Some notes on generalized Pell graphs

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Outline

Fibonacci and Lucas numbers

- The Fibonacci numbers are defined by the recurrence relation

$$F_n = F_{n-1} + F_{n-2}, \quad n \geq 2$$

  with the initial conditions $F_0 = 0$ and $F_1 = 1$.
- The Lucas numbers satisfy the same recurrence relation as Fibonacci numbers

$$L_n = L_{n-1} + L_{n-2}, \quad n \geq 2$$

  but begin with the initial conditions $L_0 = 2$ and $L_1 = 1$.

Combinatorial interpretation (Benjamin and Quinn, 2003)

- F_{n+1} counts the number of tilings of an n-board using squares and dominoes.
- 

$\rightarrow (1 \times n)$-board
- 

$\rightarrow (1 \times 1)$-square
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$\rightarrow (1 \times 2)$-domino

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$\rightarrow (1 \times n)$-board
- s $\rightarrow (1 \times 1)$-square
- d $\rightarrow (1 \times 2)$-domino
- L_n counts the number of circular tilings of an n-board using squares and dominoes.

Hypercubes

- The hypercube graph Q_n of dimension n is one of the most famous models for interconnection networks.
  $V(Q_n) = \{b_1 b_2 ... b_n | b_i \in \{0, 1\}\}$, $|V(Q_n)| = 2^n$
  $E(Q_n) = \{(u, v) | u, v \in V(Q_n), d_H(u, v) = 1\}$, $|E(Q_n)| = n2^{n-1}$

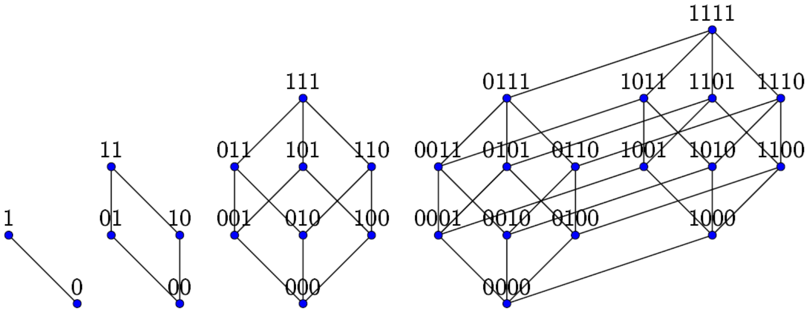


Figure: Hypercubes Q_1, Q_2, Q_3 and Q_4.

Fibonacci cubes (Hsu, 1993)

- The n dimensional Fibonacci cube Γ_n is obtained by removing vertices in that have two consecutive 1s in its binary labeling.

$$V(\Gamma_n) = \{b_1 b_2 ... b_n | b_i \in \{0,1\}, b_i b_{i+1} = 0\}$$
$$E(\Gamma_n) = \{(u,v) | u,v \in V(\Gamma_n), d_H(u,v) = 1\}$$

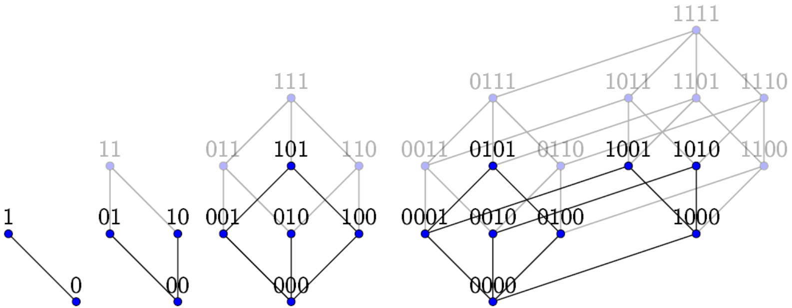
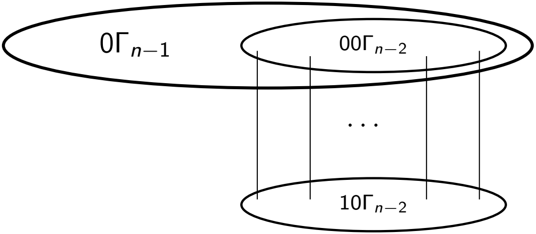


Figure: Fibonacci cubes $\Gamma_1, \Gamma_2, \Gamma_3$ and Γ_4.

Fibonacci cubes

- $|V(\Gamma_n)| = F_{n+2}$
- $|E(\Gamma_n)| = \frac{nF_{n+1}+2(n+1)F_n}{5}$
- $\Gamma_n = 0\Gamma_{n-1} \oplus 10\Gamma_{n-2}, \quad n \geq 2$



The fundamental decomposition of Γ_n

Lucas cubes (Munarini et al., 1999)

- The *Lucas cube* Λ_n is the graph with
  $V(\Lambda_n) = \{b_1 b_2 ... b_n | b_i \in \{0,1\}, b_i b_{i+1} = 0, b_1 b_n = 0\}$
  $E(\Lambda_n) = \{(u,v) | u,v \in V(\Lambda_n), d_H(u,v) = 1\}$

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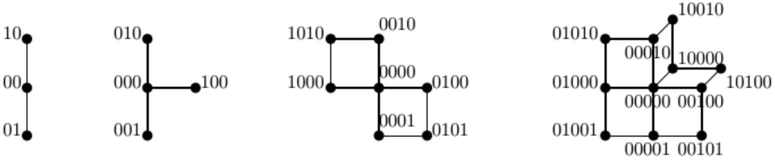
  $E(\Lambda_n) = \{(u,v) | u, v \in V(\Lambda_n), d_H(u,v) = 1\}$
- $|V(\Lambda_n)| = L_n$
- $|E(\Lambda_n)| = nF_{n-1}$
- $\Lambda_n = 0\Gamma_{n-1} \oplus 10\Gamma_{n-3}0, \quad n \geq 3$



Pell Graphs (Munarini, 2019)

- A *Pell string* is a word over the alphabet $T = \{0, 1, 2\}$ such that there are no runs of 2s of odd length. Equivalently, a *Pell string* is a word over the alphabet $T^{'} = \{0, 1, 22\}$.

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- Let $\mathcal{P}_n$ denote the set of *Pell strings* of length n. Then $\mathcal{P}_0 = \{\varepsilon\}, \mathcal{P}_1 = \{0, 1\}$ and for $n \geq 0$,

$$\mathcal{P}_{n+2} = 0\mathcal{P}_{n+1} + 1\mathcal{P}_{n+1} + 22\mathcal{P}_n$$

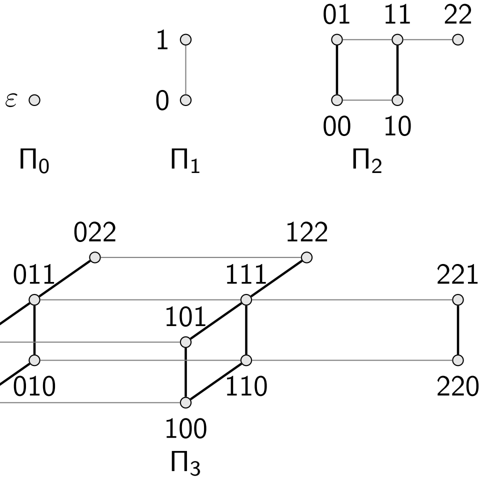
Thus $|\mathcal{P}_n| = P_{n+1}$, where P_n is the nth Pell number defined by

$$P_n = 2P_{n-1} + P_{n-2}, \quad n \geq 2$$

with the initial conditions $P_0 = 0$ and $P_1 = 1$.

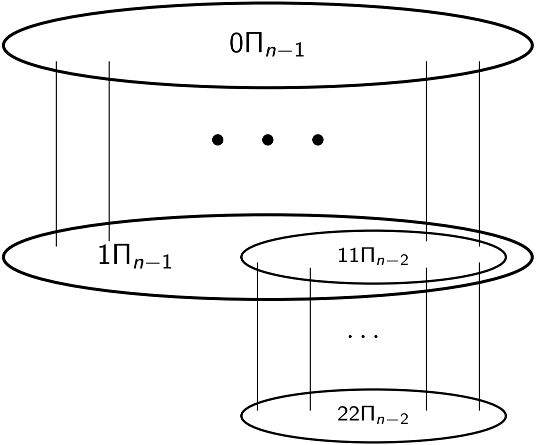
Pell Graphs

- The *Pell* graph Π_n has $V(\Pi_n) = \mathcal{P}_n$ and adjacency: $0 \leftrightarrow 1$ or $11 \leftrightarrow 22$



Pell Graphs

- $\Pi_n = 0\Pi_{n-1} \oplus 1\Pi_{n-1} \oplus 22\Pi_{n-2}, \;\; n \geq 2$



The fundamental decomposition of Π_n

Generalized Pell Graphs (Klavzar et al., 2023)

- For $k \geq 2$, *the generalized Pell string* is a string over the alphabet $\{0, 1, ..., k-1, kk\}$ such that each run of k is even length.
  When $k = 2$, we get the *Pell string*.

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  When $k = 2$, we get the *Pell string*.
- Let $\mathcal{F}_{n,k}$ denote the set of *generalized Pell strings* of length n. Then Clearly, $\mathcal{F}_{0,k} = \{\varepsilon\}$ and $\mathcal{F}_{1,k} = \{0, 1, ..., k-1\}$, while for $n \geq 2$ we have

$$\mathcal{F}_{n,k} = 0\mathcal{F}_{n-1,k} + 1\mathcal{F}_{n-1,k} + ... + (k-1)\mathcal{F}_{n-1,k} + kk\mathcal{F}_{n-2,k}.$$

- $|\mathcal{F}_{n,k}| = F_{n+1,k}$ where $F_{n,k}$ is the nth k-Fibonacci number defined by

$$F_{n,k} = kF_{n-1,k} + F_{n-2,k}, \quad n \geq 2$$

  with the initial conditions $F_{0,k} = 0$ and $F_{1,k} = 1$.

Generalized Pell Graphs

- The *Generalized Pell graph* $\Pi_{n,k}$ has $V(\Pi_{n,k}) = \mathcal{F}_{n,k}$ and adjacency: $i \leftrightarrow i+1$ for $i \in \{0, 1, ..., k-2\}$ or $(k-1)(k-1) \leftrightarrow kk$

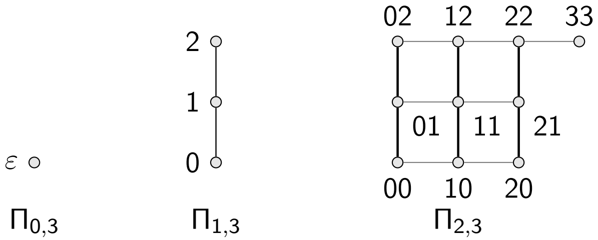


Figure: Generalized Pell graphs $\Pi_{n,3}$ for $n \in \{0, 1, 2\}$.

Generalized Pell Graphs

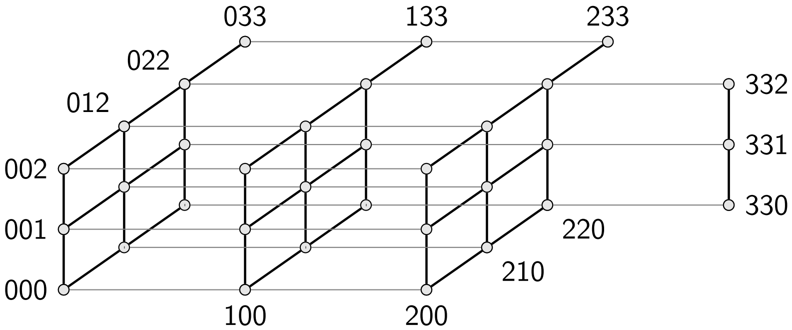


Figure: Generalized Pell graph $\Pi_{3,3}$

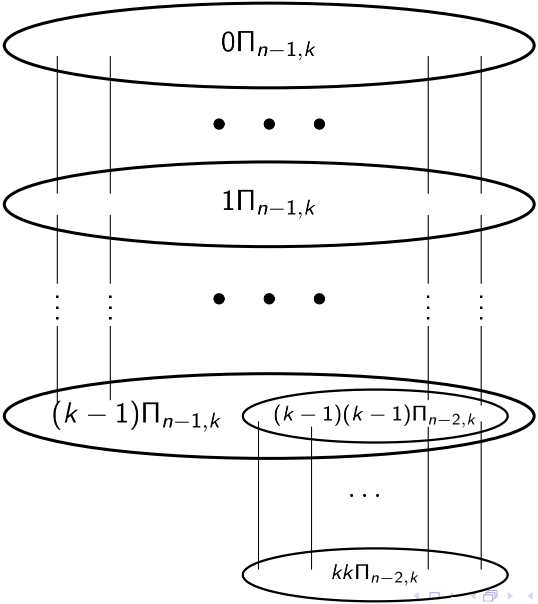
Generalized Pell Graphs

- The fundamental decomposition of the *Generalized Pell graph* $\Pi_{n,k}$ is

$$\Pi_{n,k} = 0\Pi_{n-1,k} \oplus 1\Pi_{n-1}, k \oplus ... \oplus (k-1)\Pi_{n-1,k} \oplus kk\Pi_{n-2,k}$$

with $\Pi_{0,k} = K_1$ and $\Pi_{1,k}$ is the path on k vertices.

Generalized Pell Graphs



The number of edges

- From the fundamental decomposition of $\Pi_{n,k}$, the edges of $\Pi_{n,k}$ are of the following four types:
  (i) edges from k copies of $\Pi_{n-1,k}$
  (ii) edges from $\Pi_{n-2,k}$
  (iii) the link edges between the vertices in the k copies of $\Pi_{n-1,k}$
  (iv) the link edges between the vertices in $kk\Pi_{n-2,k}$ and $(k-1)(k-1)\Pi_{n-2,k}$

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  (iv) the link edges between the vertices in $kk\Pi_{n-2,k}$ and $(k-1)(k-1)\Pi_{n-2,k}$
- For $n \geq 2$,

$$|E(\Pi_{n,k})| = k|E(\Pi_{n-1,k})| + |E(\Pi_{n-2,k})| + (k-1)F_{n,k} + F_{n-1,k}$$

with $|E(\Pi_{0,k})| = 0$ and $|E(\Pi_{1,k})| = k - 1$.

The number of edges

- The generating function of the number of edges in $\Pi_{n,k}$ is

$$\sum_{n\geq 0} |E(\Pi_{n,k})| t^n = \frac{(k-1+t)t}{(1-kt-t^2)^2}$$

- $$|E(\Pi_{n,k})| = \sum_{i=0}^{n} F_{i,k}(F_{n-i+2,k} - F_{n-i+1,k})$$

Cube Polynomial

- The cube polynomial of a graph G, denoted by $C_G(x)$, is the counting polynomial defined by the generating function

$$C_G(x) = \sum_{i \geq 0} c_i(G) x^i$$

where $c_i(G)$ counts the number of induced i-cubes in G. Clearly, $c_0(G) = |V(G)|$, $c_1(G) = |E(G)|$.

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- The first few cube polynomials of $\Pi_{n,k}$ are listed below:
- $C_{\Pi_{0,k}}(x) = 1$
- $C_{\Pi_{1,k}}(x) = k + (k-1)x$
- $C_{\Pi_{2,k}}(x) = k^2 + 1 + (2k^2 - 2k + 1)x + (k^2 - 2k + 1)x^2$
- $C_{\Pi_{3,k}}(x) = k^3 + 2k + (3k^3 - 3k^2 + 4k - 2)x + (3k^3 - 6k^2 + 5k - 2)x^2 + (\mathrm{k}^3 - 3k^2 + 3k - 1)x^3$

Cube Polynomial

- The cube polynomials $C_{\Pi_{n,k}}(x)$ satisfy the recurrence relation

$$C_{\Pi_{n,k}}(x) = (k + (k-1)x)\,C_{\Pi_{n-1,k}}(x) + (1+x)\,C_{\Pi_{n-2,k}}(x),\, n \geq 2$$

with the initial values $C_{0,k}(x) = 1$ and $C_{\Pi_{1,k}}(x) = k + (k-1)x$.

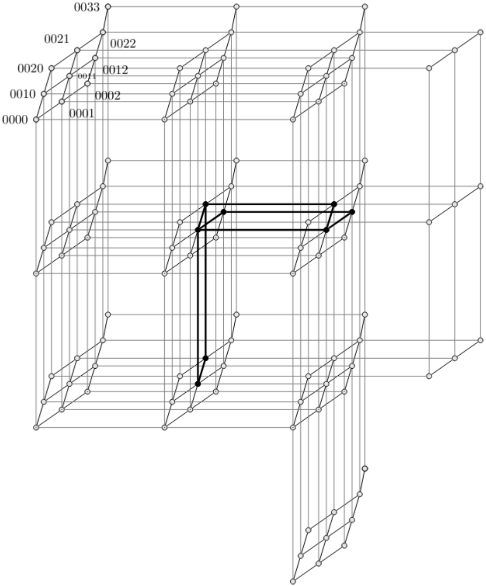
- $$C_{\Pi_{n,k}}(x) = \sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n-i}{i} \left(k + (k-1)\,x\right)^{n-2i} (1+x)^{i}.$$

Some graph theoretic properties

- The graph $\Pi_{n,k}$ has a Hamiltonian path.
- $\Pi_{n,k} \subseteq \Gamma_{(2k-2)n-1}$
- $eccentricity(u) = max\ d(u, v)$ for all $v \in V(\Pi_{n,k})$
  $diam(\Pi_{n,k}) = nk - \lfloor \frac{n}{2} \rfloor$
  $rad(\Pi_{n,k}) = \lfloor \frac{nk}{2} \rfloor$
- The center consists of vertices whose eccentricity equals the radius.

$$|C(\Pi_{n,k})| = \begin{cases} F_{n+2}; & k \text{ even}, \\ (n+4)2^{\frac{n}{2}-2}; & k \text{ odd and } n \text{ even}, \\ 2^{\frac{n-1}{2}}; & k \text{ odd and } n \text{ odd}. \end{cases}$$

Some graph theoretic properties



References

- Iršič V., Klavžar S., Tan E. Generalized Pell graphs, Turkish Journal of Mathematics, 47(7), 1955-1973, 2023.

Thank You

Thank you for your attention!
Teşekkür ederim!

The 22nd International Fibonacci Conference will be held in Istanbul in July 2026!