Character type constructions and 3-dimensional Hadamard cubes*

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5th Croatian Combinatorial Days

19 - 20 Sep 2024, Zagreb, Croatia

* This work was fully supported by the Croatian Science Foundation under the project 9752.

Motivation and definitions

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The set of all Hadamard maps on a group G is denoted by $\mathcal{H}ad(G)$.

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Theorem. ($\mathcal{H}ad(|G|, 3) \Leftrightarrow$ character type subset)

Let G be an abelian group of order $2n$ and let $K < G$ be a subgroup of index 2. Then $\psi \in \mathcal{H}ad(G)$ (modulo K) if and only if $\chi\psi$ is from character type subset of $\mathcal{H}ad(2n, 3)$, where χ be a character on G with a kernel K.

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$\{\varphi_{ig} \mid g \in G\}$ is a $(4n^2, 2n^2, 2n)$-net for any $i \in [3]$.

Theorem. ($\mathcal{H}ad(G) \Leftrightarrow$ net cube)

Let G be an abelian group of even order and let $K < G$ be a subgroup of index 2 such that $G/K = \langle aK \rangle$. Let $\psi : G^3 \to G$ be a map where $\psi(x, y, z) = a^{\varphi(x,y,z)}k(x, y, z)$, such that $\varphi(x, y, z) \in \mathbb{F}_2$ and $k(x, y, z) \in K$ for $(x, y, z) \in G^3$. Then $\psi \in \mathcal{H}ad(G)$ if and only if φ is $(4n^2, 2n^2, 2n)$-net cube.

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The End

Thank you!