 & \cos\alpha & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \qquad (4)$$

$$\mathbf{T}_x(w_2,q_2) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos w_2 & q_2\sin w_2 & 0 \\ 0 & -q_2\sin w_2 & \cos w_2 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \qquad (5)$$

$$\mathbf{T}(\pm d_i,\pm d_j,\pm d_k) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \pm d_i & \pm d_j & \pm d_k & 1 \end{pmatrix}. \qquad (6)$$

The revolutionary movement of the point $P = (x_0,y_0,z_0,1)$ about axis ${}^3o$ with angular velocity $w_3 = m_2 w_2 = m_2 m_1 v$ and in the direction determined by parameter $q_3 = \pm 1$ is represented by matrix

$$\mathbf{T}_3(w_3(v),q_3) = \mathbf{T}(0,-d_2,0)\cdot\mathbf{T}_z(w_3,q_3)\cdot\mathbf{T}(0,d_2,0), \quad (7)$$

where matrix $\mathbf{T}(0,\pm d_2,0)$ in (6) represents translation with translation vector $(0,\pm d_2,0)$, and matrix $\mathbf{T}_z(w_3,q_3)$ is for $i=3$ represented by (2).

A vector function of the cycloidal curve k created by simultaneous revolution of the point $P = (x_0,y_0,z_0,1)$ about axes ${}^3o$, ${}^2o$ and ${}^1o$ is

$$\mathbf{r}(v) = \mathbf{R}\cdot\mathbf{T}_3(w_3(v),q_3)\cdot\mathbf{T}_2(w_2(v),q_2)\cdot\mathbf{T}_1(w_1(v),q_1), \quad v\in\langle 0,2\pi\rangle, \quad (8)$$

where $\mathbf{T}_3(w_3(v),q_3)$, $\mathbf{T}_2(w_2(v),q_2)$, $\mathbf{T}_1(w_1(v),q_1)$ are matrices of particular revolutions expressed in (6), (3), (1) and $\mathbf{R}(x_0,y_0,z_0,1)$ is the positioning vector of the point P.

Let the new coordinate system be defined at the arbitrary regular point $P\in k$, identical to the trihedron (P,t,n,b) determined by tangent t, basic normal n and binormal b to the curve k with unit vectors expressed in (9).

$$\mathbf{t}(v) = (t_1,t_2,t_3) = \frac{\mathbf{r}'(v)}{|\mathbf{r}'(v)|},$$
$$\mathbf{n}(v) = (n_1,n_2,n_3) = \frac{\mathbf{r}''(v)}{|\mathbf{r}''(v)|},$$
$$\mathbf{b}(v) = (b_1,b_2,b_3) = \mathbf{t}(v)\times\mathbf{n}(v). \qquad (9)$$

The cycloidal cyclical surface can be created by movement of the circle $c' = (P,r)$ with centre P and radius r alongside the curve k so that the circle is located in the normal plane of the curve in the point $P\in k$, which is determined by basic normal n and binormal b to this curve. Vector function of this surface is

$$\mathbf{P}(u,v) = \mathbf{r}(v) + \mathbf{c}(u)\cdot\mathbf{M}(v), \quad u\in\langle 0,2\pi\rangle, v\in\langle 0,2\pi\rangle, \ (10)$$

where $\mathbf{r}(v)$ is vector function of the cycloidal curve k expressed in (8), $\mathbf{c}(u) = (0, r\cos u, r\sin u)$ is vector function of the circle c with centre in the origin of the coordinate system (O,x,y,z) and radius r, located in the coordinate plane (yz). Matrix

$$\mathbf{M}(v) = \begin{pmatrix} t_1 & t_2 & t_3 & 0 \\ n_1 & n_2 & n_3 & 0 \\ b_1 & b_2 & b_3 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad (11)$$

transforms the circle c on the circle c' with the centre in the origin of the coordinate system (P,t,n,b) and radius r located in the normal plane (nb) of the cycloidal curve k in the point P. Entries of the first row of this matrix are coordinates of the unit vector of the tangent t, entries of the second row are coordinates of the unit vector of the basic normal n and the entries of the third row are coordinates of the unit vector of the binormal b expressed in equations (9), (Fig. 13).

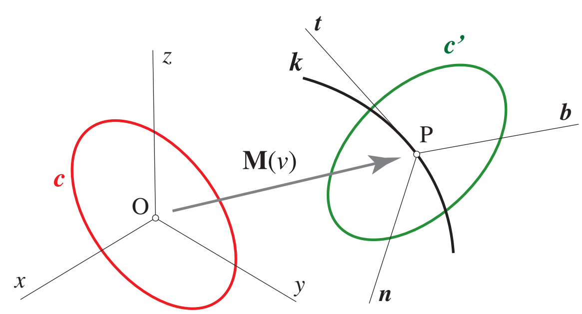


Figure 13

The form of the cycloidal curve k and created cycloidal cyclical surface changes in dependence on the relative position of the axes of revolutions that are determined by parameters $d_i, i = 1,2,3$. Surface has m_1 identical external branches, where every branch has m_2 identical internal branches. Point P revolves about axis ${}^3o$ with angular velocity w_3, which is m_2-multiple of the angular velocity w_2 of the revolution about axis ${}^2o$ and w_2 is m_1-multiple of the angular velocity w_1 of the revolution about axis ${}^1o$. Many different forms of cycloidal cyclical surfaces can be created by change of their determining parameters.

Variations of the surface form are shown by change of some parameters of the surface of type III 2 a displayed in Fig. 11. Presented surface is determined by parameters $m_1 = 6, m_2 = 3, q_1 = q_2 = q_3 = +1$, then it has 6 external and 3 internal branches, and all three revolutions are right-handed. Surface in Fig. 14 is determined by parameter $m_2 = 6$, in Fig. 15 by $m_1 = 4, m_2 = 2$, and in Fig. 16 by $m_1 = 3, m_2 = 4$, and there are changes in the number of external and internal branches.

In Fig. 17 depicted surface is determined by parameters $m_1 = 3, m_2 = 3, q_2 = -1, q_3 = +1$, in Fig. 18 by $q_2 = +1$ and $q_3 = -1$ and in Fig. 19 by $q_2 = -1$ and $q_3 = -1$, then there are changes in the orientations of particular revolutions.

In Fig. 20, there is presented surface with parameters identical to parameters of surface in Fig. 19, but the position of the point $P(x_0, y_0, z_0, 1)$ was changed from $(\frac{d_1}{2}, \frac{d_2}{2}, \frac{d_3}{3}, 1)$ to $(\frac{d_1}{2}, 3\frac{d_2}{2}, \frac{d_3}{3}, 1)$ and in Fig. 21 to $(d_1, d_2, d_3, 1)$. Surface with parameters $m_1 = 6, m_2 = 4, q_2 = +1, q_3 = +1$ is illustrated in Fig. 22, but relative position of the axes has been changed to position ${}^2o/{}^1o, {}^2o \perp {}^1o$ and ${}^2o \perp {}^3o$. Surfaces in Figures 14 - 21 are displayed by view from above, because in these views the changes of parameters are more illustrative.

As the conclusion it can be summarised that the presented family of cycloidal cyclical surfaces serves as an endlessly rich source of inspiration for artistic and design purposes. Their unusually complex forms obtained in a relatively simple way of composite spatial transformation. Special skew symmetry and harmonical periodicity reflect their simplistic generating priciple based on the naturally basic movement of our universe, revolution about an axis in the space. Several surface types from the presented classification frame are displayed in the Fig. 23 without commentary, as the most persuasive evidence.

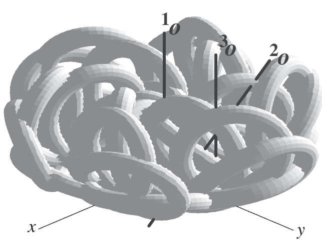


Figure 14

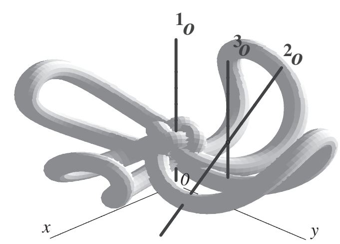


Figure 15

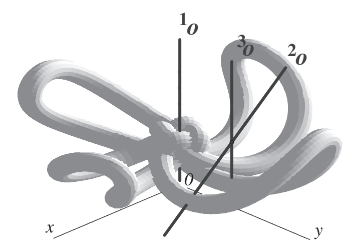


Figure 16

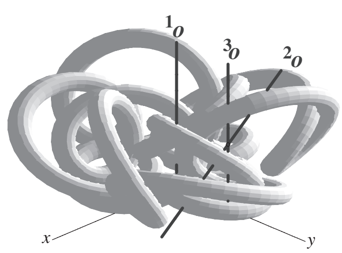


Figure 17

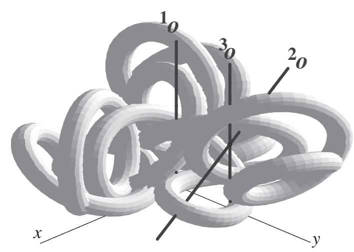


Figure 18

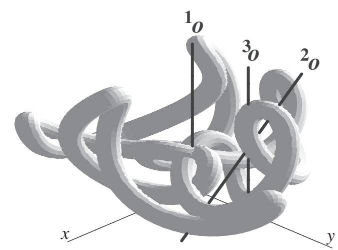


Figure 19

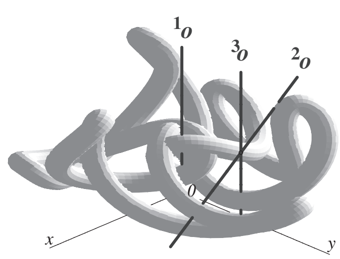


Figure 20

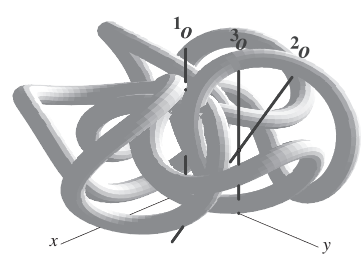


Figure 21

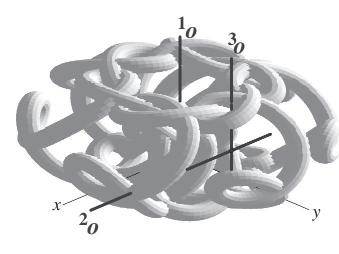


Figure 22

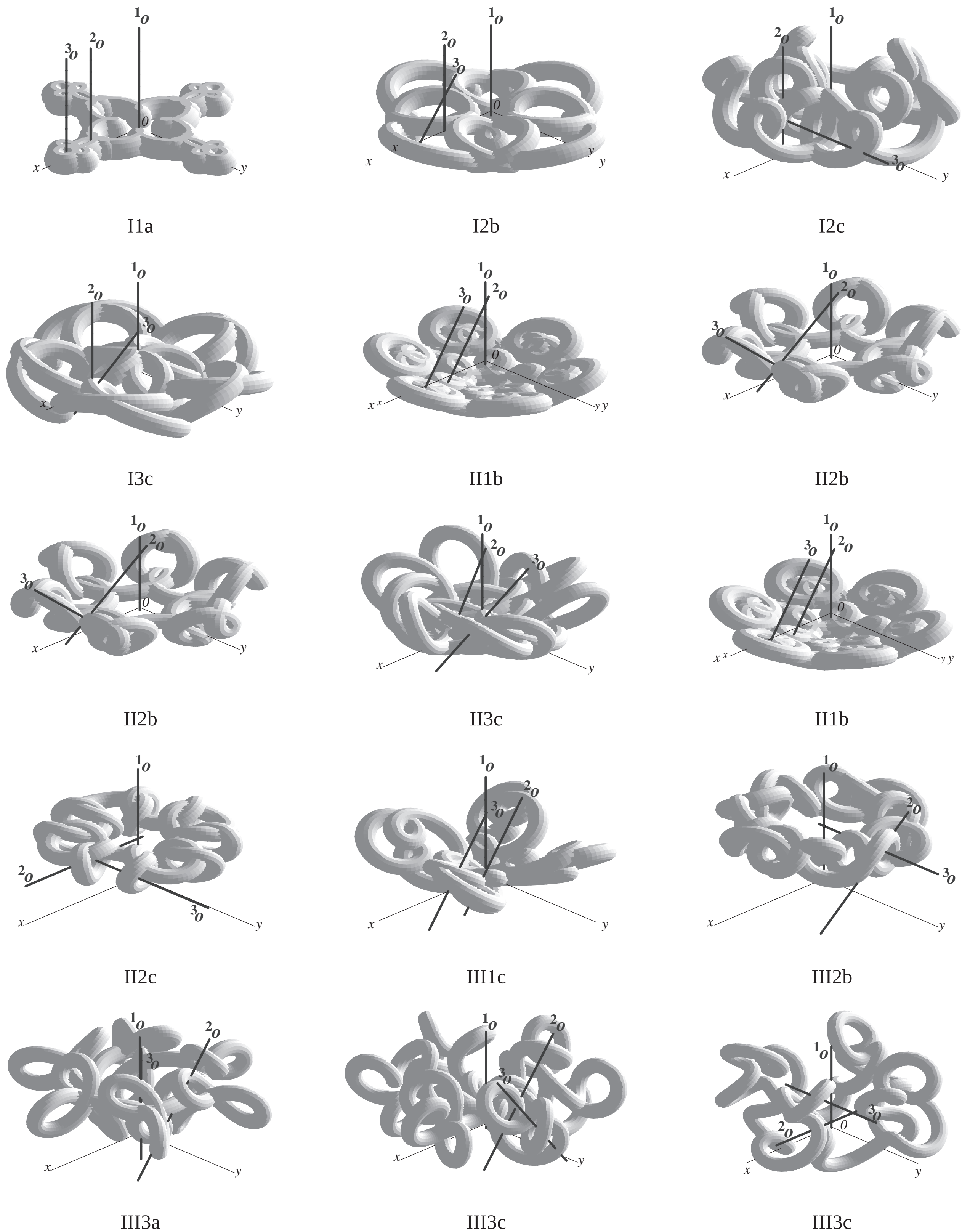

I1a
I2b
I2c
I3c
II1b
II2b
II2b
II3c
II1b
II2c
III1c
III2b
III3a
III3c
III3c

Figure 23

References

[1] D. VELICHOVÁ, Multy-Axial Revolutions of Euler Type. *Proceedings of 5th International Conference on Applied Mathematics APLIMAT 2006*, SjF STU Bratislava, 2006, 191–197

[2] D. VELICHOVÁ, Trajectories of Composite Revolutionary Movements. *G, Slovak Journal for Geometry and Graphics* **3** (2006), No.5, Slovak Society for Geometry and Graphics, SjF STU Bratislava, 47–64.

[3] D. VELICHOVÁ, Classification of Two-Axial Surfaces of Revolution. *G, Slovak Journal for Geometry and Graphics* **4** (2007), No.7, Slovak Society for Geometry and Graphics, SjF STU Bratislava, 63–82

[4] T. OLEJNÍKOVÁ, 3D Cycloidal Curves and cyclical Surfaces. *Proceedings of International Conference "70 years of SvF STU"*, Section 05, ISBN 978-80-227-2979-6, Bratislava, 2008

[5] B. BUDINSKÝ, B. KEPR, *Basic of Differetial Geometry with Technical Applications*, SNTL - Publishers of Technical Literature, Praha, 1970

[6] L. GRANÁT, H. SECHOVSKÝ, *Computer Graphics*, SNTL - Publishers of Technical Literature, Praha, 1980

[7] V. MEDEK, J. ZÁMOŽÍK, *Constructive geometry for technics*, ALFA - Publishers of Technical and Economical Literature, Bratislava, 1974

Acknowledgements
The paper was supported by the VEGA 1 / 4002 / 07.

Tatiana Olejníková,
Faculty of Civil Engineering
Tehnical University in Košice
Vysokoškolská 4, 042 00, Košice, Slovakia
e-mail: tatiana.olejnikova@tuke.sk

Original scientific paper
Accepted 30.12.2008.

Ayşe Bayar

On the Trigonometric Functions in Maximum Metric

On the Trigonometric Functions in Maximum Metric

ABSTRACT

The trigonometric functions are defined on the unit circle and the shape of the unit circle changes according to the metric. So, the values of the trigonometric functions depend on the specific metric. This paper presents definitions, rules, and identities of trigonometric functions with respect to maximum metric. Also, geometrical interpretations by using the identities of these functions are given for further work.

Key words: trigonometric functions, non- Euclidean metric, maximum metric

MSC 2000: 51K05, 51K99

O trigonometrijskim funkcijama u maksimalnoj metrici

SAŽETAK

Trigonometrijske funkcije su definirane na jediničnoj kružnici, a njezin se oblik mijenja s obzirom na metriku. Dakle, vrijednosti trigonometrijskih funkcija ovise zasebno o svakoj metrici. Ovaj članak prikazuje definicije pravila i identitete trigonometrijskih funkcija s obzirom na maksimalnu metriku. Također, za daljnje istraživanje dane su geometrijske interpretacije koje koriste identitete ovih funkcija.

Ključne riječi: trigonometrijske funkcije, neeuklidska metrika, maksimalna metrika

1 Introduction

In the plane geometry, trigonometric functions are defined as $x = \cos\theta$, $y = \sin\theta$ for all points (x,y) on the unit circle, where θ is the angle with initial side the positive $x-$axis and terminal side the radial line passing through point (x,y). The unit circle is defined as the set of all points whose distance from the origin is one and this is a different point set, a different shape in different metrics. So, the values of trigonometric functions change according to the metric which we use. The trigonometric functions on the unit circle of taxicab and Chinese Checker metrics have been defined and developed in [1,...,6]. In the present paper, the trigonometric functions are defined with angle θ in the maximum metric which is a non Euclidean metric defined in $\mathbb{R}^2$ for $X = (x_1,y_1)$, $Y = (x_2,y_2)$ as

$$\begin{aligned} d_m(X,Y) &= d_m((x_1,y_1),(x_2,y_2)) \\ &= \max\{|x_1-x_2|,|y_1-y_2|\} \end{aligned} \quad (1)$$

in section 2 and several trigonometric identities of these functions are given in section 3. Also, the definitions of trigonometric functions are developed by using the reference angle α and the change of maximum length of a line segment after rotations are given in section 4.

2 $m-$trigonometric functions

We wish to maintain the standard definitions of the trigonometric functions on $m-$unit circle in the same way one determines their Euclidean analogues. $m-$ unit circle in $\mathbb{R}^2$ is the set of points (x,y) which satisfies the equation $\max\{|x|,|y|\} = 1$.The graph of unit circle is in Figure 1.

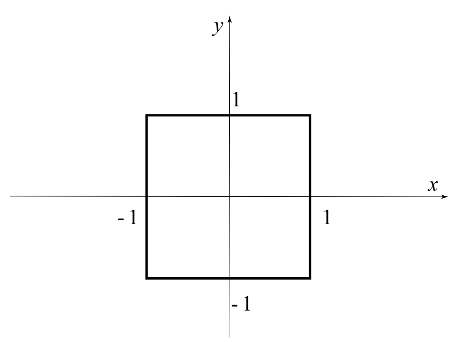


Figure 1: m – unit circle

Since the slope of radial line passing through point (x,y) does not change, tangent function does not depend on the metric. So, we can define the trigonometric functions according to the maximum metric in terms of the standard Euclidean tangent function. Consequently, the slope of the radial line goes through the point $(\cos_m\theta,\sin_m\theta)$ on $m-$ unit circle is

$$\tan\theta=\frac{\sin_m\theta}{\cos_m\theta}=\tan_m\theta. \qquad (2)$$

So, for the definitions of sine and cosine it is necessary to find only the point that will imply $(\cos_m\theta,\sin_m\theta)$ is on the line that makes an angle θ with the positive $x-$axis and on m- unit circle. The equation of the line joining (x,y) and $(0,0)$ is $y=(\tan\theta)x$. Solving the system

$$\begin{cases} y=(\tan\theta)x \\ \max\{|x|,|y|\}=1 \end{cases}$$

we have the following chain of results

$$\begin{aligned}&\max\{|x|,|y|\}=1\\ &|x|\max\{1,|\tan\theta|\}=1\\ &|x|=\tfrac{1}{\max\{1,|\tan\theta|\}}.\end{aligned}$$

If it is made appropriate choice of sign for the absolute value based on the quadrant, we have m-cosine function,

$$\cos_m\theta=\begin{cases} 1, & -\frac{\pi}{4}\leqslant\theta<\frac{\pi}{4}\\ \cot\theta, & \frac{\pi}{4}\leqslant\theta<\frac{3\pi}{4}\\ -1 & \frac{3\pi}{4}\leqslant\theta<\frac{5\pi}{4}\\ -\cot\theta & \frac{5\pi}{4}\leqslant\theta<\frac{7\pi}{4}.\end{cases} \qquad (3)$$

Also, one can obtain easily $m-$sine function by using the identity (3) as

$$\sin_m\theta=\begin{cases} \tan\theta, & -\frac{\pi}{4}\leqslant\theta<\frac{\pi}{4}\\ 1, & \frac{\pi}{4}\leqslant\theta<\frac{3\pi}{4}\\ -\tan\theta, & \frac{3\pi}{4}\leqslant\theta<\frac{5\pi}{4}\\ -1 & \frac{5\pi}{4}\leqslant\theta<\frac{7\pi}{4}.\end{cases} \qquad (4)$$

The graphs of $m-$cosine and $m-$sine are presented in Figure 2 and Figure3.

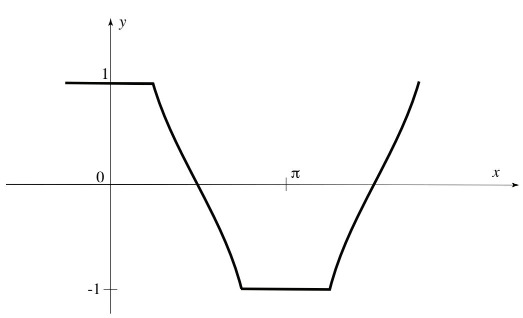


Figure 2: The graph of $y=\cos_m x$

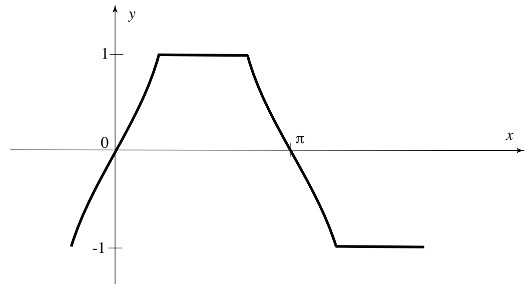


Figure 3: The graph of $y=\sin_m x$

3 Identities of the maximum trigonometric functions

The trigonometric identities for m-trigonometric functions will differ from their Euclidean counterparts in most cases. If we define the secant, cosecant and cotangent functions in maximum metric as we do in Euclidean metric, then $\csc_m\theta=\frac{1}{\sin_m\theta}$, $\sec_m\theta=\frac{1}{\cos_m\theta}$. The cotangent function does not depend on the metric as the tangent function does.

The cofunction identities of these functions are like their Euclidean counterparts, i.e. $\cos_m(\frac{\pi}{2}-\theta)=\sin_m\theta$ and $\sin_m(\frac{\pi}{2}-\theta)=\cos_m\theta$.

Using the identities $\tan(-x)=-\tan x$ and $\cot(-x)=-\cot x$ in (3) and (4),one can verify the identities $\cos_m(-\theta)=\cos_m\theta$ and $\sin_m(-\theta)=-\sin_m\theta$.

The Pythagorean identities are the fundamental identities in Euclidean trigonometry. In maximum metric case, the Pythagorean identities are different from $\cos^2\theta+\sin^2\theta=1$ in Euclidean metric. Clearly, The Pythagorean identity follows simply from equation of the $m-$ unit circle $\max\{|x|,|y|\}=1$, giving us

$$\max\{|\cos_m x|,|\sin_m x|\}=1,$$

and applying the tangent identity (2), we have

$$\max\{1,|\tan_m x|\}=|\sec_m x|.$$

Since the tangent function does not change in the maximum metric, the Euclidean identity

$$\tan(u\pm v)=\frac{\tan u\pm\tan v}{1\mp\tan u\tan v}$$

holds in the maximum metric. Applying (2) to both sides, this equation can be rewritten and simplified as

$$\frac{\sin_m(u\pm v)}{\cos_m(u\pm v)}=\frac{\frac{\sin_m u}{\cos_m u}\pm\frac{\sin_m v}{\cos_m v}}{1\mp\frac{\sin_m u\sin_m v}{\cos_m u\cos_m v}}=\frac{\sin_m u\cos_m u\pm\sin_m v\cos_m v}{\cos_m u\cos_m v\mp\sin_m u\sin_m v}$$

and since the fractions on both sides are in the lowest terms, we can equate numerator and denominator. So, the sum and difference formulas in the maximum metric is obtained:

$$\begin{aligned} \sin_m(u \pm v) &= \sin_m u \cos_m u \pm \sin_m v \cos_m v \\ \cos_m(u \pm v) &= \cos_m u \cos_m v \mp \sin_m u \sin_m v. \end{aligned} \tag{5}$$

Also, the double angles formulas are obtained by applying $u = v$ in (5).

4 m- trigonometric functions with reference angle

In Euclidean metric, an angle size and an arc length are equivalent on the unit circle. But, there is a non-uniform change in the arc length increasing the angle θ by a fixed amount in maximum metric. So, it is necessary to develop the trigonometric functions defined on the maximum unit circle by the reference angle for θ not in standard position [4]. If θ is not in standard position, it can be defined two angles in standard position. So, we can define the general trigonometric functions for the angle θ with the reference angle α on $m-$ unit circle. When θ is the angle with the reference angle α which is the angle between θ and the positive direction of the x- axis on $m-$ unit circle, we can define the general cosine and sine functions θ as follows:

$$\begin{aligned} \mathrm{mcos}\theta &= \cos_m(\alpha+\theta).\cos_m\alpha - \sin_m(\alpha+\theta).\sin_m\alpha \\ \mathrm{msin}\theta &= \sin_m(\alpha+\theta).\cos_m\alpha + \cos_m(\alpha+\theta).\sin_m\alpha\,. \end{aligned} \tag{6}$$

In these definitions, since $\alpha+\theta$ and α are in standard position, the value of $\mathrm{mcos}(\alpha+\theta)$,$\mathrm{mcos}\alpha$,$\mathrm{msin}(\alpha+\theta)$ and $\mathrm{msin}\alpha$ are calculated by using (3) and (4). If $\alpha = 0$, then $\mathrm{mcos}\theta = \cos_m\theta$ and $\mathrm{msin}\,\theta = \sin_m\theta$ since θ is in standard position. The general definitions for other trigonometric functions for the angles which are not in standard position can be given similarly.

Consequently, the general definitions of trigonometric functions can be given by defining angles with the reference angle in maximum metric.

It is well known that all rotations and translations preserve the Euclidean distance. But , if a line segment is rotated , the length of it changes in non-Euclidean metrics. Thus, the change of a line segment length after rotation is given by the following theorem:

Theorem 1 *Let OA be a line segment, not on the x-axis with reference angle α and $d_m(O,A) = k$. If OA' is the image of OA under the rotation through the angle θ, then*

$$\begin{aligned} d_m(O,A') &= k\sqrt{\frac{\cos_m^2\alpha+\sin_m^2\alpha}{\cos_m^2(\alpha+\theta)+\sin_m^2(\alpha+\theta)}} = \\ &= k.\frac{\max\{|\cos(\alpha+\theta)|,|\sin(\alpha+\theta)|\}}{\max\{|\cos\alpha)|,|\sin\alpha|\}}. \end{aligned}$$

PROOF: Since the lengths under the translations are preserved in maximum metric, it is enough to consider a line segment passing through the origin. Let $d_m(O,A)$ be k. By the rotation of OA through an angle θ, we get the line segment OA'. If α is the reference angle of θ, then $A = (k.\cos_m\alpha, k.\sin_m\alpha)$. Let $d_m(O,A')$ be k'. Then $A' = (k'.\cos_m(\alpha+\theta),\ k'.\sin_m(\alpha+\theta))$. Because of the equality of Euclidean lengths of the line segments OA and OA' we get

$$d_m(O,A) = d_m(O,A')$$

and therefore

$$(k.\cos_m\alpha)^2 + (k.\sin_m\alpha)^2 = (k'.\cos_m(\alpha+\theta))^2 + (\sin_m(\alpha+\theta))^2$$

$$k' = k\sqrt{\frac{\cos_m^2\alpha+\sin_m^2\alpha}{\cos_m^2(\alpha+\theta)+\sin_m^2(\alpha+\theta)}}.$$

and by using the identity

$$\cos_m^2\alpha+\sin_m^2\alpha = 1+\tan^2\alpha = \sec^2\alpha,$$

$$k' = k.\frac{\max\{|\cos(\alpha+\theta)|,|\sin(\alpha+\theta)|\}}{\max\{|\cos\alpha)|,|\sin\alpha|\}}$$

is obtained in terms of the Euclidean sine and cosine functions.

The following corollary shows how one can find the maximum length, after a rotation of a line segment through an angle θ in standard form.

Corollary: Let OA be a line segment on the x-axis. If OA' is the image of OA under the rotation through an angle θ then

$$d_m(O,A') = \frac{k}{\sqrt{\cos^2\theta+\sin^2\theta}} = k\max\{|\cos\theta|,|\sin\theta|\}.$$

PROOF: Using the value $\alpha = 0$ in the theorem, the corollary is obtained.

Consequently, this paper on the trigonometric functions in maximum metric provides good facilities for further works on the subjects as norm, inner product, cosine theorem and area of triangles in maximum metric.

References

[1] AKÇA, Z. AND KAYA, R., On Taxicab Trigonometry, Jour. of Inst. of Math & Comp. Sci. (Math. Ser.) 10, No 3, 151-159 (1997).

[2] BAYAR A AND EKMEKÇI S, On the Chinese-Checker Sine And Cosine Functions, International Journal of Mathematics and Analysis1, No.3, 249-259 (2006).

[3] R. BRISBIN AND P. ARTOLA, Taxicab Trigonometry, Pi Mu Epsilon J., 8, 89-95 (1984).

[4] K. THOMPSON AND T. DRAY, Taxicab Angles and Trigonometry, Pi Mu Epsilon J., 11, 87-97 (2000).

[5] ÖZCAN, M., EKMEKÇI S., BAYAR, A., A Note on the Variation of the Taxicab Lengths Under Rotations, Pi Mu Epsilon J.,11, No 7, 381-384 (2002).

[6] ÖZCAN, M.AND YÜKSEL, S., On the CC-trigonometry, International Journal of Applied Mathematics, 20, No 7, 959-974 (2007).

Ayşe Bayar
akorkmaz@ogu.edu.tr

Eskişehir Osmangazi Univ.
Department of Mathematics
26480, Eskişehir–TURKEY

Professionelle Arbeit
Angenommen am 10.12.2008.

MARCO HAMANN

Zum axonometrischen Umriss einer Kugel

On Axonometric Projection of the Contour of a Sphere

ABSTRACT

The present article refers to a question that arose during the lecture of descriptive geometry for undergraduate students in the fields of geodesy and cartography: If the axonometric projection of the contour of a sphere is constructed with the help of the cutting method (L. ECKHART, 1937), the pairs of contour points with regard to the cutting directions of two non-associated auxiliary projections determine two in general non conjugate diameters and their conjugate directions of the ellipse of contour. Conversely an ellipse is in general overdetermined by two given diameters and their conjugate directions. The theorem of PASCAL can be deduced from the figure of construction whereby the construction of the ellipse of contour is considered from a planar point of view.

Key words: Axonometric projection of a sphere, ellipse of contour, theorem of PASCAL, conic

MSC 2000: 51N05, 51A05

O konturi kugle u aksonometriji

SAŽETAK

Ovaj se članak bavi pitanjem koje se javlja u nastavi nacrtne geometrije za studente preddiplomskog studija u području geodezije i kartografije: Ako se aksonometrijska slika konture kugle konstruira pomoću metode presjeka (L. ECKHART, 1937), tada parovi konturnih točaka, promatranih s obzirom na smjerove presjeka dviju nepridruženih pomoćnih projekcija, određuju općenito dva nekonjugirana promjera konturne elipse i njima konjugirane smjerove. Obratno, elipsa je općenito određena s dva zadana promjera i njima konjugiranim smjerovima. Promatra li se konstrukcija konturne elipse s ravninskog aspekta, iz ovog se načina konstrukcije može izvesti PASCALOV teorem.

Ključne riječi: aksonometrija kugle, konturna elipsa, PASCALOV teorem, konika

1 Einleitung

Die Konstruktion des axonometrischen Umriss einer Kugel gehört zu den klassischen Aufgaben in Darstellender Geometrie in der Ausbildung von Studentinnen und Studenten der Architektur sowie in einigen ingenieurwissenschaftlichen Studiengängen. Sie wird üblicherweise mit Hilfe des nach L. ECKHART (1937) benannten Einschneideverfahrens durchgeführt: Der (scheinbare) Umriss k^α einer Kugel Φ ist dabei axonometrisches Bild jenes Großkreises $k \subset \Phi$, dessen Trägerebene orthogonal zur Projektionsrichtung s liegt. Er ist ein Kreis respektive eine Ellipse, je nachdem, ob s orthogonal zur Bildebene π steht oder nicht. Wird o. B. d. A. der Mittelpunkt M von Φ als Ursprung eines kartesischen Koordinatensystems $(O;E_x,E_y,E_z)$ gewählt, so schneidet k die in den Koordinatenebenen liegenden Großkreise $k_i := \pi_i \cap \Phi$ $(i=1,2,3)$ in den Gegenpunkten K_i, $\bar{K}_i$. Sind beispielsweise Grund- und Aufrissfigur von Φ als Einschneidehilfsrisse gegeben, so lassen sich die Paare $(K_1^\alpha,\bar{K}_1^\alpha)$ bzw. $(K_2^\alpha,\bar{K}_2^\alpha)$ von Konturpunkten unter Verwendung perspektiver Affinitäten zwischen dem Einschneidegrundriss bzw. -aufriss und dem axonometrischen Grundriss bzw. - Aufriss konstruieren. Die Tangenten an k^α in K_1^α, $\bar{K}_1^\alpha$ respektive K_2^α, $\bar{K}_2^\alpha$ sind durch die Einschneiderichtungen s' bzw. s'' bestimmt, vgl. etwa [1].

Von k^α sind hierdurch zwei Durchmesser $[K_1^\alpha,\bar{K}_1^\alpha]$ und $[K_2^\alpha,\bar{K}_2^\alpha]$ mit den Paaren s'- bzw. s''-paralleler Tangenten in den Durchmesserendpunkten und damit ein Tangentenparallelogramm gegeben. Die beiden Durchmesser von k^α sind dabei im Allgemeinen nicht konjugiert. Die Umrissellipse k^α wird in [1] mithilfe einer geeignet gewählten perspektiven Affinität ϕ aus einem Kreis $k^\kappa = \phi^{-1}(k^\alpha)$ konstruiert, deren Achse a mit einer Seite des Tangentenparallelogramms zusammenfällt (Abb. 1), vgl. [1]: Es sei o. B. d. A. $a = t_1^\alpha$ mit $t_1^\alpha := K_1^\alpha \parallel s'$ gewählt. Damit besitzen k^κ und k^α das gemeinsame Linienelement $(K_1^\alpha,t_1^\alpha) = (K_1^\kappa,t_1^\kappa)$. Zur Bestimmung des Mittelpunktes O^κ von k^κ und damit eines Angabepaares (O^κ,O^α) von ϕ ist zu beachten, dass die Diagonalen eines Tangentenparallelogramms einer Ellipse Richtungen konjugierter Durchmesser sind. Der Mittelpunkt O^κ ergibt sich danach in zweideutiger Weise aus dem Schnitt der Geraden $n := K_1^\kappa \perp t_1^\kappa$ mit dem Thaleskreis k_0 über der zu t_1^α gehörenden Seite des Tangentenparallelogramms.

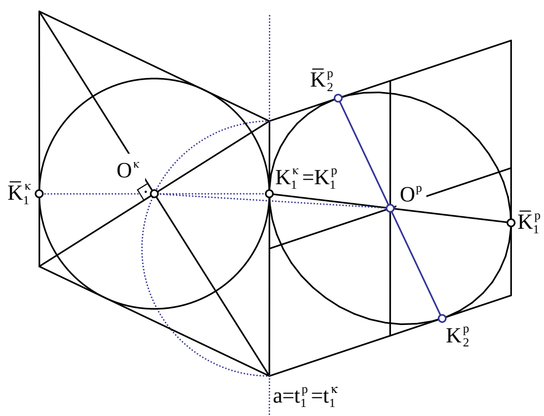


Abbildung 1: *Perspektive Affinität*

Bemerkt werden soll, dass das zweite Paar von Konturpunkten $(K_2^\alpha, \overline{K}_2^\alpha)$ für die Festlegung von $\phi\big(a,(O^\kappa,O^\alpha)\big)$ mit $\phi(k^\kappa) = k^\alpha$ nicht benötigt wird. Lediglich die Tangenten in K_2^α, $\overline{K}_2^\alpha$ als Seiten des Tangentenparallelogramms gehen in die Konstruktion ein, vgl. [1]. Hieraus ist erkennbar, dass die Ellipse k^α durch Angabe der tangentialen Linienelemente (K_j^α, t_j^α), $(\overline{K}_j^\alpha, \overline{t}_j^\alpha)$ $(j = 1,2)$ in den Endpunkten zweier Durchmesser grundsätzlich überbestimmt ist. Umgekehrt können die Punkte K_2^α, $\overline{K}_2^\alpha$ auf t_2^α, $\overline{t}_2^\alpha$ bei vorgegebener Figur $(K_1^\alpha, \overline{K}_1^\alpha, t_j^\alpha, \overline{t}_j^\alpha)$ nicht beliebig gewählt werden.

Wird nun obige Konstruktion der Umrissellipse k^α als planare Aufgabe aufgefasst, so scheint die Ableitung jener Bedingung aufschlussreich, wonach die überzähligen Punkte K_2^α, $\overline{K}_2^\alpha$ auf k^α liegen. In Folge dieser ergibt sich die Frage, in welchen Fällen durch Angabe zweier verschiedener Durchmesser $[L_1, \overline{L}_1]$ und $[L_2, \overline{L}_2]$ mit den zugehörigen Paaren paralleler Tangenten in den Durchmesserendpunkten eine Ellipse, allgemein ein Kegelschnitt, bestimmt ist. Der Fall eines Paares konjugierter Durchmesser, durch den bekanntlich eine Ellipse festgelegt ist, ist hierin einzubetten.

Es ist zu vermuten, dass speziell die zweite Frage bereits Gegenstand ausführlicher Untersuchungen war. Ihre Beantwortung erfolgt an dieser Stelle vielmehr aus didaktischen Überlegungen im Zusammenhang mit der ersten.

2 Der Satz von Pascal

Bekanntlich ist ein projektiver Kegelschnitt c durch fünf Punkte P_l $(l = 1,\dots,5)$ der Ebene festgelegt. Sind nicht drei Punkte P_{l_1}, P_{l_2}, P_{l_3} $(l_1,l_2,l_3 \in \{1,\dots,5\}, l_1 \neq l_2 \neq l_3 \neq l_1)$ kollinear, so ist der Kegelschnitt nicht entartet. Nach dem Satz von Pascal liegt ein weiterer Punkt P genau dann auf c, wenn die Schnittpunkte $R := P_1P_2 \cap P_4P_5$, $S := P_2P_3 \cap P_5P$ und $T := P_3P_4 \cap PP_1$ von Gegenseiten des Sechsecks $P_1 \dots P_5P$ kollinear liegen (Abb. 2).

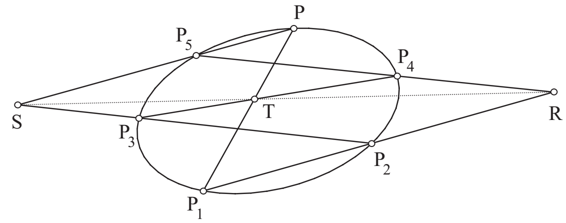


Abbildung 2: *Satz von* Pascal

Wird diese Aussage ins Duale übertragen, so gehört eine weitere Gerade p einem durch fünf Geraden p_l $(l = 1,\dots,5)$ bestimmten Klassenkegelschnitt $c^\star$ genau dann an, wenn die Verbindungsgeraden $r := p_1p_2 \vee p_4p_5$, $s := p_2p_3 \vee p_5p$ und $t := p_3p_4 \vee pp_1$ von Gegenecken des Sechsseits $p_1 \dots p_5p$ kopunktal liegen. Werden schließlich in der Angabe $(K_1^\alpha, \overline{K}_1^\alpha, t_j^\alpha, \overline{t}_j^\alpha)$ die Linienelemente (K_1^α, t_1^α), $(\overline{K}_1^\alpha, \overline{t}_1^\alpha)$ sowie t_2^α als infinitesimal benachbarte Tangenten an k^α gedeutet, so lassen sich die Berührpunkte auf t_2^α und $\overline{t}_2^\alpha$ nachträglich konstruieren. Für $K_2^\alpha \in t_2^\alpha$ ergibt sich beispielsweise

$$r = t_1^\alpha t_2^\alpha \vee \overline{K}_1^\alpha,\ t = t_2^\alpha \overline{t}_1^\alpha \vee K_1^\alpha,\ s = rt \parallel t_1^\alpha,\ K_2^\alpha = st_2^\alpha.$$

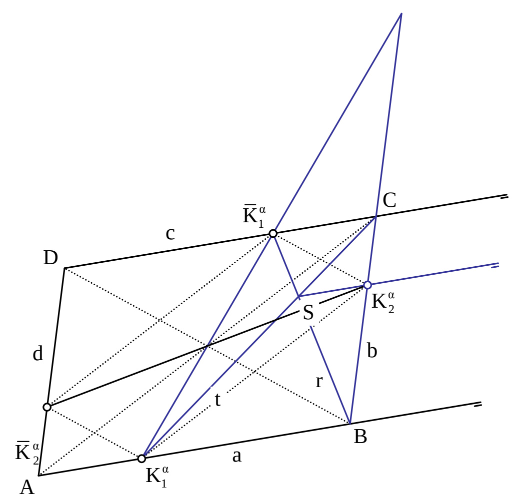


Abbildung 3: *Konstruktion des Berührpunktes*

Die Konstruktion von $K_2^\alpha \in t_2^\alpha$ ist in Abb. 3 visualisiert. Sie veranschaulicht ferner den Satz von Pascal, welcher der Konstruktion zugrunde liegt. Entsprechendes gilt für die Konstruktion des Punktes $\overline{K}_2^\alpha$. Er lässt sich außerdem über die Punktsymmetrie zu $\overline{K}_2^\alpha$ bezüglich O^α bestimmen.

Wegen $t_j^\alpha \parallel \overline{t}_j^\alpha$ ergeben sich in der Figur von Abb. 3 Strahlensatzfiguren mit den Scheiteln $S := rt$ und $B := t_1^\alpha t_2^\alpha$. Über diese kann ein Vergleich von Teilverhältnissen vorgenommen werden, die den Tripeln von Punkten zugeordnet sind, welche auf den Seiten des Tangentenparallelogramms liegen. Für deren Angabe werden die Ecken bzw. Seitenlängen des Parallelogramms entsprechend Abb. 3

mit A, B, C und D bzw. a, b, c und d bezeichnet. Für die übrigen Abschnitte werden die Schreibweisen $\overline{BK_2^\alpha}=b_1$, $\overline{K_2^\alpha C}=b_2$, $\overline{CK_1^\alpha}=c_1$, $\overline{K_1^\alpha D}=c_2$, dergleichen a_1, a_2, d_1, d_2 vereinbart. Es ergeben sich die nachstehenden Verhältnisgleichheiten.

$$\frac{c_2}{c_1}=\frac{\overline{SK_1^\alpha}}{\overline{SC}}=\frac{b_1}{b_2},\ \frac{c_2}{c}=\frac{b_1}{b}.$$

Fernerhin gelten die augenscheinlichen Gleichheiten $a_2/a=c_2/c$ und $b_1/b=d_1/d$. Werden schließlich die Quotienten a_2/a, b_1/b, c_2/c und d_1/d als Teilverhältnisse aufgefasst, so gelten folgende Gleichheiten:

$$\begin{aligned} TV(K_1^\alpha;B,A) &= TV(\bar{K}_1^\alpha;D,C)\\ &= TV(K_2^\alpha;B,C)\\ &= TV(\bar{K}_2^\alpha;D,A). \end{aligned} \tag{1}$$

Die Gleichung $TV(K_2^\alpha;B,C)=TV(K_1^\alpha;B,A)$ beschreibt eine notwendige und hinreichende Bedingung, wonach die Punkte K_2^α, $\bar{K}_2^\alpha$ auf der durch $(K_1^\alpha,\bar{K}_1^\alpha,t_j^\alpha,\bar{t}_j^\alpha)$ festgelegten Ellipse k^α liegen. Diese impliziert insbesondere, dass die Parallelitäten $K_1^\alpha K_2^\alpha \parallel \bar{K}_1^\alpha\bar{K}_2^\alpha \parallel AC$ sowie $K_1^\alpha\bar{K}_2^\alpha \parallel \bar{K}_1^\alpha K_2^\alpha \parallel BD$ gelten, die Verbindungsgeraden der Konturpunkte demnach parallel zu den Diagonalen des Tangentenparallelogramms, d. h. einem Paar konjugierter Richtungen, liegen. Die erhaltene Bedingung ist räumlich sofort einsichtig, da jenes zu den Durchmessern $[K_1^\alpha,\bar{K}_1^\alpha]$ und $[K_2^\alpha,\bar{K}_2^\alpha]$ von k^α gehörende Tangentenparallelogramm $ABCD$ axonometrisches Bild eines k umschließenden Rhombus ist, bezüglich dessen obige Teilverhältnisgleichheiten und damit die Orthogonalitäten $K_1K_2 \perp BD$, $\bar{K}_1\bar{K}_2 \perp BD$, des weiteren $K_1\bar{K}_2 \perp AC$ und $\bar{K}_1K_2 \perp AC$, der Verbindungsgeraden entsprechender Berührpunkte gelten. Die axonometrische Abbildung erhält Parallelitäten und Teilverhältnisse, so dass sich die Eigenschaften wie angegeben übertragen lassen. Daneben lässt sich die erhaltene Bedingung in gleicher Weise mithilfe der in Abschnitt 1 beschriebenen perspektiven Affinität $\phi\big(a,(O^\kappa,O^\alpha)\big)$ zwischen k^α und k^κ bestätigen, da unter perspektiven Affinitäten Parallelitäten und Teilverhältnisse erhalten bleiben.

3 Die Konstruktion der Umrissellipse als planimetrische Aufgabe

Wird die Konstruktion der Ellipse k^α unter Verwendung der Einschneidehilfsrisse, unabhängig ihrer räumlichen Deutbarkeit, als ebene Aufgabe aufgefasst, so ist die Frage zu beantworten, wodurch sich die in (1) genannten Teilverhältnisgleichheiten ergeben. Die Beantwortung scheint insbesondere aufschlussreich, da die Lagen und Maßstäbe der Einschneidehilfsrisse in zwei Koordinatenebenen, bis auf den Fall paralleler Einschneiderichtungen, frei gewählt werden können.

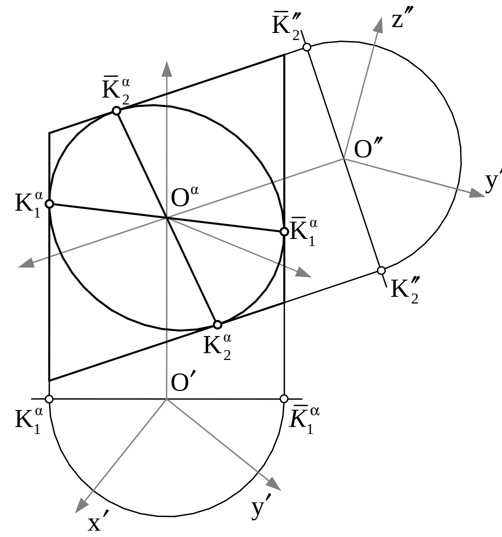


Abbildung 4: *Konturpunkte*

Dem Nachweis wird eine Konstruktionsfigur entsprechend Abb. 4 zugrunde gelegt, in welcher Grund- und Aufrissfigur von Φ als Einschneidehilfsrisse sowie die Einschneiderichtungen s', s'' bezüglich des Grund- respektive Aufriss gewählt werden. Die Paare von Konturpunkten $(K_j,\bar{K}_j)$ bezüglich s', s'' sind im jeweiligen Einschneidehilfsriss durch den Schnitt $\{K_1',\bar{K}_1'\}=k_1\cap(O'\perp s')$ bzw. durch $\{K_2'',\bar{K}_2''\}=k_2\cap(O''\perp s'')$ festgelegt. Ihre axonometrischen Bilder $(K_j^\alpha,\bar{K}_j^\alpha)$ lassen sich unter Verwendung der perspektiven Affinitäten $\phi_1:\pi_1'\rightarrow\pi_1^\alpha$ und $\phi_2:\pi_2''\rightarrow\pi_2^\alpha$ zwischen den Einschneidehilfsrissen und dem axonometrischen Grund- bzw. Aufriss konstruieren. Die Tangenten an k^α in K_1^α, $\bar{K}_1^\alpha$ respektive K_2^α, $\bar{K}_2^\alpha$ sind durch die Einschneiderichtungen s' bzw. s'' gegeben.

Um die Bedingung der Teilverhältnisgleichheit auf den Seiten des hierdurch festgelegten Tangentenparallelogramms zu überprüfen, scheint es sinnvoll, die $(K_1',\bar{K}_1')$, $(K_2'',\bar{K}_2'')$ bezüglich $(K_j^\alpha,\bar{K}_j^\alpha)$ ergänzenden Hilfsrisse $(K_1'',\bar{K}_1'')$, $(K_2',\bar{K}_2')$ zu konstruieren. Hierfür kann benutzt werden, dass

$$\phi_2^{-1}\circ\phi_1|_{y'}:y'\rightarrow y'',\ \phi_1^{-1}\circ\phi_2|_{y''}:y''\rightarrow y'$$

teilverhältnistreu operieren.

Die Konturpunkte lassen sich durch Übertragen ihrer y-Koordinaten im jeweils anderen Riss bestimmen. Sind y'- und y''-Achse nicht parallel gewählt, so erlaubt die in Abb. 5 dargestellte Vorschrift die zeichnende Bestimmung der Risse $(K_1'', \bar{K}_1'')$ und $(K_2', \bar{K}_2')$. Sie folgt aus der Tatsache, dass die Schnittpunkte der jeweils x'-parallelen Geraden durch den Grundriss P' mit der z''-parallelen Geraden durch deren Aufriss P'' für beliebige Punkte P kollinear liegen, vgl. [1].

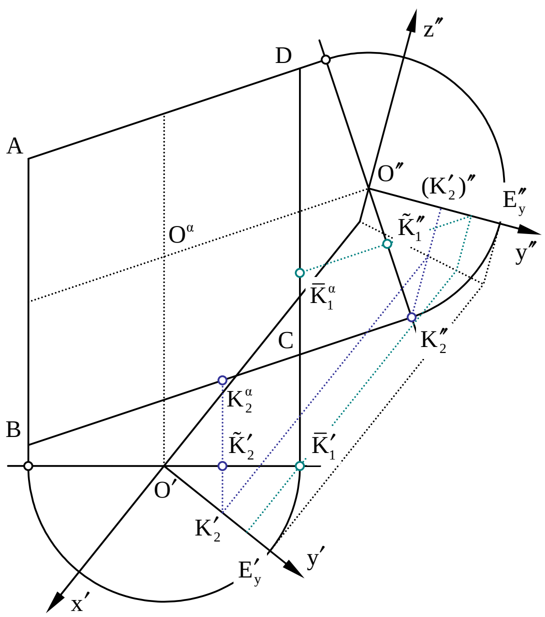


Abbildung 5: *Übertragen von Teilverhältnissen*

Hiernach folgen sodann die Gleichheiten der Teilverhältnisse $TV\left((\bar{K}_1'')';E_y',O'\right)$ und $TV\left(\bar{K}_1'';E_y'',O''\right)$ sowie $TV\left((K_2')'';E_y'',O''\right)$ und $TV\left(K_2';E_y',O'\right)$, die jeweils dem Kosinus des Winkelmaß entsprechen, der zwischen der orientierten Geraden $K_1'\bar{K}_1'$ bzw. $K_2''\bar{K}_2''$ und der positiven Richtung der y'-Achse bzw. y''-Achse eingeschlossen wird.

$$TV\left((\bar{K}_1'')';E_y',O'\right) = TV\left(\bar{K}_1'';E_y'',O''\right) = \cos\gamma$$
$$TV\left((K_2')'';E_y'',O''\right) = TV\left(K_2';E_y',O'\right) = \cos\varepsilon.$$

Werden nun $\bar{K}_1''$ und K_2' in Richtung der Einschneiderichtungen s'' bzw. s' projiziert, so ergeben sich die Punkte $\tilde{K}_1'' := (\bar{K}_1'' \| s'') \cap K_2''\bar{K}_2''$ beziehungsweise $\tilde{K}_2' := (K_2') \| s' \cap K_1'\bar{K}_1'$. Für die Punkte $\tilde{K}_1''$ und $\tilde{K}_2'$ gelten dann offenbar $TV(\tilde{K}_1'';K_2'',O'') = TV(\tilde{K}_2';\bar{K}_1',O') = \cos\gamma\cos\varepsilon$, woraus schließlich die gesuchte Gleichheit folgt:

$$TV(\bar{K}_1^{\alpha};C,D) = TV(K_2^{\alpha};C,B). \qquad (2)$$

4 Weitere Folgerungen aus der Anwendung des Satzes von PASCAL

Im letzten Abschnitt wurde planimetrisch begründet, dass die Konstruktion der Konturpunkte bezüglich der Einschneiderichtungen in Grund- und Aufriss tatsächlich zwei Durchmesser sowie die dazu konjugierten Richtungen einer Ellipse angibt. Diese ist durch Bedingung (2) wohldefiniert. Die Bedingungen (1) bzw. (2) sind hierbei affingeometrische Folgerungen aus der Anwendung des Satzes von PASCAL, der zur projektiven Geometrie zu zählen ist. Da diese nicht ausschließlich auf Ellipsen zu beziehen sind bzw. sich vergleichbar für Hyperbeln formulieren lassen, scheint die Frage berechtigt, wie (1) geeignet zu spezifizieren sind. Darüber hinaus stellt sich die Frage, wie sich hierin durch Anwendung des Satzes von PASCAL auch der Fall einer Parabel einbetten lässt.

Bekanntlich lassen sich Ellipse, Parabel und Hyperbel in ihrer Lage bezüglich der uneigentlichen Geraden affingeometrisch kennzeichnen. In den Abb. 6 und 7 sind die Fälle einer Hyperbel und Ellipse in einem projektiv abgeschlossenen affinen Raum (nach Anwendung einer Kollineation) schematisch dargestellt.

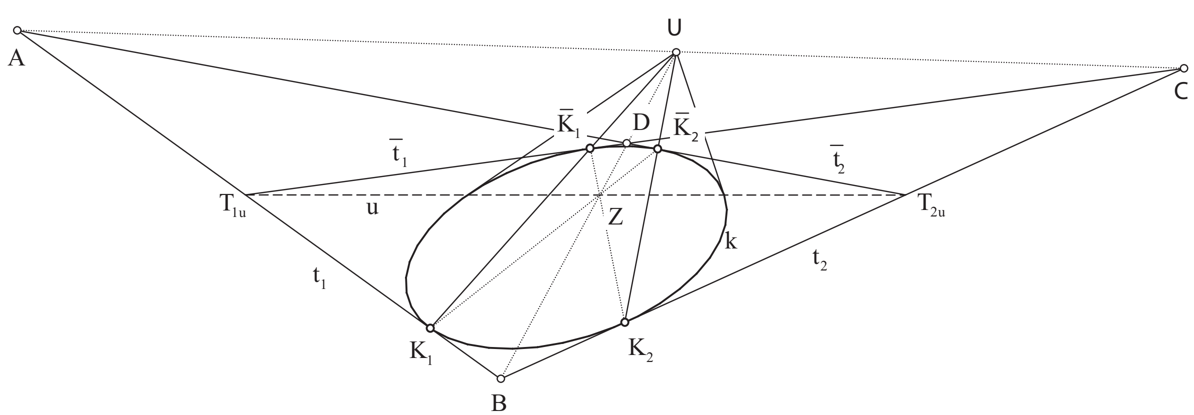


Abbildung 6: *Hyperbel*

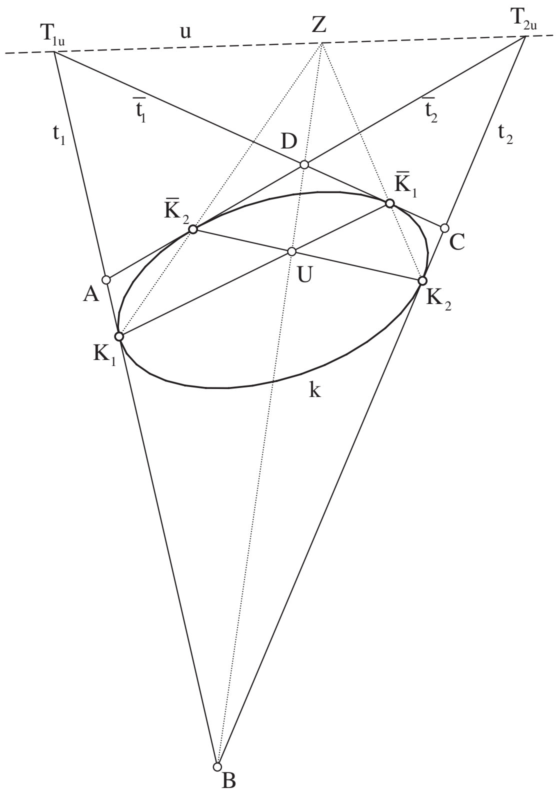


Abbildung 7: *Ellipse*

Die in Abschnitt 2 entwickelte Figur eines Tangentenparallelogramms lässt sich hierin kennzeichnen: Die von den uneigentlichen Punkten T_{ju} an den Kegelschnitt k gelegten Tangenten t_j, $\overline{t_j}$ bilden ein Tangentenvierseit. Die zu T_{ju} bezüglich k polaren Verbindungsgeraden $K_j\overline{K_j}$ der Berührungspunkte inzidieren mit dem Pol U der uneigentlichen Geraden u bezüglich k, verlaufen demnach durch dessen Mittelpunkt und sind somit Durchmesser von k. Die von T_{ju} verschiedenen Schnittpunkte der Tangenten t_j, $\overline{t_j}$ bilden das Viereck $ABCD$. Mit jeder der Seiten t_j, $\overline{t_j}$ inzidieren damit genau zwei Eckpunkte von $ABCD$ sowie jeweils genau einer der Punkte T_{ju}. Demnach sind auf den Geraden t_j, $\overline{t_j}$ Punktequadrupel bestimmt.

Wegen der Polaritätsbeziehungen bezüglich k gelten ersichtlich $U \in AC, BD$ sowie $Z \in u$ mit $Z := K_1\overline{K_2} \cap \overline{K_1}K_2$. Die Punktequadrupel auf t_j, $\overline{t_j}$ lassen sich sodann vermöge der Geradenbüschel $\mathcal{G}_U$ bzw. $\mathcal{G}_Z$ perspektiv respektive projektiv aufeinander abbilden, wonach das Doppelverhältnis der Punktequadrupel erhalten bleibt. Da jeweils $T_{ju} \mapsto T_{ju}$ bzw. $T_{1u} \mapsto T_{2u}$ gelten, lassen sich aus

$$\begin{aligned} DV(K_1;B,A,T_{1u}) &=_{\pi_U} DV(\overline{K_1};D,C,T_{1u}) \\ &=_{\pi_Z} DV(K_2;B,C,T_{2u}) \\ &=_{\pi_U} DV(\overline{K_2};D,A,T_{2u}) \end{aligned} \tag{3}$$

mit den Vereinbarungen $DV(K_1;B,A,T_{1u}) =: TV(K_1;B,A)$ etc. die Teilverhältnisgleichheiten (1) erhalten. Im Unterschied zueinander trennen im Fall der Ellipse die Punktepaare (A,B) und (K_1,T_{1u}) etc. einander, im Fall der Hyperbel jedoch nicht. Infolgedessen genügen die Werte der Doppelverhältnisse und mithin die entsprechenden Teilverhältnisse im Fall der Ellipse der Relation

$$0 < TV(K_1;B,A) < 1, \tag{4}$$

wogegen im Fall der Hyperbel

$$TV(K_1;B,A) > 1 \tag{5}$$

gilt. Des Weiteren ergibt sich für den Fall der konjugierten Lage der Durchmesser $[K_1,\overline{K_1}]$ und $[K_2,\overline{K_2}]$ einer Ellipse ein vollständiges Viereck, bezüglich dessen sich die in (3) genannten Punktequadrupel in harmonischer Lage befinden. Das Teilverhältnis $TV(K_1;B,A)$ nimmt mit der getroffenen Vereinbarung den Wert $1/2$ an, der sich als spezieller Teilverhältniswert in (4) ergibt.

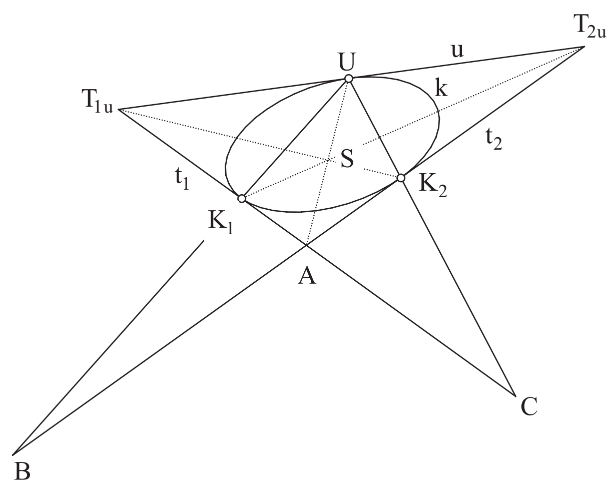


Abbildung 8: *Parabel*

Im Fall einer Parabel k ist die uneigentliche Gerade u Tangente der Kurve, weswegen der Pol U bezüglich k mit seiner Polaren u inzidiert. Es gilt also $U \in u$. Die Menge aller Geraden durch U bildet bekanntlich ein Parallelgeradenbüschel von Durchmessern, welches auch die Achse von k enthält. Es lässt sich entsprechend den vorangegangenen Fällen untersuchen, unter welcher Bedingung durch die Angabe zweier nichtparalleler Linienelemente (K_1,t_1)

und (K_2,t_2) sowie einer Geradenrichtung d eine Parabel festgelegt ist, für welche (K_1,t_1) und (K_2,t_2) tangential liegen und d die Durchmesserrichtung angibt.

Nach Anwendung des Satzes von PASCAL entsprechend Abb. 8 ergeben sich hierfür vollständige Vierecke, deren Ecken neben U, K_1, bzw. K_2 die Punkte $A := t_1t_2$, T_{2u} bzw. T_{1u} bilden. Die Nebenecken sind durch (S,T_{1u},B) bzw. (S,T_{2u},C) mit $B := (K_1\|d)\cap t_2$ und $C := (K_2\|d)\cap t_1$ gegeben. Da in einem vollständigen Viereck bekanntlich die Nebenecken von den Schnittpunkten ihrer Verbindungsgeraden mit den Seiten durch die dritte Nebenecke harmonisch getrennt werden, gilt dies unter Verwendung einer geeigneten Perspektivität auch für die zugeordneten Punktepaare auf t_j. Wird erneut vereinbart, dass $DV(K_1;C,A,T_{1u}) = TV(K_1;C,A)$ und entsprechend $DV(K_2;B,A,T_{2u}) = TV(K_2;B,A)$ gelten, so folgt einsichtig

$$TV(K_1;C,A) = TV(K_2;B,A) = 2. \qquad (6)$$

5 Eine Konstruktion der Umrissellipse

Ziel dieses abschließenden Abschnittes ist es, eine punkt- und tangentenweise Konstruktion des axonometrischen Umriss k^α einer Kugel Φ vorzuschlagen, der durch die Konturpunkte K_1^α, $\bar{K}_1^\alpha$ und K_2^α, $\bar{K}_2^\alpha$ bezüglich der Einschneiderichtungen s' und s'' gegeben ist. Alternativ kann eine Ellipse k nach Abschnitt 3 durch zwei Durchmesser $[K_1,\bar{K}_1]$ und $[K_2,\bar{K}_2]$ sowie die dazu konjugierten Richtungen gegeben sein, falls zusätzlich Bedingung (2) erfüllt ist.

Wird über dem Durchmesser eines Kreises ein beliebiges Aufsatzdreieck errichtet, so liegen die Höhenfußpunkte auf den vom Durchmesser verschiedenen Seiten nach Umkehrung des Satz von THALES auf dem Kreis. Übertragen auf Ellipsen als perspektiv affine Bilder von Kreisen lässt sich eine punktweise Konstruktion von k vorschlagen, vgl. [2] (Abb. 9): O.B.d.A ist eine Gerade q in zu $[K_1,\bar{K}_1]$ konjugierter Richtung gewählt, die $[K_1,\bar{K}_1]$ innen schneidet. Mit $T := K_1K_2 \cap q$ und $H := \bar{K}_1K_2 \cap q$ folgt entsprechend $P := T\bar{K}_1 \cap K_1H$ mit $P \in k$. Da die Strecke $[H,T]$ von den Tangenten an k in P und K_2 in ihrem Mittelpunkt M_{12} geteilt wird, lässt sich auch die Tangente t_P an k in P konstruieren. Es gilt: $t_P = M_{12}P$. Die Ellipse k kann mithin auch als Ortskurve der Punkte P bzw. als Enveloppe der Tangentenschar erzeugt werden.

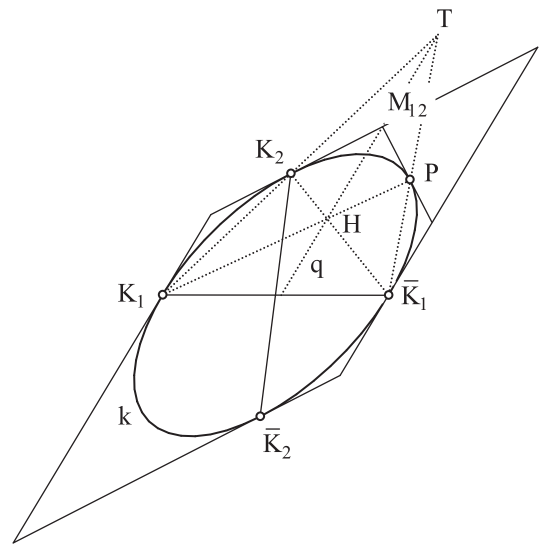


Abbildung 9: *Ellipse als Ortskurve*

Literatur

[1] BRAUNER, H.: *Lehrbuch der Konstruktiven Geometrie*. Springer-Verlag Wien, Fachbuchverlag Leipzig, 1986.

[2] JEGER, M.: *Konstruktive Abbildungsgeometrie*. Räber Verlag, 1964.

Marco Hamann
Technical University of Dresden
e-mail: Marco.Hamann@tu-dresden.de

Stručni rad
Prihvaćeno 28. 12. 2008.

VEDRAN CAR
DINO DRAGUN
JELENA BEBAN - BRKIĆ

Multimedijski pristup u vizualizaciji polarne stereografske projekcije

Multimedia Concept in Visualization of the Polar Stereographic Projection

ABSTRACT

The paper deals with topic of *polar stereographic projection* with emphasis on the ICT application in teaching. The formulae for the above mentioned projection have been derived, which made it possible for it to be visualized by writing the codes in the program Mathematica, and the multimedia component has been achieved by implementing it into video record by means of the program *Bulent's Screen Recorder* from the group of *Desktop Screen Recorders*.

Key words: information and communication technologies (ICT), e-learning, cartographic projection, polar stereographic projection, visualization

MSC 2000: 97U80, 51N20

Multimedijski pristup i vizualizacija polarne stereografske projekcije

SAŽETAK

U radu je obrađena tema *polarna stereografska projekcija* s naglaskom na primjenu ICT-a u nastavi. Dan je izvod formula polarne stereografske projekcije što je omogućilo njenu vizualizaciju u programu *Mathematica*, dok je multimedijska komponenta ostvarena implementiranjem u video zapis uz pomoć programa *Bulent's Screen Recorder* iz skupine *Desktop Screen Recorders-a*.

Ključne riječi: informacijske i komunikacijske tehnologije (ICT), e-učenje, kartografske projekcije, polarna stereografska projekcija, vizualizacija

Akademske godine 2007/08 Vedran Car i Dino Dragun, tadašnji studenti treće godine preddiplomskog studija geodezije i geoinformatike na Geodetskom fakultetu Sveučilišta u Zagrebu, napisali su rad *Učiti na drugi način – upotreba multimedijskog i interaktivnog sadržaja*. Voditeljica rada bila je Jelena Beban Brkić. Studenti su za svoj rad dobili Nagradu dekana Geodetskog fakultea. Ovaj članak je izvadak iz tog rada.

1 Uvod

E-učenje je jedan od brojnih pojmova s prefiksom "e-" koji se u posljednje vrijeme sve češće spominju. Općenito, prefiks "e-" (elektroničko, eng. electronic) označava izvođenje određenih djelatnosti uz pomoć informacijsko-komunikacijske tehnologije (ICT).

Postoje različite definicije e-učenja. Mi ćemo navesti onu koja po našem mišljenju dobro ocrtava smisao i ideju e-učenja: *E-učenje je proces obrazovanja (učenja i podučavanja) uz uporabu informacijsko-komunikacijske tehnologije u svrhu unapređenja kvalitete samog procesa i ishoda obrazovanja* [6].

Budući da su nove informatičke tehnologije obilježile našu epohu, tako se i očekuje da će u visokoškolskom obrazovanju postati standardom, dok će poznavanje i sposobnost uporabe tehnologija e-učenja biti sastavni dio osnovne pismenosti svakog člana akademske zajednice. Jednako tako možemo reći da se e-učenje pojavljuje i kao rješenje u novoj situaciji u kojoj broj studenata na sveučilištima ubrzano raste, jer postaje sve teže osigurati "staru" neposrednu komunikaciju između nastavnika i studenata. Stoga je *Strategija e-učenja Sveučilišta u Zagrebu* 2007. - 2010., na sjednici Senata Sveučilišta održanoj 12. lipnja 2007. godine, prihvaćena konsenzusom svih sveučilišnih sastavnica, gdje je jasno da e-učenje postaje sinonim za novo, moderno i kvalitetno obrazovanje.

Iz navedenog se očituje da e-učenje treba predstavljati visokokvalitetni proces obrazovanja u kojem nastavnici i studenti aktivno surađuju, sa svrhom postizanja zadanih obrazovnih ciljeva. Pri tome trebaju intenzivno koristiti informacijsku i komunikacijsku tehnologiju za stvaranje prilagodljivog virtualnog okruženja u kojem se razvijaju i koriste multimedijski interaktivni obrazovni materijali, ostvaruje međusobna komunikacija i suradnja, studenti izvršavaju pojedinačne ili grupne zadatke i projekte, te provode kontinuiranu samoprovjeru i provjeru znanja.

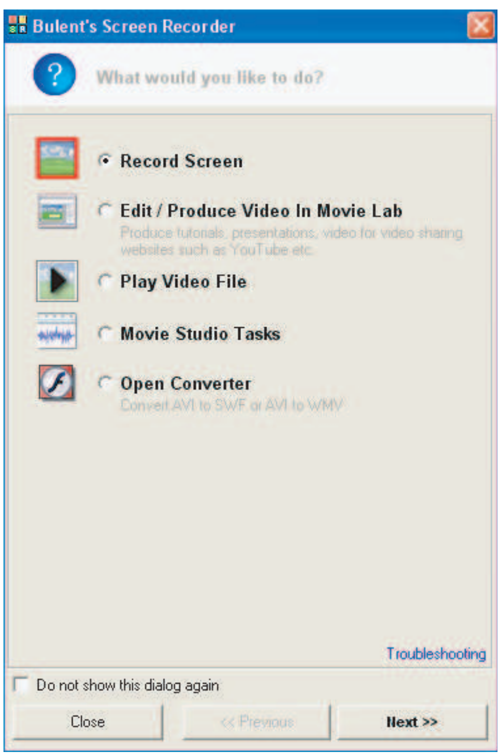
Upravo zbog korištenja ICT-a kao poboljšanja kvalitete prijenosa znanja, u ovom radu dajemo primjer kako upotrebom programa *Desktop Screen Recorders* multimedijska komponenta može biti ostvarena.

Glavno obilježje ovih programa je da omogućuju snimiti svaku aktivnost na grafičkom zaslonu, istovremeno snimajući i zvuk, te istu aktivnost spremaju u video datoteku.

Iz skupine navedenih programa, nakon provedenih početnih testiranja, izabrali smo program *Bulent's Screen Recorder (Version 4)* [12].

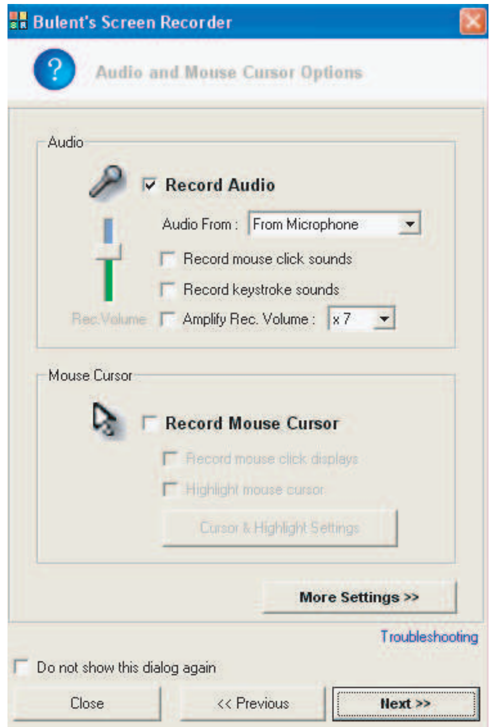
Može se snimati cijeli ekran, određeni prozor, ili dio ekrana. Moguće je i podešavati kvalitetu i intenzitet zvuka, te smo sukladno radu s navedenim programom došli do najoptimalnije konfiguracije navedenih mogućnosti.

Kako bi prikazali način na koji dolazimo do konačnog proizvoda, tj. video clipa, slijedi dio postupka u postavkama programa, od samog uključivanja do izlaznog podatka. Za početak odabiremo snimanje zaslona jednom od prikazanih mogućnosti :



Slika 1: *Odabir postavki snimanja*

Zatim, odabiremo snimanje zvuka pomoću mikrofona, dok isključujemo pokazivač miša na ekranu prilikom snimanja videa:



Slika 2: *Odabir načina snimanja zvuka*

Konačno, izlazni je podatak, video zapis određenog sadržaja (u ovom se slučaju radi o datoteci naziva: "Stereografska projekcija.avi")

2 Programski paket *Mathematica*

Softverski alati moćno su oružje u inoviranju pristupa nastavi. Istaknuto mjesto među matematičkim softverom ima upravo programski paket *Mathematica*, softver tvrtke Wolfram Research. Davno prepoznat kao jedan od najboljih svjetskih softverskih matematičkih alata, ali također jedan od najboljih softverskih proizvoda uopće, *Mathematica* nam uz minimalan utrošak resursa omogućuje odlične rezultate i visok stupanj inovacije nastave, te mijenja sam pristup matematici i predmetima vezanim uz matematiku. Implementacija sadržaja moguća je od najnižih do najviših razina matematike, ovisno o potrebama. Ono što *Mathematica* omogućuje u nastavi je prevođenje problema iz jezika u matematički jezik, uči kako eksperimentirati s matematikom, uči kako spojiti matematiku i programiranje, matematiku sa drugim znanstvenim disciplinama, omogućuje shvaćanje rekurzija i kako ih praktično upotrijebiti. Ukratko, *Mathematica* je:

- kalkulator proizvoljne točnosti,
- kalkulator za simboličko računanje,
- alat za rješavanje različitih algebarskih, računskih i drugih problema,
- sustav za vizualiziranje funkcija, podataka i složenih objekata,
- generator zadataka, provjera znanja, vježbi... sa i bez odgovora,
- aplikacija za izradu interaktivnih prezentacija. [7]

Mathematica postoji već više od 20 godina. Postala je standard u mnogim organizacijama, tvrtkama i sveučilištima. Ministarstvo znanosti i tehnologije RH je krajem 1994. godine opremilo s *Mathematicom* hrvatska sveučilišta, a 1996. godine većinu istraživačkih instituta. Broj nastavnika i studenata koji ju koriste neprestano raste. Mi smo uz pomoć *Mathematice* izvršili vizualizaciju stereografske projekcije, a zatim sve to uspješno implementirali u video zapis.

U nastavku, prije izvoda formula i prikaza kodova i ilustracija, dajemo kratki uvod u kartografske projekcije (vidi [1], [3], [9] i [11]) i nekoliko povijesnih notica iz tog područja ([5], [9], [10]).

3 O kartografskim projekcijama

Grana kartografije koja proučava načine preslikavanja zakrivljene površine Zemlje i ostalih nebeskih tijela na ravninu često se naziva matematičkom kartografijom. Budući da danas matematika sve više prodire i u ostale grane kartografije, naziv matematička kartografija nije više prikladan, pa se ta grana kartografije naziva *Kartografske projekcije*. Cilj izučavanja kartografskih projekcija je stvaranje matematičke osnove za izradu karata i rješavanje teorijskih i praktičnih zadataka u kartografiji, geodeziji, geografiji, astronomiji, navigaciji i ostalim srodnim znanostima. Pri izradi karata najprije se točke s fizičke površine Zemlje prenose po određenim pravilima na plohu elipsoida ili sfere, a zatim se elipsoid odnosno sfera preslikavaju u ravninu. U tu svrhu služe kartografske projekcije. To su načini preslikavanja plohe elipsoida ili sfere u ravninu. Na plohi elipsoida ili sfere točke su određene presjekom koordinatnih linija meridijana i paralela. Svaka mreža koordinatnih linija preslikana u ravninu naziva se kartografska mreža. Zadatak kartografskog preslikavanja je da ustanovi ovisnost između koordinata točaka na Zemljinom elipsoidu ili sferi i koordinata tih točaka u projekciji. Ta ovisnost najčešće se određuje jednadžbama:

$$x = f_1(\varphi,\lambda), \quad y = f_2(\varphi,\lambda). \tag{1}$$

Budući da točku na plohi Zemljinog elipsoida ili sfere najčešće određujemo geografskim koordinatama φ i λ, a u ravnini pravokutnim koordinatama x i y, to jednadžbe (1) nazivamo osnovnim jednadžbama kartografskih projekcija. Te jednadžbe odnosno funkcije f određuju svojstva kartografskih projekcija i može ih biti beskonačno mnogo. U praksi se međutim koristi nekoliko stotina kartografskih projekcija. Prikaz Zemljine plohe u ravnini u bilo kojoj projekciji je deformiran. Dužine, površine i kutovi pri preslikavanju se mijenjaju, tj. deformiraju. Prema obliku deformacija kartografske projekcije dijelimo u ove četiri grupe:

1. konformne ili istokutne,
2. ekvivalentne ili istopovršinske,
3. ekvidistantne ili istodužinske,
4. uvjetne (sve koje ne spadaju pod 1, 2 odnosno 3). [3]

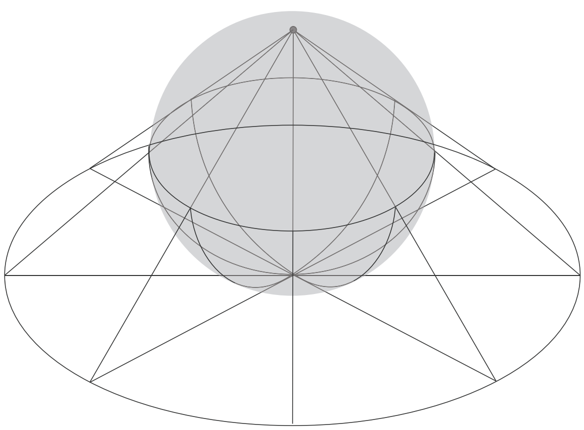
4 Stereografska projekcija

Stereografska projekcija ubraja se među najstarije projekcije, a njen pronalazak se pripisuje grčko astronomu Hiparhu oko 150. godine pr.Kr. prilikom izrade karata nebeske sfere. Njenu upotrbu opisao je grčki matematičar, astronom i geograf *Ptolomej* (2. st. n.e.). Još od Hiparha, Ptolomeja, a najvjerojatnije još i od starih Egipćana, *polarna* stereografska projekcija korištena je za karte zvijezda, uključujući i prve tiskane karte; *'Imagines coeli septentrionales cum duodecim imaginibus zodiaci'* i *'Imagines coeli meridionales'*, od strane njemačkog slikara i grafičara *Albrechta Dürera* 1515. godine, koji je iskoristio polarnu stereografsku projekciju za izradu karte *sjeverne ekliptičke polusfere* [9]. Godine 1507. njemački znanstvenik *Gaulterius Lud* (1448. - 1547.) izradio je vjerojatno najstariju poznatu kartu svijeta, koja je bazirana na stereografskoj projekciji. *Franois d'Aiguillon* 1613.

uvodi pojam stereografske projekcije umjesto dotadašnjeg naziva *planisfera*: "To je projekcija u ravninu iz točke na zemaljskoj sferi koja se nalazi točno nasuprot točke u kojoj ravnina tangira sferu." Upravo zbog toga ne možemo prikazati cijelu zemaljsku kuglu iz jednog projekcijskog središta.

Stereografska projekcija spada u konformna preslikavanja, iz tog razloga su mali elementi (zapravo beskrajno mali djelovi) prikazani bez deformacije oblika, dok s druge strane kod većih područja dolazi do deformacija oblika. Ove osobine je prvi pokazao engleski matematičar i astronom *Edmond Halley* (1656. - 1742.), poznat po izradi različitih tematskih karata, te karata putanja kometa. U svojim djelima bavio se dokazom konformnosti, pa je tako u jednom od njih objavio i sljedeću tvrdnju: "U stereografskoj projekciji, kut pod kojima kružnice sijeku jedna drugu u ravnini projekcije, ima istu vrijednost kao i kut pod kojim se te kružnice sijeku na sferi. Ovo može biti vrlo važno svojstvo ove projekcije ukoliko se ono i dokaže!", što je Halley nekoliko godina kasnije i dokazao.

Dakle, točka gledanja je u točki na površini sfere iz koje je promjer sfere okomit na ravninu projiciranja.



Slika 3: *Princip projiciranja pri polarnoj stereografskoj projekciji sfere na ravninu*

5 Izvod formula stereografske projekcije

Neka je $Oxyz$ pravokutni Kartezijev koordinatni sustav. Postavimo sferu polumjera R tako da u južnom polu dodiruje $xy-$ravninu i to u ishodištu koordinatnog sustava. Projiciramo točke sfere iz sjevernog pola $N(0,0,2R)$ na $xy-$ ravninu. Točki Z sfere biti će pridružena točka z koja se dobije kao probodište pravca NZ i ravnine xy.

Neka je sfera zadana jednadžbom

$$x^2+y^2+(z-R)^2=R^2. \tag{2}$$

Točkama $Z(\xi,\eta,\zeta)$ na sferi odgovarat će točke $z(x,y)$ ravnine xy. Pokazat ćemo da su koordinate (ξ,η,ζ) i (x,y) vezane relacijama:

$$\xi=\frac{4R^2x}{x^2+y^2+4R^2}\qquad \eta=\frac{4R^2y}{x^2+y^2+4R^2}\qquad \zeta=\frac{4R^2z}{x^2+y^2+4R^2} \tag{3}$$

$$x=\frac{2R\xi}{2R-\zeta},\quad y=\frac{2R\eta}{2R-\zeta}. \tag{4}$$

Kao prvo ćemo provjeriti da li relacije (3) vrijede za jednadžbu zadane sfere (2):

$$\xi^2+\eta^2+(\zeta-R)^2=R^2$$

$$\left(\frac{4R^2x}{x^2+y^2+4R^2}\right)^2+\left(\frac{4R^2y}{x^2+y^2+4R^2}\right)^2+\left(\frac{4R^2z}{x^2+y^2+4R^2}-R\right)^2=R^2$$

$$\cdots\cdots$$

$$\frac{R^2\left(16R^2x^2+16R^2y^2+\left(x^2+y^2-4R^2\right)^2\right)}{\left(x^2+y^2+4R^2\right)^2}$$

$$\cdots\cdots$$

$$\frac{R^2\left(x^2+y^2+4R^2\right)^2}{\left(x^2+y^2+4R^2\right)^2}$$

$$R^2=R^2.$$

Promotrimo sada radijvektore točaka Z i z u odnosu na pol N:

$$\overrightarrow{NZ}=\xi\vec{i}+\eta\vec{j}+(\zeta-2R)\vec{k},\quad \overrightarrow{Nz}=x\vec{i}+y\vec{j}+2R\vec{k}. \tag{5}$$

Kako su oni kolinearni treba biti:

$$\frac{\xi}{x}=\frac{\eta}{y}=\frac{\zeta-2R}{-2R}. \tag{6}$$

Iz relacije (6) slijedi da je $x=\frac{2R\xi}{2R-\zeta}$, $y=\frac{2R\eta}{2R-\zeta}$, čime je pokazano da vrijede relacije (4).

Dalje, uvedemo li parametar t u (6) dobivamo:

$$\xi=xt,\quad \eta=yt,\quad \zeta=2R-2Rt. \tag{7}$$

Kako se točka $Z(\xi,\eta,\zeta)$ nalazi na sferi, mora zadovoljavati njenu jednadžbu. Uvrstimo stoga (7) u jednadžbu (2). Nakon sređivanja dobivamo da je

$$t=\frac{4R^2}{x^2+y^2+4R^2}, \tag{8}$$

što uvršteno natrag u (7) daje koordinate točaka Z na sferi:

$$\xi=\frac{4R^2x}{x^2+y^2+4R^2}\quad \eta=\frac{4R^2y}{x^2+y^2+4R^2}\quad \zeta=\frac{4R^2z}{x^2+y^2+4R^2}.$$

Ovime su i formule (3) dokazane. Detaljniji račun može se naći u [4].

Napomenimo da se formule (3) i (4) mogu dobiti i rabeći sličnost trokuta.

Međutim, u geodeziji se točke na sferi najčešće zadaju u *geografskim koordinatama* φ, λ. Upotrijebimo li parametre φ, λ, sfera zadana implicitnom jednadžbom (2) ima vektorsku jednadžbu

$$\vec{r}(\varphi,\lambda) = (R\cos\lambda\cos\varphi, R\sin\lambda\cos\varphi, R(\sin\varphi+1)), \qquad (9)$$
$$\lambda\in[-\pi,\pi],\ \varphi\in[-\pi/2,\pi/2],$$

odnosno parameterske jednadžbe

$$\begin{aligned}\xi &= R\cos\lambda\cos\varphi\\ \eta &= R\sin\lambda\cos\varphi \qquad (10)\\ \zeta &= R(\sin\varphi+1).\end{aligned}$$

Uvrstimo li jednadžbe (10) u formule (4) dobivamo traženu vezu:

$$x = \frac{2R}{1-\sin\varphi}(\cos\lambda\cos\varphi),\ y = \frac{2R}{1-\sin\varphi}(\sin\lambda\cos\varphi). \quad (11)$$

6 *Mathematica* vizualizacije

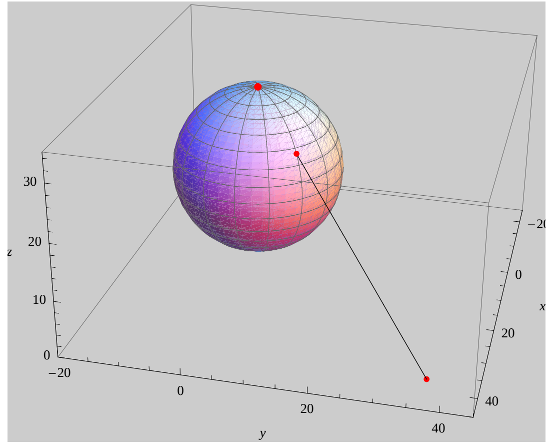
Prilozi rada [2] sadrže i *Mathematica bilježnicu* u kojoj se vizualizira stereografska projekcija točaka i krivulja na sferi. Ovdje prikazujemo neke od *Mathematica* vizualizacija iz te bilježnice.

Na temelju jednadžbi (10) i (11) definirane su liste koordinata točaka na sferi i njihovih stereografskih projekcija.

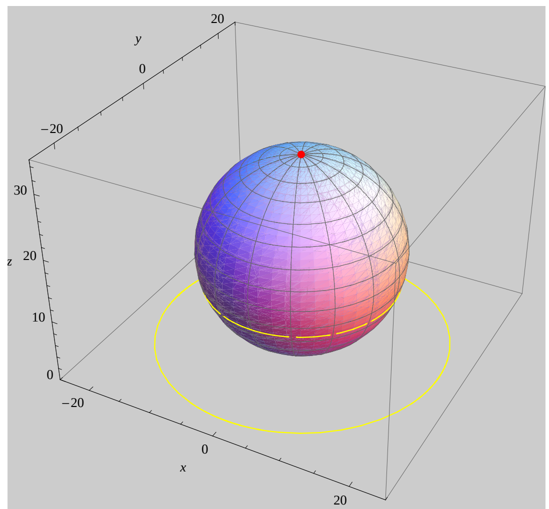
```
sfera[R_, λ_, φ_] :=
{R * Cos[λ] Cos[φ], R * Sin[λ] Cos[φ], R * Sin[φ] + R}

stereografska[R_, λ_, φ_] :=
{ 2 R * Cos[λ] Cos[φ]    2 * R Sin[λ] Cos[φ]
  ------------------- , ------------------- , 0}
      1 - Sin[φ]             1 - Sin[φ]
```

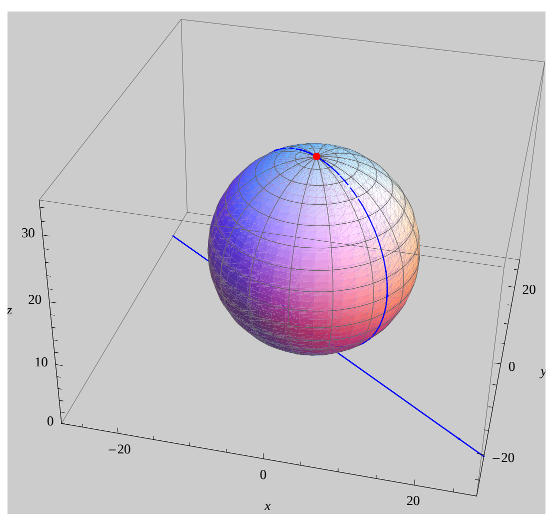
Tako definirane funkcije omogućuju omogućuju prikaze koji slijede.



Slika 4: *Sfera, pol, točka na sferi, njezina projekcija i zraka projiciranja.*

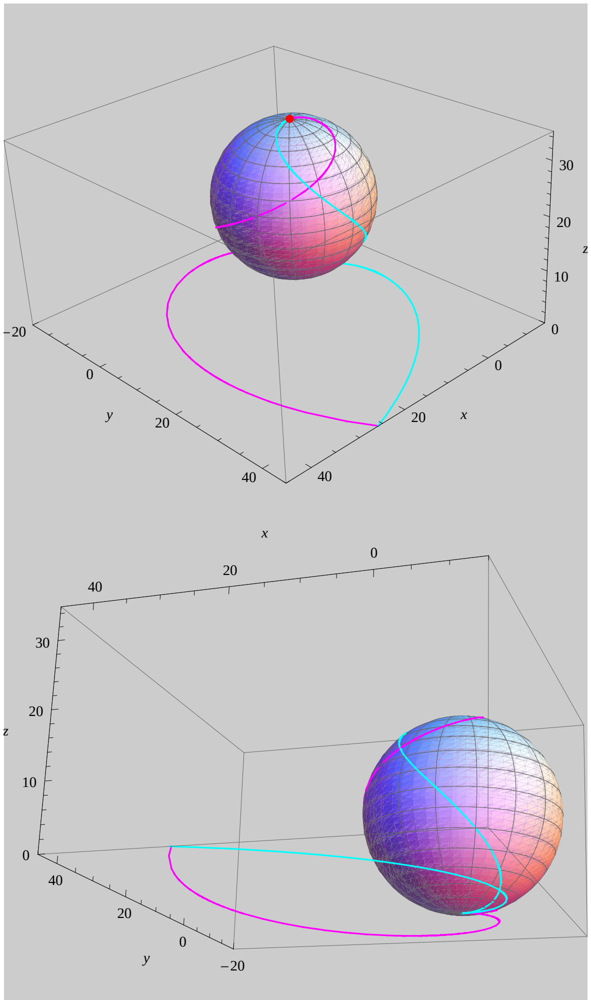


Slika 5: *Paralela i njezina projekcija.*



Slika 6: *Meridijan i njegova projekcija.*

Na slici 7 prikazane su dvije krivulje k_1 i k_2 na sferi i njihove projekcije. Krivulja k_1 određena je relacijom $\lambda = 2\varphi$, a krivulja k_2 relacijom $\lambda = \pi/2 - \varphi$.

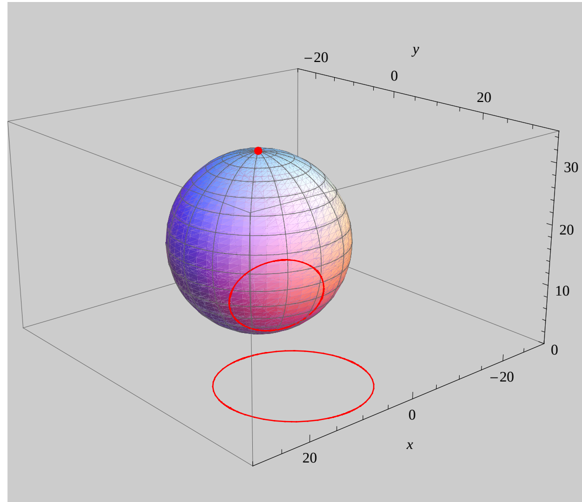


Slika 7: *Krivulje k_1 (magenta) i k_2 (cyan) na sferi i njihove projekcije prikazane s dva različita pogleda.*

Ako je krivulja u *xy*–ravnini zadana svojim parametarskim jednadnadžbama, tada se pomoću jednadžbi (3) mogu odrediti parametarske jednadžbe njenog originala na sferi. Na taj su način, za kružnicu određenu parametarskim jednadžbama

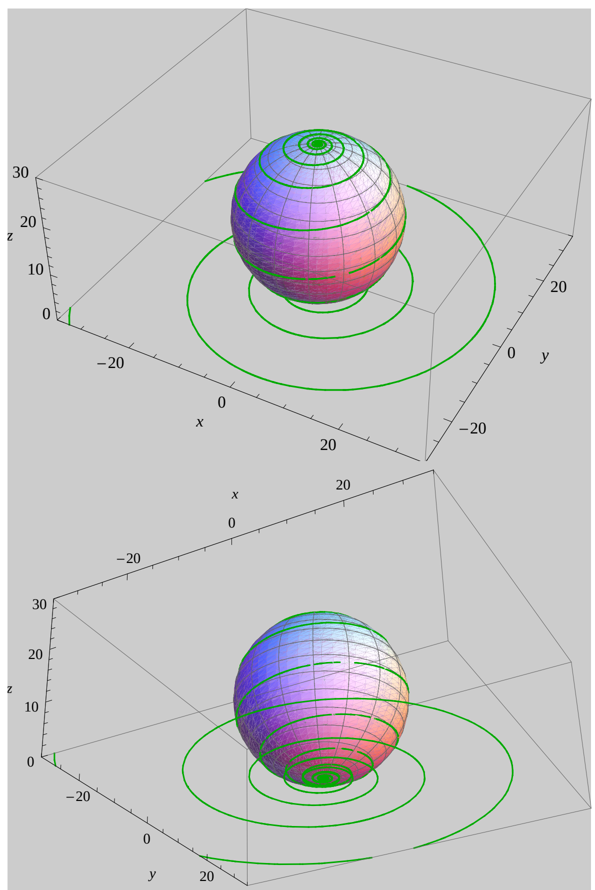
$$x = 12\cos t - 10, \quad y = 12\sin t - 20, \quad t \in [0, 2Pi],$$

određene parametarske jednadžbe njezina originala na sferi polumjera $R = 15$. Obje su krivulje prikazane na slici 8.

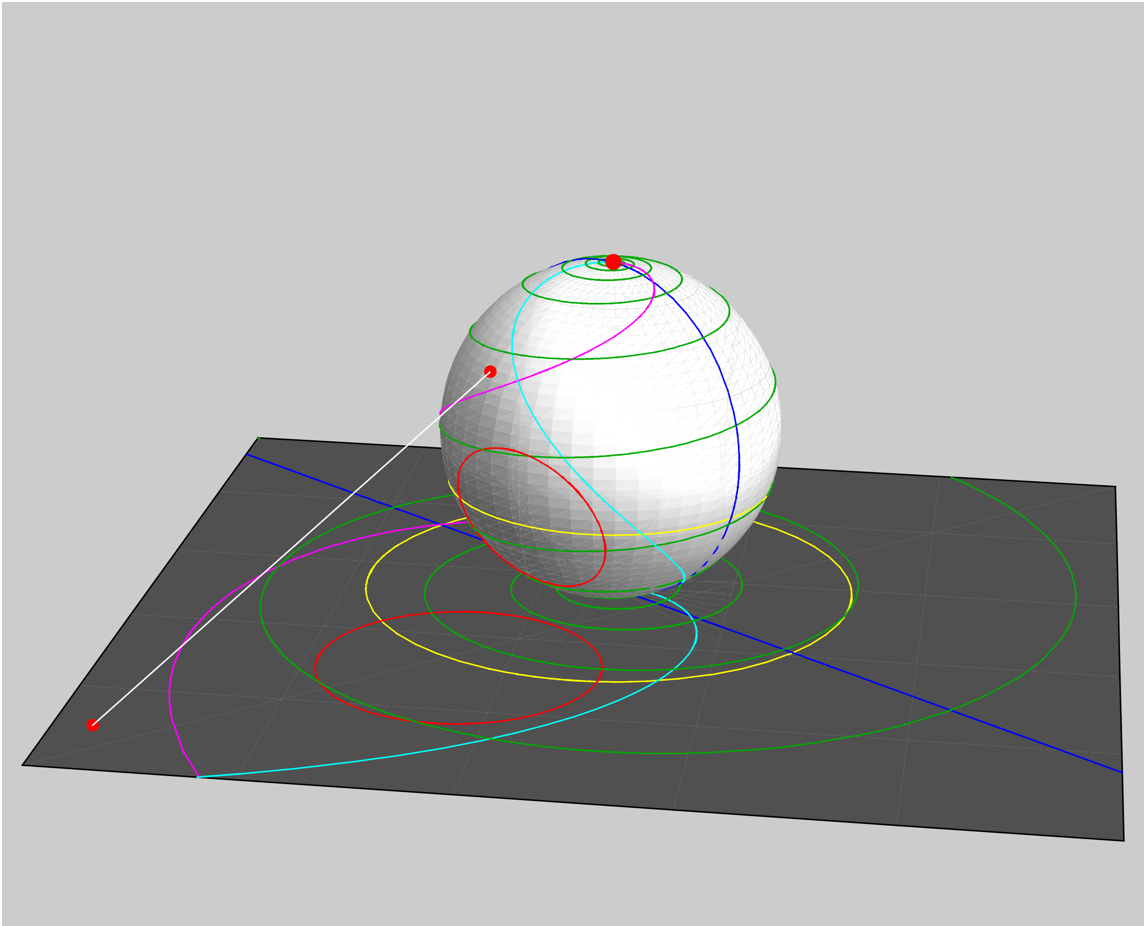


Slika 8: *Kružnica u xy–ravnini i njezin original na sferi.*

Loksodroma je krivulja na sferi koja sve njezine meridijane siječe pod istim kutom.



Slika 9: *Loksodroma koja meridijane siječe pod kutom* $\alpha = \arctan 10$ *i njezina projekcija u xy–ravnini prikazane s dva različita pogleda.*



Slika 10: *Sfera, ravnina projekcije i sve ranije nacrtane krivulje prikazane na istom Mathematica crtežu.*

7 Zaključak

Kod kartografskih projekcija, u ovom slučaju stereografske polarne projekcije govorimo o preslikavanju točke s fizičke površine Zemlje na trodimenzionalni oblik (elipsoid ili sferu), kojeg dalje preslikavamo u ravninu. Za prikaz navednog preslikavanja korišten je programski paket *Mathematica* 6.0 koji pruža 3D- vizualizaciju problema s mogućnošću promjene kuta gledanja. Uz *Mathematicu*, koristimo i program iz skupine *Desktop Screen Recorders-a* u svrhu dodatnog približavanja sadržaju, čineći shvaćanje i savladavanje teme *stereografska polarna projekcija* jednostavnijim i efikasnijim.

Prisutan je i pojam *samoučenja* jer se uz implementaciju obrađenog sadržaja na web, što se bez većih problema može realizirati, studentu omogućuje samostalno pristupanje, analiziranje i konačno shvaćanje problema s kojim se susreće u rješavanju zadatka.

Ovaj oblik prikupljanja novih znanja, spoznaja, je relativno novi u sustavu obrazovanja jer se radi o jednoj vrsti osuvremenjivanja načina učenja, sukladno tome kako Bolonjski proces i nalaže. Očituje se važna primjena ICT-a u nastavnim aktivnostima gdje je važno ispravno korištenje raznih vještina i tehnologija.

Ovaj rad nije isključivo nametanje novog načina pristupa učenju, više sugestija uz prikaz prednosti koje donosi.

Literatura

[1] BORČIĆ, D. , *Matematička kartografija (Kartografske projekcije)*, udžbenik, Tehnička knjiga, Zagreb, 1995.

[2] CAR, V., DRAGUN, D., *Učiti na drugi način - upotreba multimedijskog i interaktivnog sadržaja*, studentski rad nagrađen Dekanovom nagradom, Geodetski fakultet, Zagreb, 2008.

[3] FRANČULA, N., *Kartografske projekcije*, skripta Geodetskog fakulteta, Sveučilište u Zagrebu, 2004.

[4] JOVIČIĆ, D., *Praktikum - Matematika III*, web skripta Geodetskog fakulteta, Sveučilište u Zagrebu, 2003.

[5] SNYDER, J. P., *Flattening the Earth*, The University of Chicago Press, 1997.

[6] *Strategija e - učenja* 2007. - 2010., Sveučilište u Zagrebu, 2007.

[7] WOLFRAM, S., *Mathematica*–Second edition, Addison-Wesley, 1993.

[8] WOLFRAM, S., *MathWorld*, web enciklopedija (http://mathworld.wolfram.com/)

[9] http://en.wikipedia.org/wiki/Stereographic_projection

[10] http://digilander.libero.it/diogenes99/Cartografia/Cartografia02.htm

[11] http://www.progonos.com/furuti/MapProj/Normal/ProjAz/projAz.html

[12] http://www.thesilver.net/

Vedran Car
e-mail: 7vcar7@gmail.com

Dino Dragun
e-mail: ddragun@geof.hr

Jelena Beban-Brkić
e-mail: jbeban@geof.hr

Geodetski fakultet Sveučilišta u Zagrebu
Kačićeva 26, 10000 Zagreb

Zahvala: Najljepše se zahvaljujemo mr. sc. Draženu Tutiću i dr. sc. Sonji Gorjanc na korisnim stručnim savjetima.

GUIDE FOR AUTHORS

SCOPE. "KoG" publishes scientific and professional papers from the fields of geometry, applied geometry and computer graphics.

SUBMISSION. Scientific papers submitted to this journal should be written in English or German, professional papers should be written in Croatian, English or German. Only unpublished material can be accepted.
The manuscript should be delivered electronically to the one of the editors.

Sonja Gorjanc	Jelena Beban - Brkić
sgorjanc@grad.hr	jbeban@geof.hr
Faculty of Civil Engineering	Faculty of Geodesy

Kačićeva 26, 10000 Zagreb, Croatia

The first page should contain the article title, author and coauthor names, affiliation, a short abstract in English, a list of keywords and the Mathematical subject classification.

ELECTRONIC FORMATS. Accepted papers should be sent by electronic mail as ASCII files (LaTeX format is recommended) to the address: sgorjanc@grad.hr

OFFPRINTS. The total of 20 reprints of each contribution will be sent to its first mentioned author (if not otherwise desired) free of charge.

How to get KoG?

The easiest way to get your copy of KoG is by contacting the editor's office:

Nikoleta Sudeta
nsudeta@arhitekt.hr
Faculty of Architecture
Kačićeva 26, 10 000 Zagreb, Croatia
Tel: (+385 1) 4639 219
Fax: (+385 1) 4639 465

The price of the issue is €15 + mailing expenses €5 for European countries and €10 for other parts of the world.

The amount is payable to:

ACCOUNT NAME: Hrvatsko društvo za geometriju i grafiku
Kačićeva 26, 10000 Zagreb, Croatia

IBAN: HR862360000-1101517436

UPUTE ZA AUTORE

PODRUČJE. "KoG" objavljuje znanstvene i stručne radove iz područja geometrije, primijenjene geometrije i računalne grafike.

UPUTSTVA ZA PREDAJU RADA. Znanstveni radovi trebaju biti napisani na engleskom ili njemačkom jeziku, a stručni na hrvatskom, engleskom ili njemačkom. Rad treba biti neobjavljen.
Rukopis treba biti poslan u elektroničkom obliku na adresu jedne od urednica:

Sonja Gorjanc	Jelena Beban - Brkić
sgorjanc@grad.hr	jbeban@geof.hr
Građevinski fakultet	Geodetski fakultet

Kačićeva 26, 10000 Zagreb

Prva stranica treba sadržavati naslov rada, podatke o autoru i koautorima, sažetak na hrvatskom i engleskom, ključne riječi i MSC broj.

ELEKTRONIČKI FORMATI. Prihvaćene radove autori dostavljaju elektronskom poštom kao ASCII datoteke (preporučuje se LaTeX format) na adresu: sgorjanc@grad.hr

POSEBNI OTISCI. Autori dobivaju 20 otisaka svog rada koji se šalju prvom koautoru, ukoliko nije drukčije dogovoreno.

Kako nabaviti KoG?

KoG je najjednostavnije nabaviti u uredništvu časopisa:

Nikoleta Sudeta
nsudeta@arhitekt.hr
Arhitektonski fakultet
Kačićeva 26, 10 000 Zagreb
Tel: (01) 4639 219
Fax: (01) 4639 465

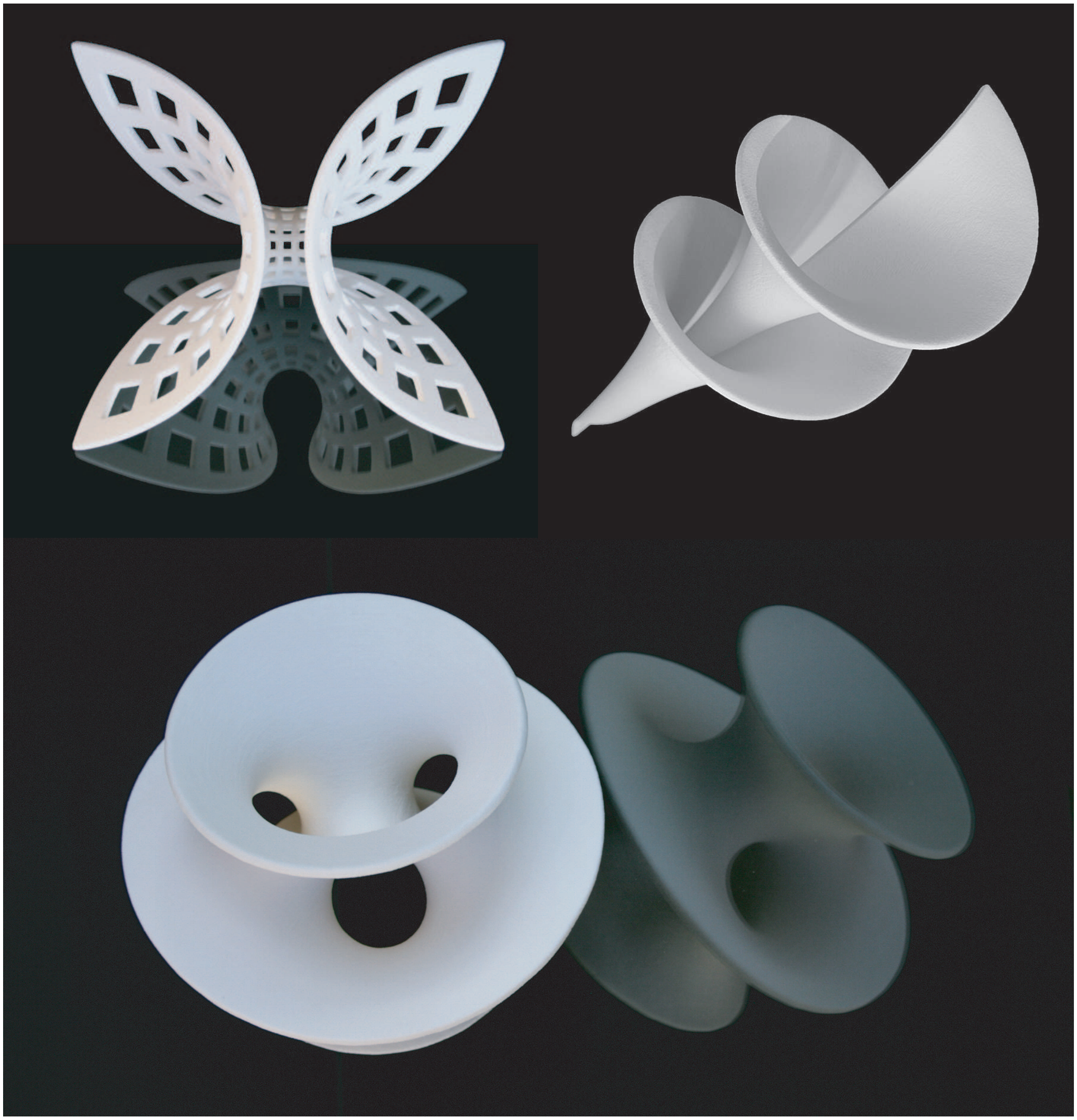
Za Hrvatsku je cijena primjerka 100 KN + 10 KN za poštarinu.

Nakon uplate za:

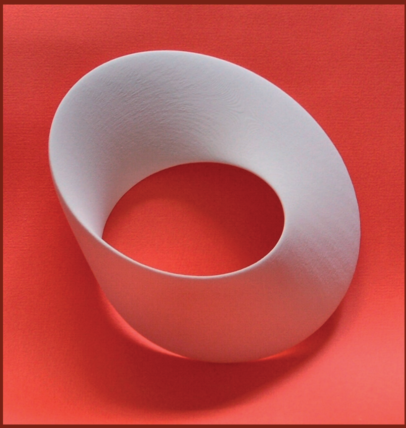
HDGG (za KoG), Kačićeva 26, 10000 Zagreb
žiro račun broj **2360000-1101517436**

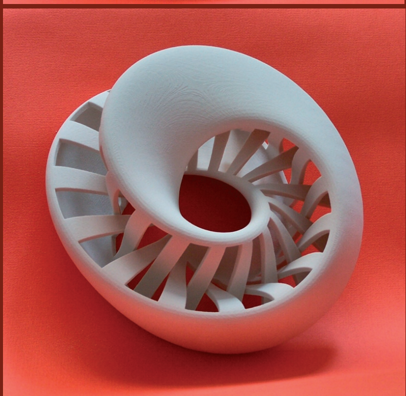
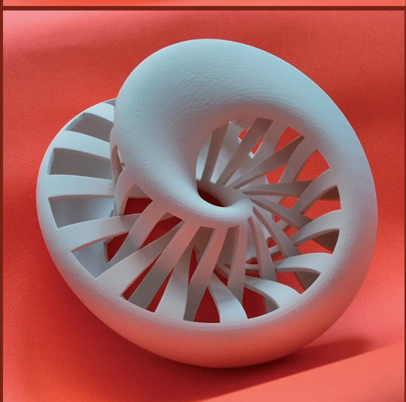
poslat ćemo časopis na Vašu adresu.

Ako Vas zanima tematika časopisa i rad našega društva, preporučamo Vam da postanete članom HDGG (godišnja članarina iznosi 100 KN). Za članove društva časopis je besplatan.



The editors express their gratitude to Prof. Gregorio Franzoni from the University of Cagliari
for providing beautiful photos of his 3D prints for this issue of KoG.



FROM MÖBIUS TO KLEIN

ISSN 1331-1611

9 771331 161005