The Golden Ratio within Regular Polyhedra and Geometrical Construction

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Abstract

We say that line segment is divided into the golden ratio if the ratio between the length of its longer part and its smaller part is equal to the ratio between segment's entire length and the longer part. This well known value, usually denoted by the Greek letter φ, is the irrational number $\frac{1+\sqrt{5}}{2}$. In addition to the golden rectangle, the golden ratio appears in regular pentagon and regular icosahedron. Here we present a construction named golden window where one can find a few golden proportions. Some further constructions are also discussed. While the golden ratio is well understood in natural sciences and mathematics, it also appears in social areas including economy. Possibly it plays an important role in the stock market by means to determine critical points that cause an asset's price to reverse. Presented figures are calculated and drawn in GeoGebra by the author.

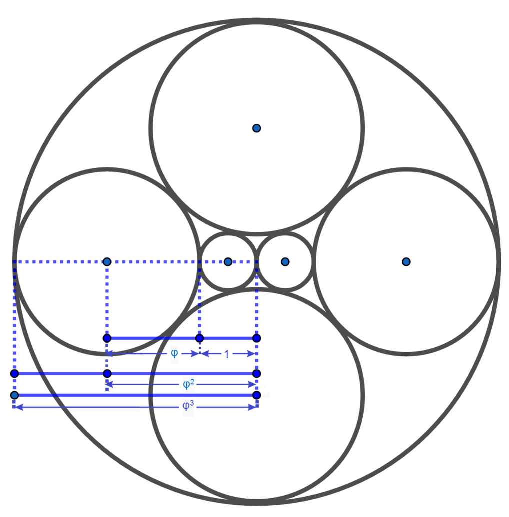
Golden window

Steps of construction:

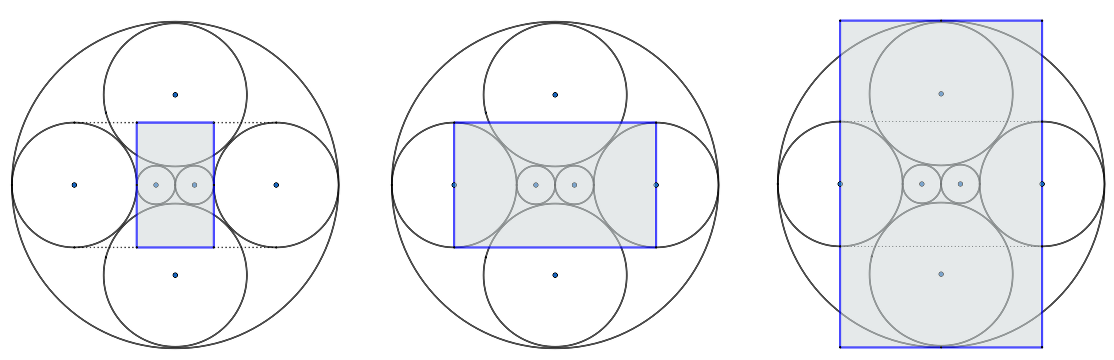
1. start with two small central tangential circles of unit diameter
2. draw two pairs of circles on their left and right and their top and bottom side, such that there exist a pair of congruent circles that are simultaneously tangent to all other circles

Which radius R two circles on their left and right must have such that external circle is simultaneously tangential to all four circles?

The answer is $R = \varphi$ and only for that R this construction with tangential circles of unit diameter is possible.



configuration of circles that is replete with the golden ratio and its powers named golden window



the occurence of various golden rectangles in the construction

Tilings

Tiling is a way of covering a flat surface with smaller shapes or tiles, with no gaps or overlaps, subject to a couple of rules. In creating a set of two symmetrical tiles, each of which is the combination of the two triangles found in the geometry of the pentagon, the relationship of the sides of the pentagon and also the tiles, is φ, 1 and $\frac{1}{\varphi}$. Some examples of tilings with occurence of the golden ratio one can find in the Tilings Encyclopedia.

References

[1] R. Dunlap, *The Golden Ratio and Fibonacci Numbers*, Singapore, World Scientific, 1997.

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[3] D. Pedoe, *Geometry, A comprehensive course*, Dover Publications, inc.,1988.

[4] D. Perkins, *Phi, pi, e and i*, MAA Press, 2017.

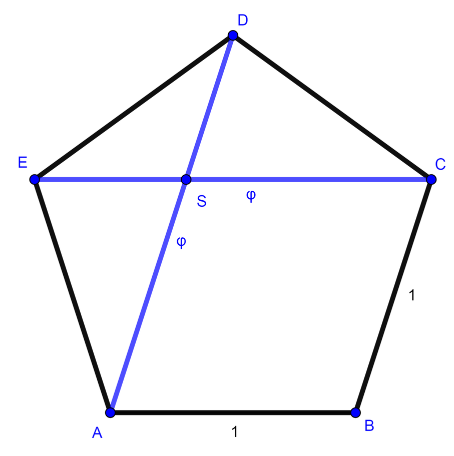
Properties of the golden ratio

It has the unique characteristic in that it differs from its reciprocal by 1, that is, $\varphi - \frac{1}{\varphi} = 1$. Fundamental properties of the golden ratio are

$$\varphi^n = \varphi^{n-1} + \varphi^{n-2} \quad \text{and} \quad \varphi^n = F_n\varphi + F_{n-1},$$

where F_n denotes the n-th Fibonacci number.

Golden geometric figure



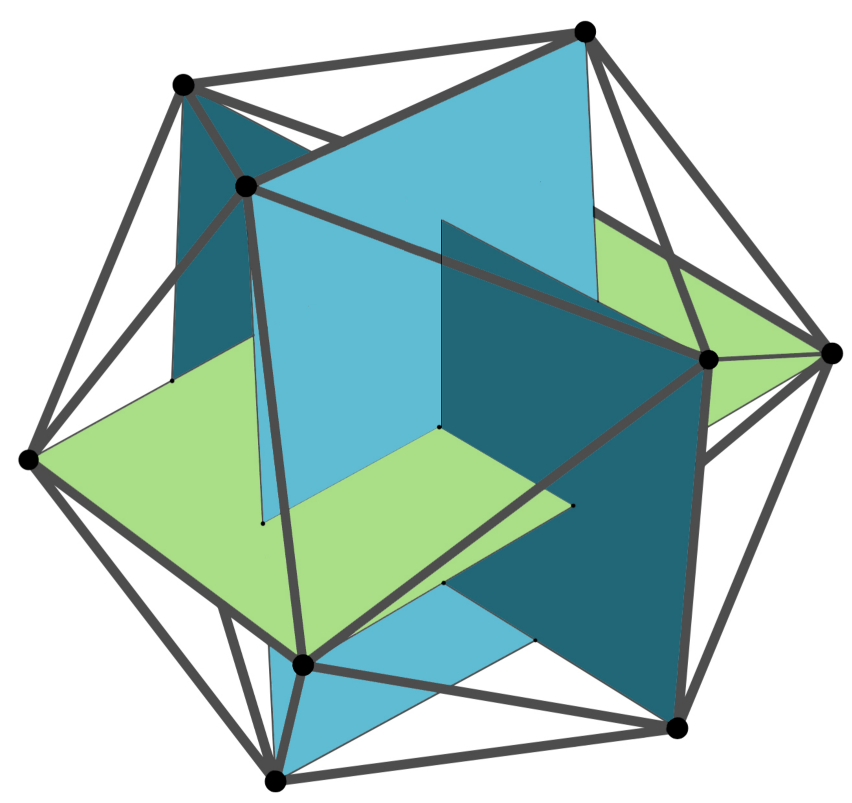
regular pentagon

Theorem
If a regular pentagon has sides of length 1, then φ is the length of each of its diagonals.

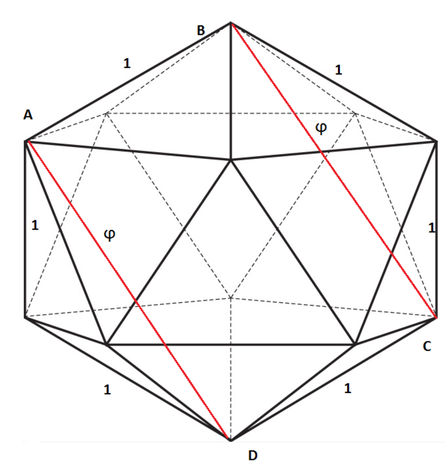
Proof:
Because the pentagon is regular, diagonal EC is parallel to side AB and also is AD parallel to BC. Thus $ABCS$ is a rhombus. Because sides of rhombus have length 1, we know $ES = EC - 1$. Because $\triangle ESD \sim \triangle ECD$ we know $ES : 1 = 1 : EC$. Thus, $ES \cdot EC = 1$. By supstituting $ES = EC - 1$, we have $EC(EC - 1) = 1$. Considering only a positive root, diagonal EC has a length $EC = \frac{1+\sqrt{5}}{2}$.

The Golden Ratio within Regular Polyhedra

Three golden rectangles may be arranged so that they are mutually perpendicular and their centers are coincident. Four vertices for each of the three rectangles lie at the vertices of an regular icosahedron and longer sides of the rectangles coincide with the diagonals of the regular planar pentagons. The regular icosahedron contains 15 such golden rectangles, each one meeting two others at right angles. Thus the golden ratio here is the ratio of the length of diagonal in a regular pentagon to length of edge of regular icosahedron.

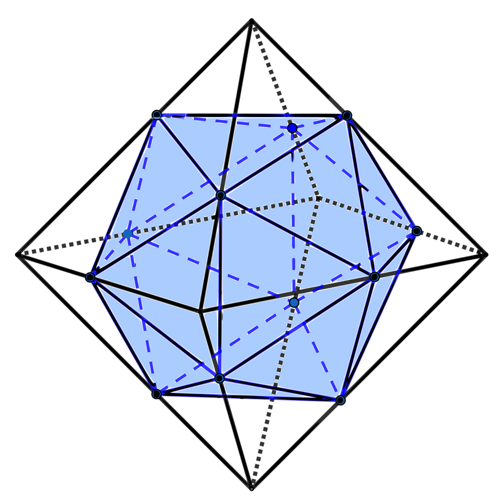


regular icosahedron with three mutually perpendicular golden rectangles

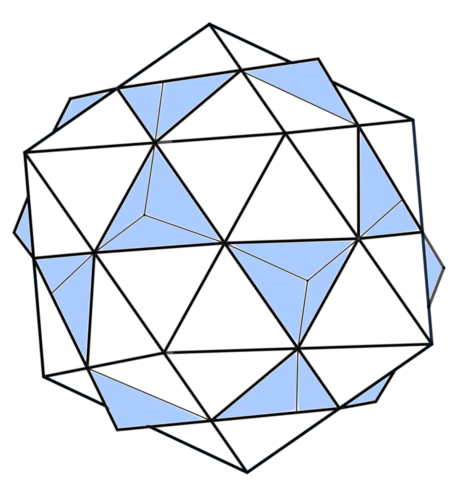


If regular icosahedron has an edge length of 1, then resulting golden rectangles have dimensions 1 by φ.

Any polyhedron can be associated with a second dual figure such that vertices of one correspond to the faces of the other and the edges between pairs of vertices of one correspond to the edges between pairs of faces of the other.



vertex-to-edge relationship between regular icosahedron and regular octahedron



edge-to-edge relationship between regular dodecahedron and regular icosahedron

Vertices of the regular icosahedron divides the edges of the regular octahedron into the golden ratio. Also, value of the edge length ratio obtained in edge-to-edge relationship between regular dodecahedron and regular icosahedron is $\frac{1}{\varphi}$.