Geometry of point particles. Symbolic computer verification of the Atiyah's Conjecture for five points in the Euclidean plane

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Outline

Abstract

In 2001 Sir M. F. Atiyah formulated a conjecture $C1$ and later with P. Sutcliffe two stronger conjectures $C2$ and $C3$. These conjectures, inspired by physics (spin-statistics theorem of quantum mechanics), are geometrically defined for any configuration of n points in the Euclidean three space. The conjecture $C1$ is proved for $n = 3$ in [1] and for $n = 4$ in [2], and $C1 - C3$ in [3]. After two decades we succeeded in verifying $C1$ for arbitrary five points in the Euclidean plane. The computer symbolic certificate produces a new remarkable universal ('hundred pages long') positive polynomial invariant (for any five planar points), in terms of newly discovered shear coordinates. This refines the original Atiyah's conjecture and we are optimistic for its verification for n greater than five (less optimistic variant ... 'It remains a conjecture for 300 years (like Fermat)', see Atiyah: Edinburgh Lectures..2010). In 2013. Atiyah's conjectures were put on the new list of nine open problems [4] (hopefully easier than remaining nine millennium problems!).

[1] M. Atiyah. The geometry of classical particles. Surveys in Differential Geometry (International Press) 7 (2001).
[2] M. Eastwood and P. Norbury, A proof of Atiyah's conjecture on configurations of four points in Euclidean three-space. Geometry & Topology 5 (2001), 885-893.
[3] D. Svrtan, A proof of All three Euclidean Four Point Atiyah-Sutcliffe Conjectures, https://emis.de/journals/SLC/wpapers/s73vortrag/svrtan.pdf
[4] Open problems in Honor of Wilfried Schmied
https://legacy-www.math.harvard.edu/conferences/schmid_2013/problems/index.html

Introduction 1/3

In 2001. Sir Michael Atiyah, inspired by physics (Berry–Robbins problem related to spin statistics theorem of quantum mechanics) associated a remarkable determinant to any n distinct points in Euclidean 3–space, via elementary construction.
More generally, let $(x_1, x_2, \ldots, x_n)$ be n distinct points inside the ball of radius R in Euclidean 3–space.

Definition.

Let the oriented line $x_i x_j$ meet the boundary 2–sphere in a point (direction) u_{ij} regarded as a point of the complex Riemann sphere $(\mathbb{C} \cup \{\infty\})$.

Form a complex polynomial p_i of degree $n-1$ whose roots are $u_{ij}, j \neq i$ (p_i is determined up to a scalar factor). The Atiyah's conjecture C_1 now reads

Conjecture C_1

For all $(x_1, x_2, \ldots, x_n)$ the n polynomials p_i are linearly independent.

Conjecture C_1 $\Leftrightarrow$ nonvanishing of the determinant D of the matrix of coefficients of the polynomials p_i.
The determinant D can be normalized so that D becomes a continuous function of $(x_1, x_2, \ldots, x_n)$ which is $SL(2, \mathbb{C})$–invariant (using the ball model or upper half space model of hyperbolic 3–space).

Introduction 2/3

The more refined conjectures of Atiyah and Sutcliffe C_2 and C_3 relate D to products of 2 and $n-1$–subsequences of points $x_1, x_2, \ldots, x_n$.

The conjecture C_1 is proved for $n = 3, 4$ and for general n only for some special configurations (M.F. Atiyah, M. Eastwood and P. Norbury, D. Đoković).

In a lengthy preprint [5] we have verified the conjectures C_2 and C_3 for parallelograms, cyclic quadrilaterals and some infinite families of tetrahedra.

Also we proved C_2 for Đoković's dihedral configurations. In [8] a proof of C_1 is given for convex planar quadrilaterals. We have also proposed a strengthening of the conjecture C_3 for configurations of four points (Four Points Conjectures, stronger then some new conjectures in [8]) and a number of conjectures for almost collinear configurations, and proved them for n up to 10.

In 2001. Eastwood and Norbury [3] found an intrinsic formula for the four point Atiyah determinant (a polynomial of sixth degree in six distances having several hundreds of terms) and gave a proof of C_1.

Introduction 3/3

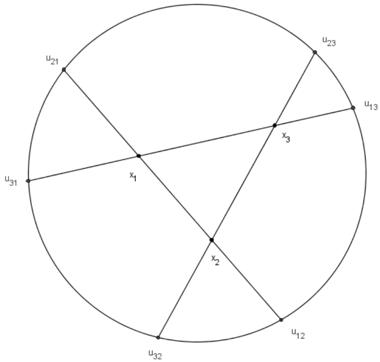
The present author found a new geometric fact for arbitrary tetrahedra which leads to a proof of C_2 and C_3 for arbitrary four points in the euclidean three space (and also a proof of stronger Four Points Conjecture of Svrtan and Urbiha).

Later we obtain another intrinsic polynomial formula a la Eastwood and Norbury for four points (and for five "planar" points – having one hundred thousand terms) and have an existence proof of a polynomial formula for all planar configurations what was conjectured in [3].

This approach produces also trigonometric formulas for four points Atiyah determinants (not involving so called Crelle angles which are used in [8]). Some work is done in the hyperbolic case by finding a hyperbolic analogue of the Eastwood and Norbury formula (in the planar case- spacial case is quite a challenge!).

We also introduce (mixed) Atiyah type energies associated to any graph (on given points) and can prove that Conjecture C_1 is true, for arbitrary n, for some of these energies (work in progress).

3 points inside circle



- Three points x_1, x_2, x_3 inside disk $(|z| \leq R)$
- Three point-pairs on circle
- $P_1 \quad (u_{12})(u_{13})$
- $P_2 \quad (u_{21})(u_{23})$
- $P_3 \quad (u_{31})(u_{32})$
- Point–pair u_{12}, u_{13} define quadratic with roots $p_1 = (z - u_{12})(z - u_{13})$
- 3 point–pairs $\rightarrow 3$ quadratics
- $P_1, P_2, P_3 \rightarrow \{p_1, p_2, p_3\}$

Theorem (Atiyah 2001.)

For any triple x_1, x_2, x_3 of distinct points inside the disk the three quadratics $\{p_1, p_2, p_3\}$ are linearly independent.

Remark: Atiyah's proof, which is synthetic, does not generalize to more than three points.

Normalized determinant D_3

Theorem 1.

3-by-3 determinant of the coefficient matrix:

$$|M_3| = \begin{vmatrix} 1 & -u_{12}-u_{13} & u_{12}u_{13} \\ 1 & -u_{21}-u_{23} & u_{21}u_{23} \\ 1 & -u_{31}-u_{32} & u_{31}u_{32} \end{vmatrix} \neq 0, \quad D_3 = \frac{|M_3|}{(u_{12}-u_{21})(u_{13}-u_{31})(u_{23}-u_{32})}$$

Remark: $D_3 = 1$ only for collinear points.

Theorem 2.

$D_3 \geq 1$.

Remark: Theorem 2. $\Leftrightarrow$ Theorem 1.

Points on the "circle at ∞" are directions on a plane.

Remark: Theorem 1. and Theorem 2. are also true for $R = \infty$.

Explicit formulas for D_3

Extrinsic formula:
$$D_3 = 1 + \frac{(u_{21}-u_{31})(u_{13}-u_{23})(u_{12}-u_{32})}{(u_{12}-u_{21})(u_{13}-u_{31})(u_{23}-u_{32})}$$

Intrinsic formula for hyperbolic triangles $(0 < A+B+C < \pi)$:
$$D_3 = \frac{1}{2}(\cos^2(A/2)+\cos^2(B/2)+\cos^2(C/2)) - \frac{1}{4}\Phi,$$

where
$$\Phi^2 = 4\cos\left(\tfrac{A+B+C}{2}\right)\cos\left(\tfrac{-A+B+C}{2}\right)\cos\left(\tfrac{A-B+C}{2}\right)\cos\left(\tfrac{A+B-C}{2}\right)$$
$$= -1+\cos^2(A)+\cos^2(B)+\cos^2(C)+2\cos(A)\cos(B)\cos(C)$$

Intrinsic formula involving side lengths

$a,\ b,\ c,\ p=(a+b+c)/2,\ p_a=p-a,\ p_b=p-b,\ p_c=p-c$:

$$D_3 = 1+e^{-p}\frac{\sinh(p_a)\sinh(p_b)\sinh(p_c)}{\sinh(a)\sinh(b)\sinh(c)} \quad \left(\to 1+\frac{(-a+b+c)(a-b+c)(a+b-c)}{8abc}\text{ Eucl. case}\right)$$
$$= 1+e^{-(p_a+p_b+p_c)}\frac{(e^{p_a}-e^{-p_a})(e^{p_b}-e^{-p_b})(e^{p_c}-e^{-p_c})}{(e^{p_a+p_b}-e^{-(p_a+p_b)})(e^{p_a+p_c}-e^{-(p_a+p_c)})(e^{p_b+p_c}-e^{-(p_b+p_c)})}$$
$$= 1+\frac{(e^{2p_a}-1)(e^{2p_b}-1)(e^{2p_c}-1)}{(e^{2(p_a+p_b)}-1)(e^{2(p_a+p_c)}-1)(e^{2(p_b+p_c)}-1)}$$

$$D_3 = 1 + \frac{(e^{2p_a}-1)(e^{2p_b}-1)(e^{2p_c}-1)}{(e^{2(p_a+p_b)}-1)(e^{2(p_a+p_c)}-1)(e^{2(p_b+p_c)}-1)}$$

Lemma.

For $0 < a < b$ the function $f(x) = \dfrac{e^{\frac{a}{x}}-1}{e^{\frac{b}{x}}-1}$ $(0 < x < \infty)$ is strictly increasing and $\lim_{x\to\infty} f(x) = \dfrac{a}{b}$.

By using this lemma the recent monotonicity conjecture of Atiyah (in case $n = 3$) follows immediately (if a is replaced by a/R etc... in previous formulas).

$$D_3 = 1 + e^{-p}\frac{\sinh(p_a)\sinh(p_b)\sinh(p_c)}{\sinh(a)\sinh(b)\sinh(c)} = 1 + \frac{e^{-p_a-p_b-p_c}\sinh(p_a)\sinh(p_b)\sinh(p_c)}{\sinh(p_a+p_b)\sinh(p_a+p_c)\sinh(p_b+p_c)}$$

$$= 1 + \frac{(\cosh(p_a+p_b+p_c)-\sinh(p_a+p_b+p_c))\sinh(p_a)\sinh(p_b)\sinh(p_c)}{\sinh(p_a+p_b)\sinh(p_a+p_c)\sinh(p_b+p_c)}$$

$$= 1 + \frac{(1-\tanh(p_a))(1-\tanh(p_b))(1-\tanh(p_c))\tanh(p_a)\tanh(p_b)\tanh(p_c)}{(\tanh(p_a)+\tanh(p_b))(\tanh(p_a)+\tanh(p_c))(\tanh(p_b)+\tanh(p_c))}$$

Equations for Atiyah 3pt energies (1/4)

$$\begin{aligned}
\Delta_3 &= D_3 - 1 = e^{-p}\frac{\sinh(p-a)\sinh(p-b)\sinh(p-c)}{\sinh(a)\sinh(b)\sinh(c)}, \quad \left(p = \frac{a+b+c}{2}\right)\\
\Delta_3^+ &= D_3^+ - 1 = e^{p}\frac{\sinh(p-a)\sinh(p-b)\sinh(p-c)}{\sinh(a)\sinh(b)\sinh(c)}\\
P_1 &= (X-\Delta_3)(X-\Delta_3^+) =\\
&= X^2 - 2\frac{\cosh(p)\sinh(p-a)\sinh(p-b)\sinh(p-c)}{\sinh(a)\sinh(b)\sinh(c)}X + \frac{\sinh^2(p-a)\sinh^2(p-b)\sinh^2(p-c)}{\sinh^2(a)\sinh^2(b)\sinh^2(c)}\\
\Delta_3^{(1)} &= D_3^{001} - 1 = -e^{-(p-c)}\frac{\sinh(p)\sinh(p-a)\sinh(p-b)}{\sinh(a)\sinh(b)\sinh(c)} = -e^{-p+c}\frac{\sinh(p)}{\sinh(c)}\sin^2(A)\\
\Delta_3^{(6)} &= D_3^{110} - 1 = -e^{p-c}\frac{\sinh(p)\sinh(p-a)\sinh(p-b)}{\sinh(a)\sinh(b)\sinh(c)} = -e^{+p-c}\frac{\sinh(p)}{\sinh(c)}\sin^2(A)\\
P_2 &= (X-\Delta_3^{(1)})(X-\Delta_3^{(6)}) =\\
&= X^2 - 2\frac{\sinh(p)\sinh(p-a)\sinh(p-b)\cosh(p-c)}{\sinh(a)\sinh(b)\sinh(c)}X + \frac{\sinh^2(p)\sinh^2(p-a)\sinh^2(p-b)}{\sinh^2(a)\sinh^2(b)\sinh^2(c)}\\
P_3 &= (X-\Delta_3^{(2)})(X-\Delta_3^{(5)}) =\\
&= X^2 - 2\frac{\sinh(p)\sinh(p-a)\cosh(p-b)\sinh(p-c)}{\sinh(a)\sinh(b)\sinh(c)}X + \frac{\sinh^2(p)\sinh^2(p-a)\sinh^2(p-c)}{\sinh^2(a)\sinh^2(b)\sinh^2(c)}\\
P_4 &= (X-\Delta_3^{(3)})(X-\Delta_3^{(4)}) =\\
&= X^2 - 2\frac{\sinh(p)\cosh(p-a)\sinh(p-b)\sinh(p-c)}{\sinh(a)\sinh(b)\sinh(c)}X + \frac{\sinh^2(p)\sinh^2(p-b)\sinh^2(p-c)}{\sinh^2(a)\sinh^2(b)\sinh^2(c)}
\end{aligned}$$

Equations for Atiyah 3pt energies (2/4)

$$P(X) = P_1P_2P_3P_4 = \prod_{i=0}^{7}\left(X - \Delta^{(i)}\right) = X^8 + \cdots + e_7X - e_8$$

$$e_8 = \prod_{i=0}^{7}\Delta^{(i)} = \frac{\sinh^6(p)\sinh^6(p-a)\sinh^6(p-b)\sinh^6(p-c)}{\sinh^6(a)\sinh^6(b)\sinh^6(c)}$$

$$e_8 = \frac{(1-c_1^2)^2(1-c_2^2)^2(1-c_3^2)^2}{4096} = \frac{\sin^4(A)\sin^4(B)\sin^4(C)}{4096}$$

$$e_7 = \frac{1}{256}(1-c_1^2)(1-c_2^2)(1-c_3^2)(-1+2c_1c_2c_3+c_1^2+c_2^2+c_3^2) = \frac{1}{256}\sin^2(A)\sin^2(B)\sin^2(C)\Phi^2$$

$$P(X) = X^8 + 2X^7 + \frac{1}{4}(4+\sigma_1-\Phi^2)X^6 + \frac{1}{4}(\sigma_1 - 2\Phi^2)X^5 - \frac{1}{32}(3\sigma_3 - 2\sigma_2 + (2\sigma_1 - 8)\Phi^2)X^4 -$$
$$- \tfrac{1}{32}(\sigma_3 + 2\sigma_1\Phi^2)X^3 - \tfrac{1}{256}(\sigma_1\sigma_3 + (4\sigma_2 - \sigma_3)\Phi^2)X^2 - \tfrac{1}{256}\sigma_3\Phi^2X + \tfrac{1}{4096}\sigma_3^2$$

where $\sigma_1 = \sin^2(A) + \sin^2(B) + \sin^2(C)$, $\sigma_2 = \sin^2(A)\sin^2(B) + \sin^2(A)\sin^2(C) + \sin^2(B)\sin^2(C)$,
$\sigma_3 = \sin^2(A)\sin^2(B)\sin^2(C)$, $\Phi^2 = 4\cos(\sigma)\cos(\sigma - A)\cos(\sigma - B)\cos(\sigma - C)$, $\sigma = \frac{A+B+C}{2}$

$$\prod_{i=0}^{7} D_3^{(i)} = \frac{1}{128}(\sigma_3 - 2\sigma_2)\Phi^2 + \frac{1}{4096}(\sigma_3^2 - 16\sigma_1\sigma_3 + 256(\sigma_2 - \sigma_3))$$

Equations for Atiyah 3pt energies (3/4)

$$
\begin{aligned}
P_-(X) &= X^4 + (1+\Phi)X^3 + \frac{1}{4}(c_1^2 + c_2^2 + c_3^2 + 3c_1c_2c_3 + 3\Phi)X^2 + \frac{1}{8}(-1 + 2c_1c_2c_3 + c_1^2 + c_2^2 + c_3^2 + \\
&\quad + (1 + c_1c_2c_3)\Phi)X - \frac{1}{64}(1-c_1^2)(1-c_2^2)(1-c_3^2) \\
&= \frac{1}{8}X(2X+\Phi)\left[(2X+1)(2X+1+\Phi) + c_1c_2c_3\right] - \frac{1}{64}(1-c_1^2)(1-c_2^2)(1-c_3^2) \\
&= \frac{1}{8}X(2X+\Phi)\left[(2X+1)(2X+1+\Phi) + \frac{1+\Phi^2-(c_1^2+c_2^2+c_3^2)}{2}\right] - \frac{1}{64}(1-c_1^2)(1-c_2^2)(1-c_3^2) \\
&= \frac{1}{16}X(2X+\Phi)\left[2(2X+1)(2X+1+\Phi) - 2 + s_1^2 + s_2^2 + s_3^2\right] - \frac{1}{64}s_1^2s_2^2s_3^2 \\
P(X) &= P_-(X)P_+(X), \text{ where } P_+(X) = P_-(X)\Big|_{\Phi\to-\Phi}
\end{aligned}
$$

Equations for Atiyah 3pt energies (4/4)

$$
\begin{aligned}
P_{-}(X)\Big|_{\Phi=0} &= \frac{1}{16}X\cdot 2X\left[2(2X+1)^2-2+s_1^2+s_2^2+s_3^2\right]-\frac{1}{64}s_1^2s_2^2s_3^2\\
&= X^3(X+1)+\frac{1}{8}(s_1^2+s_2^2+s_3^2)X^2-\frac{1}{64}s_1^2s_2^2s_3^2\\
P(0) &= \frac{1}{4096}s_1^4s_2^4s_3^4\\
\prod_{i=0}^{7}D_3^{(i)} &= P(-1)=\left[\frac{1}{16}(-1)(-2+\Phi)[-2(-1+\Phi)-2+s_1^2+s_2^2+s_3^2]-\frac{1}{64}s_1^2s_2^2s_3^2\right]\cdot\\
&\quad\cdot\left[\frac{1}{16}(-1)(-2-\Phi)[-2(-1-\Phi)-2+s_1^2+s_2^2+s_3^2]-\frac{1}{64}s_1^2s_2^2s_3^2\right]\\
&= \left[\frac{1}{16}(2-\Phi)[-2\Phi+\sigma_1]-\frac{1}{64}\sigma_3\right]\left[\frac{1}{16}(2+\Phi)[2\Phi+\sigma_1]-\frac{1}{64}\sigma_3\right]\\
&= \left[\frac{1}{16}(2\Phi^2-4\Phi+2\sigma_1-\Phi\sigma_1)-\frac{1}{16}\sigma_3\right]\left[\frac{1}{16}(2\Phi^2+4\Phi+2\sigma_1+\Phi\sigma_1)-\frac{1}{16}\sigma_3\right]\\
&= \left[\frac{1}{16}(2\Phi^2+2\sigma_1)-\frac{1}{64}\sigma_3-\frac{1}{16}(4+\sigma_1)\Phi\right]\left[\frac{1}{16}(2\Phi^2+2\sigma_1)-\frac{1}{64}\sigma_3+\frac{1}{16}(4+\sigma_1)\Phi\right]\\
&= \frac{1}{64^2}\left\{\left[4\left(\Phi^2+\sigma_1\right)-\sigma_3\right]^2-4^2\left(4+\sigma_1\right)^2\Phi^2\right\}.
\end{aligned}
$$

4 points inside a ball

- Four points x_1, x_2, x_3, x_4 in a ball $(|z| \leq R)$
- 4 point-triples on the boundary 2–sphere
- $P_1 \quad (u_{12})(u_{13})(u_{14})$
- $P_2 \quad (u_{21})(u_{23})(u_{24})$
- $P_3 \quad (u_{31})(u_{32})(u_{34})$
- $P_4 \quad (u_{41})(u_{42})(u_{43})$
- point-triple u_{12}, u_{13}, u_{14} defines a cubic (polynomial):

  $p_1 := (z - u_{12})(z - u_{13})(z - u_{14})$

  $$p_1 = z^3 - (u_{12} + u_{13} + u_{14})z^2 + (u_{12}u_{13} + u_{12}u_{14} + u_{13}u_{14})z - \\ - u_{12}u_{13}u_{14}$$
- 4 point–triples $\rightarrow$ 4 cubics
- $P_1, P_2, P_3, P_4 \rightarrow \{p_1, p_2, p_3, p_4\}$

Normalized 4–point Atiyah's determinant D_4

Determinant of the coefficient matrix of polynomials:

$$|M_4| = \begin{vmatrix} 1 & -u_{12}-u_{13}-u_{14} & u_{12}u_{13}+u_{12}u_{14}+u_{13}u_{14} & -u_{12}u_{13}u_{14} \\ 1 & -u_{21}-u_{23}-u_{24} & u_{21}u_{23}+u_{21}u_{24}+u_{23}u_{24} & -u_{21}u_{23}u_{24} \\ 1 & -u_{31}-u_{32}-u_{34} & u_{31}u_{32}+u_{31}u_{34}+u_{32}u_{34} & -u_{31}u_{32}u_{34} \\ 1 & -u_{41}-u_{42}-u_{43} & u_{41}u_{42}+u_{41}u_{43}+u_{42}u_{43} & -u_{41}u_{42}u_{43} \end{vmatrix},$$

$$D_4 = \frac{|M_4|}{(u_{12}-u_{21})(u_{13}-u_{31})(u_{14}-u_{41})(u_{23}-u_{32})(u_{24}-u_{42})(u_{34}-u_{43})}$$

Conjectures ($n = 4$)

C_1 (Atiyah): $D_4 \neq 0 \quad (\Leftrightarrow p_1, p_2, p_3, p_4$ lin. indep.)

C_2 (Atiyah–Sutcliffe): $|D_4| \geq 1$

C_3 (Atiyah–Sutcliffe): $|D_4|^2 \geq D_3(1,2,3) \cdot D_3(1,2,4) \cdot D_3(1,3,4) \cdot D_3(2,3,4)$

Eastwood-Norbury formulas for euclidean D4

In 2001 they proved, by tricky use of MAPLE, that for $n = 4$ points in Eucl. 3-space

$$Re(\boldsymbol{D_4}) = \begin{array}{l} 64abca'b'c' \\ -4 \cdot d3(aa', bb', cc') \\ +SUM \\ +288 \cdot VOLUME^2, \end{array}$$

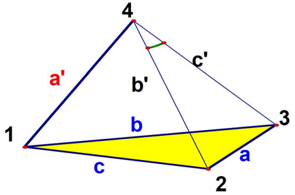
where

$SUM := a'[(b' + c')^2 - a^2)]d3(a, b, c) + ...$

$\boldsymbol{D_4}/(64abca'b'c') = D_4$

(=>eucl. Conjecture 1, and "almost"

$(= 60/64)$ of euclidean Conjecture 2)



a'((b' + c')^2- a^2)* d3(a,b,c)

New proof of the Eastwood–Norbury formula

The four points:

$P_i : x_i = (z_i, r_i),\ z_i \in \mathbb{C},\ r_i \in \mathbb{R}$

$R_{ij} := r_{ij} + r_i - r_j,\ z_{ij} := z_i - z_j$

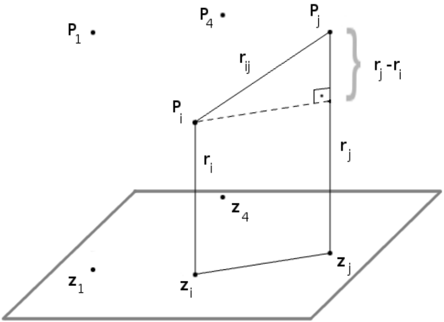
$$R_{ij}R_{ji} = r_{ij}^2 - (r_i - r_j)^2 = |z_{ij}|^2 = -z_{ij}z_{ji}$$

$$p_1 = \left(z + \frac{\overline{z_{12}}}{R_{12}}\right)\left(z + \frac{\overline{z_{13}}}{R_{13}}\right)\left(z + \frac{\overline{z_{14}}}{R_{14}}\right)$$

$$p_2 = \left(z + \frac{\overline{z_{21}}}{R_{21}}\right)\left(z + \frac{\overline{z_{23}}}{R_{23}}\right)\left(z + \frac{\overline{z_{24}}}{R_{24}}\right)$$

$$p_3 = \left(z + \frac{\overline{z_{31}}}{R_{31}}\right)\left(z + \frac{\overline{z_{32}}}{R_{32}}\right)\left(z + \frac{\overline{z_{34}}}{R_{34}}\right)$$

$$p_4 = \left(z + \frac{\overline{z_{41}}}{R_{41}}\right)\left(z + \frac{\overline{z_{42}}}{R_{42}}\right)\left(z + \frac{\overline{z_{43}}}{R_{43}}\right)$$



Matrix of coefficients of $\{p_1, p_2, p_3, p_4\}$

$$M_4 = \begin{vmatrix} \boxed{1} & \cdot & \cdot & \frac{\bar z_{12}}{R_{12}}\frac{\bar z_{13}}{R_{13}}\frac{\bar z_{14}}{R_{14}} \\ 1 & \boxed{\frac{\bar z_{21}}{R_{21}} + \frac{\bar z_{23}}{R_{23}} + \frac{\bar z_{24}}{R_{24}}} & \cdot & \cdot \\ 1 & \cdot & \boxed{\frac{\bar z_{31}}{R_{31}}\frac{\bar z_{32}}{R_{32}} + \frac{\bar z_{31}}{R_{31}}\frac{\bar z_{34}}{R_{34}} + \frac{\bar z_{32}}{R_{32}}\frac{\bar z_{34}}{R_{34}}} & \cdot \\ 1 & \cdot & \cdot & \boxed{\frac{\bar z_{41}}{R_{41}}\frac{\bar z_{42}}{R_{42}}\frac{\bar z_{43}}{R_{43}}} \end{vmatrix} \begin{matrix} \\ \cdot A \\ \cdot B \\ \cdot C \end{matrix}$$

$$A = z_{21},\ B = z_{31}z_{32},\ C = z_{41}z_{42}z_{43}$$

Normalized Atiyah determinant

$$D_4 = \underbrace{\det(M_4)}_{antisym.} \cdot \underbrace{z_{21} \cdot z_{31}z_{32} \cdot z_{41}z_{42}z_{43}}_{antisym.} = \sum 1 \cdot \left(\frac{z_{21}\bar z_{21}}{R_{21}} + \frac{z_{21}\bar z_{23}}{R_{23}} + \frac{z_{21}\bar z_{24}}{R_{24}} \right) \cdot$$

$$\cdot \left(\frac{z_{31}\bar z_{31}z_{32}\bar z_{32}}{R_{31}R_{32}} + \frac{z_{31}\bar z_{31}z_{32}\bar z_{34}}{R_{31}R_{34}} + \frac{z_{32}\bar z_{32}z_{32}\bar z_{34}}{R_{32}R_{34}} \right) R_{14}R_{24}R_{34} =$$

$$= \sum (R_{12}R_{24} + \underbrace{z_{21}z_{24}})(R_{13}R_{23}R_{34} + R_{13}\underbrace{z_{32}\bar z_{34}} + R_{23}\underbrace{z_{31}\bar z_{34}})R_{14} +$$

$$+ (R_{13}R_{24}R_{34}\underbrace{z_{21}\bar z_{23}} + R_{13}R_{24}R_{32}\underbrace{z_{21}\bar z_{34}} + R_{24}\underbrace{z_{21}\bar z_{23}z_{31}\bar z_{34}})R_{14}$$

(where summations are over all permutations of indices).

By writing $z_{ij}\bar{z}_{kl} = C[i,j,k,l] + \sqrt{-1}\, S[i,j,k,l]$ and using a Lagrange identity (involving the dot product of two cross products; a fact mentioned by N. Wildberger to the author) we have
$S[i,j,k,l]S[p,q,r,s] = C[i,j,p,q]C[k,l,r,s] - C[i,j,r,s]C[k,l,p,q]$
(we have discovered this identity independently) and using the formula

$$C[i,j,k,l] = \mathrm{Re}(z_{ij}\bar{z}_{kl}) = \frac{1}{2}[|z_{il}|^2 + |z_{jk}|^2 - |z_{ik}|^2 - |z_{jl}|^2] =$$
$$= \frac{1}{2}[r_{il}^2 + r_{jk}^2 - r_{ik}^2 - r_{jl}^2] - (r_i - r_j)(r_k - r_l)$$

we obtain our derivation of the Eastwood–Norbury formula.

By this new method we obtained a polynomial formula for the planar configurations of 5 points (by S_5–symmetrization of a "one page" expression) and a rational formula for the spatial 5 point configuration (this last formula has almost 100000 terms).

This settles one of the Eastwood–Norbury conjectures. We do not yet have definite geometric interpretations for the "nonplanar" part of the formula involving heights r_i, $i = 1, \ldots, 5$.

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> *We use M ..Eastwood* and *P.Norbury notations* from *Geometry* and *Topology* 2001*'s paper*

> $C[i,j,k,l] = \frac{1}{2}\left(r[\min(i,l),\max(i,l)]^2 + r[\min(j,k),\max(j,k)]^2 - r[\min(i,\mathrm{k}),\max(i,\mathrm{k})]^2 - r[\min(j,l),\max(j,1)]^2\right)$ ($=$ *inner product of vectors* $(z[i],z[j])$ **and** $(z[k],z[l])$) , $r[i,j]$ *is the distance of* $P[i]=(z[i],r[i])$ **and** $P[j]=(z[j],r[j])$, $R[i,j]=r[i,j]+r[i]-r[j]$,

> *In our approach the basic quantity, AS5real4(= the main diagonal product of the extended Atiyah 's matrix) which just follows, schould be symmetrized over the symmetric group S_5 to get AS5r0 (see bellow).*

> $AS5real4 := R_{1,2} R_{1,3} R_{1,4} R_{1,5} R_{2,3} R_{2,4} R_{2,5} R_{3,4} R_{3,5} R_{4,5} - C_{3,4,4,5} R_{1,2} R_{1,3} R_{1,4} R_{1,5} R_{2,3} R_{2,4} R_{2,5} R_{3,5}$

$$- C_{2,4,4,5} R_{1,2} R_{1,3} R_{1,4} R_{1,5} R_{2,3} R_{2,5} R_{3,4} R_{3,5} - C_{2,3,3,5} R_{1,2} R_{1,3} R_{1,4} R_{1,5} R_{2,4} R_{2,5} R_{3,4} R_{4,5}$$
$$+ C_{2,3,4,5} R_{1,2} R_{1,3} R_{1,4} R_{1,5} R_{2,4} R_{2,5} R_{3,5} R_{4,3} - C_{2,3,3,4} R_{1,2} R_{1,3} R_{1,4} R_{1,5} R_{2,4} R_{2,5} R_{3,5} R_{4,5}$$
$$- C_{1,4,4,5} R_{1,2} R_{1,3} R_{1,5} R_{2,3} R_{2,4} R_{2,5} R_{3,4} R_{3,5} - C_{1,3,3,5} R_{1,2} R_{1,4} R_{1,5} R_{2,3} R_{2,4} R_{2,5} R_{3,4} R_{4,5}$$
$$+ C_{1,3,4,5} R_{1,2} R_{1,4} R_{1,5} R_{2,3} R_{2,4} R_{2,5} R_{3,5} R_{4,3} - C_{1,3,3,4} R_{1,2} R_{1,4} R_{1,5} R_{2,3} R_{2,4} R_{2,5} R_{3,5} R_{4,5}$$
$$- C_{1,2,2,5} R_{1,3} R_{1,4} R_{1,5} R_{2,3} R_{2,4} R_{3,4} R_{3,5} R_{4,5} + C_{1,2,4,5} R_{1,3} R_{1,4} R_{1,5} R_{2,3} R_{2,5} R_{3,4} R_{3,5} R_{4,2}$$
$$- C_{1,2,2,4} R_{1,3} R_{1,4} R_{1,5} R_{2,3} R_{2,5} R_{3,4} R_{3,5} R_{4,5} + C_{1,2,3,5} R_{1,3} R_{1,4} R_{1,5} R_{2,4} R_{2,5} R_{3,2} R_{3,4} R_{4,5}$$
$$- C_{1,2,4,5} R_{1,3} R_{1,4} R_{1,5} R_{2,4} R_{2,5} R_{3,2} R_{3,5} R_{4,3} + C_{1,2,3,4} R_{1,3} R_{1,4} R_{1,5} R_{2,4} R_{2,5} R_{3,2} R_{3,5} R_{4,5}$$
$$- C_{1,2,2,3} R_{1,3} R_{1,4} R_{1,5} R_{2,4} R_{2,5} R_{3,4} R_{3,5} R_{4,5} + (C_{3,4,3,5} C_{2,3,4,5} + C_{2,3,3,5} C_{3,4,4,5}$$
$$- C_{2,3,3,4} C_{3,5,4,5}) R_{1,2} R_{1,3} R_{1,4} R_{1,5} R_{2,4} R_{2,5} + (-C_{2,3,2,4} C_{3,5,4,5} + C_{2,4,3,5} C_{2,3,4,5}$$
$$+ C_{2,3,3,5} C_{2,4,4,5}) R_{1,2} R_{1,3} R_{1,4} R_{1,5} R_{2,5} R_{3,4} + (C_{2,4,3,4} C_{2,3,4,5} - C_{2,3,2,4} C_{3,4,4,5}$$
$$+ C_{2,3,3,4} C_{2,4,4,5}) R_{1,2} R_{1,3} R_{1,4} R_{1,5} R_{2,5} R_{3,5} + (C_{2,3,3,5} C_{1,4,4,5} + C_{1,4,3,5} C_{2,3,4,5}$$
$$- C_{1,4,2,3} C_{3,5,4,5}) R_{1,2} R_{1,3} R_{1,5} R_{2,4} R_{2,5} R_{3,4} + (C_{1,4,3,4} C_{2,3,4,5} - C_{1,4,2,3} C_{3,4,4,5}$$
$$+ C_{2,3,3,4} C_{1,4,4,5}) R_{1,2} R_{1,3} R_{1,5} R_{2,4} R_{2,5} R_{3,5} + (C_{3,4,3,5} C_{1,3,4,5} - C_{1,3,3,4} C_{3,5,4,5}$$
$$+ C_{1,3,3,5} C_{3,4,4,5}) R_{1,2} R_{1,4} R_{1,5} R_{2,3} R_{2,4} R_{2,5} + (C_{2,4,3,5} C_{1,3,4,5} - C_{1,3,2,4} C_{3,5,4,5}$$
$$+ C_{1,3,3,5} C_{2,4,4,5}) R_{1,2} R_{1,4} R_{1,5} R_{2,3} R_{2,5} R_{3,4} + (-C_{1,3,2,4} C_{3,4,4,5} + C_{2,4,3,4} C_{1,3,4,5}$$
$$+ C_{1,3,3,4} C_{2,4,4,5}) R_{1,2} R_{1,4} R_{1,5} R_{2,3} R_{2,5} R_{3,5} - (-C_{1,3,2,3} C_{3,5,4,5} + C_{2,3,3,5} C_{1,3,4,5}$$
$$+ C_{2,3,4,5} C_{1,3,3,5}) R_{1,2} R_{1,4} R_{1,5} R_{2,4} R_{2,5} R_{4,3} - (-C_{2,3,3,5} C_{1,3,3,4} - C_{2,3,3,4} C_{1,3,3,5}$$
$$+ C_{1,3,2,3} C_{3,4,3,5}) R_{1,2} R_{1,4} R_{1,5} R_{2,4} R_{2,5} R_{4,5} + (-C_{1,3,1,4} C_{3,5,4,5} + C_{1,3,3,5} C_{1,4,4,5}$$
$$+ C_{1,4,3,5} C_{1,3,4,5}) R_{1,2} R_{1,5} R_{2,3} R_{2,4} R_{2,5} R_{3,4} + (C_{1,4,3,4} C_{1,3,4,5} + C_{1,3,3,4} C_{1,4,4,5}$$
$$- C_{1,3,1,4} C_{3,4,4,5}) R_{1,2} R_{1,5} R_{2,3} R_{2,4} R_{2,5} R_{3,5} + (C_{3,4,4,5} C_{1,2,2,5} - C_{1,2,3,4} C_{2,5,4,5}$$
$$+ C_{2,5,3,4} C_{1,2,4,5}) R_{1,3} R_{1,4} R_{1,5} R_{2,3} R_{2,4} R_{3,5} - (C_{1,2,3,4} C_{2,4,4,5} - C_{1,2,4,5} C_{2,4,3,4}$$
$$- C_{1,2,2,4} C_{3,4,4,5}) R_{1,3} R_{1,4} R_{1,5} R_{2,3} R_{2,5} R_{3,5} + (C_{1,2,2,5} C_{2,4,4,5} + C_{1,2,4,5} C_{2,4,2,5}$$
$$- C_{1,2,2,4} C_{2,5,4,5}) R_{1,3} R_{1,4} R_{1,5} R_{2,3} R_{3,4} R_{3,5} - (-C_{1,2,3,4} C_{3,5,4,5} + C_{3,4,4,5} C_{1,2,3,5}$$
$$+ C_{3,4,3,5} C_{1,2,4,5}) R_{1,3} R_{1,4} R_{1,5} R_{2,4} R_{2,5} R_{3,2} - (-C_{1,2,4,5} C_{2,3,3,4} + C_{1,2,3,4} C_{2,3,4,5}$$
$$- C_{3,4,4,5} C_{1,2,2,3}) R_{1,3} R_{1,4} R_{1,5} R_{2,4} R_{2,5} R_{3,5} - (-C_{2,3,3,5} C_{1,2,2,5} - C_{1,2,3,5} C_{2,3,2,5}$$
$$+ C_{1,2,2,3} C_{2,5,3,5}) R_{1,3} R_{1,4} R_{1,5} R_{2,4} R_{3,4} R_{4,5} - (-C_{1,2,2,3} C_{2,5,4,5} + C_{1,2,2,5} C_{2,3,4,5}$$
$$+ C_{1,2,4,5} C_{2,3,2,5}) R_{1,3} R_{1,4} R_{1,5} R_{2,4} R_{3,5} R_{4,3} - (-C_{1,2,3,4} C_{2,3,2,5} - C_{2,3,3,4} C_{1,2,2,5}$$
$$+ C_{1,2,2,3} C_{2,5,3,4}) R_{1,3} R_{1,4} R_{1,5} R_{2,4} R_{3,5} R_{4,5} - (C_{2,4,3,5} C_{1,2,4,5} - C_{1,2,2,4} C_{3,5,4,5}$$
$$+ C_{2,4,4,5} C_{1,2,3,5}) R_{1,3} R_{1,4} R_{1,5} R_{2,5} R_{3,2} R_{3,4} - (-C_{1,2,2,4} C_{3,4,4,5} + C_{1,2,3,4} C_{2,4,4,5}$$
$$+ C_{1,2,4,5} C_{2,4,3,4}) R_{1,3} R_{1,4} R_{1,5} R_{2,5} R_{3,2} R_{3,5} - (-C_{1,2,2,3} C_{2,4,4,5} + C_{1,2,2,4} C_{2,3,4,5}$$

$$- C_{1,2,4,5} C_{2,3,2,4}) R_{1,3} R_{1,4} R_{1,5} R_{2,5} R_{3,4} R_{3,5} - (C_{1,2,4,5} C_{2,3,3,5} + C_{1,2,3,5} C_{2,3,4,5}$$
$$- C_{1,2,2,3} C_{3,5,4,5}) R_{1,3} R_{1,4} R_{1,5} R_{2,5} R_{3,4} R_{4,2} - (C_{1,2,2,3} C_{2,4,3,5} - C_{2,3,3,5} C_{1,2,2,4}$$
$$- C_{1,2,3,5} C_{2,3,2,4}) R_{1,3} R_{1,4} R_{1,5} R_{2,5} R_{3,4} R_{4,5} - (C_{1,2,4,5} C_{2,3,3,4} + C_{1,2,3,4} C_{2,3,4,5}$$
$$- C_{3,4,4,5} C_{1,2,2,3}) R_{1,3} R_{1,4} R_{1,5} R_{2,5} R_{3,5} R_{4,2} - (C_{1,2,4,5} C_{2,3,2,4} + C_{1,2,2,4} C_{2,3,4,5}$$
$$- C_{1,2,2,3} C_{2,4,4,5}) R_{1,3} R_{1,4} R_{1,5} R_{2,5} R_{3,5} R_{4,3} - (-C_{1,2,3,4} C_{2,3,2,4} - C_{2,3,3,4} C_{1,2,2,4}$$
$$+ C_{1,2,2,3} C_{2,4,3,4}) R_{1,3} R_{1,4} R_{1,5} R_{2,5} R_{3,5} R_{4,5} + (C_{1,4,2,5} C_{1,2,4,5} + C_{1,4,4,5} C_{1,2,2,5}$$
$$- C_{1,2,1,4} C_{2,5,4,5}) R_{1,3} R_{1,5} R_{2,3} R_{2,4} R_{3,4} R_{3,5} + (C_{1,2,4,5} C_{1,4,2,4} + C_{1,2,2,4} C_{1,4,4,5}$$
$$- C_{1,2,1,4} C_{2,4,4,5}) R_{1,3} R_{1,5} R_{2,3} R_{2,5} R_{3,4} R_{3,5} - (-C_{1,2,1,4} C_{3,5,4,5} + C_{1,4,4,5} C_{1,2,3,5}$$
$$+ C_{1,4,3,5} C_{1,2,4,5}) R_{1,3} R_{1,5} R_{2,4} R_{2,5} R_{3,2} R_{3,4} - (-C_{1,2,1,4} C_{3,4,4,5} + C_{1,4,4,5} C_{1,2,3,4}$$
$$+ C_{1,4,3,4} C_{1,2,4,5}) R_{1,3} R_{1,5} R_{2,4} R_{2,5} R_{3,2} R_{3,5} + (C_{1,2,4,5} C_{1,4,2,3} - C_{1,2,1,4} C_{2,3,4,5}$$
$$+ C_{1,2,2,3} C_{1,4,4,5}) R_{1,3} R_{1,5} R_{2,4} R_{2,5} R_{3,4} R_{3,5} + (-C_{1,2,1,3} C_{2,5,3,5} + C_{1,2,3,5} C_{1,3,2,5}$$
$$+ C_{1,3,3,5} C_{1,2,2,5}) R_{1,4} R_{1,5} R_{2,3} R_{2,4} R_{3,4} R_{4,5} - (C_{1,2,2,5} C_{1,3,4,5} - C_{1,2,1,3} C_{2,5,4,5}$$
$$+ C_{1,2,4,5} C_{1,3,2,5}) R_{1,4} R_{1,5} R_{2,3} R_{2,4} R_{3,5} R_{4,3} + (-C_{1,2,1,3} C_{2,5,3,4} + C_{1,2,3,4} C_{1,3,2,5}$$
$$+ C_{1,3,3,4} C_{1,2,2,5}) R_{1,4} R_{1,5} R_{2,3} R_{2,4} R_{3,5} R_{4,5} - (C_{1,2,3,5} C_{1,3,4,5} - C_{1,2,1,3} C_{3,5,4,5}$$
$$+ C_{1,2,4,5} C_{1,3,3,5}) R_{1,4} R_{1,5} R_{2,3} R_{2,5} R_{3,4} R_{4,2} + (C_{1,3,3,5} C_{1,2,2,4} + C_{1,2,3,5} C_{1,3,2,4}$$
$$- C_{1,2,1,3} C_{2,4,3,5}) R_{1,4} R_{1,5} R_{2,3} R_{2,5} R_{3,4} R_{4,5} - (C_{1,2,4,5} C_{1,3,3,4} - C_{1,2,1,3} C_{3,4,4,5}$$
$$+ C_{1,2,3,4} C_{1,3,4,5}) R_{1,4} R_{1,5} R_{2,3} R_{2,5} R_{3,5} R_{4,2} - (C_{1,2,2,4} C_{1,3,4,5} - C_{1,2,1,3} C_{2,4,4,5}$$
$$+ C_{1,2,4,5} C_{1,3,2,4}) R_{1,4} R_{1,5} R_{2,3} R_{2,5} R_{3,5} R_{4,3} + (-C_{1,2,1,3} C_{2,4,3,4} + C_{1,3,3,4} C_{1,2,2,4}$$
$$+ C_{1,2,3,4} C_{1,3,2,4}) R_{1,4} R_{1,5} R_{2,3} R_{2,5} R_{3,5} R_{4,5} + (C_{1,2,3,5} C_{1,3,4,5} - C_{1,2,1,3} C_{3,5,4,5}$$
$$+ C_{1,2,4,5} C_{1,3,3,5}) R_{1,4} R_{1,5} R_{2,4} R_{2,5} R_{3,2} R_{4,3} + (C_{1,2,1,3} C_{3,4,3,5} - C_{1,2,3,4} C_{1,3,3,5}$$
$$- C_{1,2,3,5} C_{1,3,3,4}) R_{1,4} R_{1,5} R_{2,4} R_{2,5} R_{3,2} R_{4,5} - (-C_{1,2,3,5} C_{1,3,2,3} + C_{1,2,1,3} C_{2,3,3,5}$$
$$- C_{1,3,3,5} C_{1,2,2,3}) R_{1,4} R_{1,5} R_{2,4} R_{2,5} R_{3,4} R_{4,5} - (C_{1,2,4,5} C_{1,3,2,3} + C_{1,2,2,3} C_{1,3,4,5}$$
$$- C_{1,2,1,3} C_{2,3,4,5}) R_{1,4} R_{1,5} R_{2,4} R_{2,5} R_{3,5} R_{4,3} - (C_{1,2,1,3} C_{2,3,3,4} - C_{1,3,3,4} C_{1,2,2,3}$$
$$- C_{1,2,3,4} C_{1,3,2,3}) R_{1,4} R_{1,5} R_{2,4} R_{2,5} R_{3,5} R_{4,5} - (C_{2,3,3,5} C_{1,3,3,4} C_{2,4,4,5} - C_{2,3,2,4} C_{3,5,4,5} C_{1,3,3,4}$$
$$- C_{1,3,2,3} C_{3,4,3,5} C_{2,4,4,5} - C_{1,3,2,4} C_{3,4,4,5} C_{2,3,3,5} + C_{2,3,3,4} C_{1,3,3,5} C_{2,4,4,5} + C_{2,4,3,5} C_{2,3,4,5} C_{1,3,3,4}$$
$$+ C_{2,4,3,4} C_{1,3,4,5} C_{2,3,3,5}) R_{1,2} R_{1,4} R_{1,5} R_{2,5} + (-C_{1,4,3,5} C_{2,3,4,5} C_{1,3,3,4} - C_{2,3,3,5} C_{1,3,3,4} C_{1,4,4,5}$$
$$+ C_{1,4,2,3} C_{3,5,4,5} C_{1,3,3,4} - C_{1,4,3,4} C_{1,3,4,5} C_{2,3,3,5} - C_{2,3,3,4} C_{1,3,3,5} C_{1,4,4,5} + C_{1,3,2,3} C_{3,4,3,5} C_{1,4,4,5}$$
$$+ C_{1,3,1,4} C_{3,4,4,5} C_{2,3,3,5}) R_{1,2} R_{1,5} R_{2,4} R_{2,5} - (-C_{1,2,3,4} C_{2,5,4,5} C_{2,3,3,5} + C_{3,4,4,5} C_{1,2,2,5} C_{2,3,3,5}$$
$$- C_{2,3,3,4} C_{3,5,4,5} C_{1,2,2,5} + C_{2,5,3,4} C_{1,2,4,5} C_{2,3,3,5} + C_{3,4,3,5} C_{2,3,4,5} C_{1,2,2,5} + C_{2,3,2,5} C_{3,4,4,5} C_{1,2,3,5}$$
$$- C_{2,5,3,5} C_{3,4,4,5} C_{1,2,2,3}) R_{1,3} R_{1,4} R_{1,5} R_{2,4} - (C_{1,2,2,4} C_{3,4,4,5} C_{2,3,3,5} - C_{1,2,3,4} C_{2,4,4,5} C_{2,3,3,5}$$
$$- C_{2,4,3,5} C_{3,4,4,5} C_{1,2,2,3} + C_{2,3,2,4} C_{3,4,4,5} C_{1,2,3,5} - C_{2,3,3,4} C_{3,5,4,5} C_{1,2,2,4} + C_{1,2,4,5} C_{2,4,3,4} C_{2,3,3,5}$$
$$+ C_{3,4,3,5} C_{2,3,4,5} C_{1,2,2,4}) R_{1,3} R_{1,4} R_{1,5} R_{2,5} - (-C_{2,3,2,4} C_{3,5,4,5} C_{1,2,2,5} + C_{2,4,3,5} C_{2,3,4,5} C_{1,2,2,5}$$
$$+ C_{1,2,2,5} C_{2,4,4,5} C_{2,3,3,5} + C_{2,3,2,5} C_{2,4,4,5} C_{1,2,3,5} - C_{2,5,4,5} C_{1,2,2,4} C_{2,3,3,5} + C_{1,2,4,5} C_{2,4,2,5} C_{2,3,3,5}$$
$$- C_{2,5,3,5} C_{2,4,4,5} C_{1,2,2,3}) R_{1,3} R_{1,4} R_{1,5} R_{3,4} - (C_{2,3,2,5} C_{2,4,4,5} C_{1,2,3,4} - C_{2,5,4,5} C_{1,2,2,4} C_{2,3,3,4}$$
$$+ C_{1,2,2,5} C_{2,4,4,5} C_{2,3,3,4} + C_{2,4,2,5} C_{1,2,4,5} C_{2,3,3,4} + C_{2,4,3,4} C_{2,3,4,5} C_{1,2,2,5} - C_{2,3,2,4} C_{3,4,4,5} C_{1,2,2,5}$$
$$- C_{2,5,3,4} C_{2,4,4,5} C_{1,2,2,3}) R_{1,3} R_{1,4} R_{1,5} R_{3,5} + (C_{1,2,1,4} C_{2,5,4,5} C_{2,3,3,5} - C_{1,4,2,5} C_{1,2,4,5} C_{2,3,3,5}$$
$$- C_{1,4,3,5} C_{2,3,4,5} C_{1,2,2,5} - C_{2,3,2,5} C_{1,4,4,5} C_{1,2,3,5} + C_{1,4,2,3} C_{3,5,4,5} C_{1,2,2,5} + C_{2,5,3,5} C_{1,4,4,5} C_{1,2,2,3}$$
$$- C_{1,4,4,5} C_{1,2,2,5} C_{2,3,3,5}) R_{1,3} R_{1,5} R_{2,4} R_{3,4} + (C_{1,4,2,3} C_{3,4,4,5} C_{1,2,2,5} - C_{1,4,3,4} C_{2,3,4,5} C_{1,2,2,5}$$
$$- C_{2,3,2,5} C_{1,4,4,5} C_{1,2,3,4} + C_{2,5,3,4} C_{1,4,4,5} C_{1,2,2,3} + C_{1,2,1,4} C_{2,5,4,5} C_{2,3,3,4} - C_{1,4,4,5} C_{1,2,2,5} C_{2,3,3,4}$$
$$- C_{1,4,2,5} C_{1,2,4,5} C_{2,3,3,4}) R_{1,3} R_{1,5} R_{2,4} R_{3,5} + (-C_{2,3,2,4} C_{1,4,4,5} C_{1,2,3,5} + C_{2,4,3,5} C_{1,4,4,5} C_{1,2,2,3}$$
$$- C_{1,2,4,5} C_{1,4,2,4} C_{2,3,3,5} - C_{1,4,3,5} C_{2,3,4,5} C_{1,2,2,4} + C_{1,4,2,3} C_{3,5,4,5} C_{1,2,2,4} - C_{1,2,2,4} C_{1,4,4,5} C_{2,3,3,5}$$
$$+ C_{1,2,1,4} C_{2,4,4,5} C_{2,3,3,5}) R_{1,3} R_{1,5} R_{2,5} R_{3,4} + (-C_{1,2,4,5} C_{1,4,2,4} C_{2,3,3,4} + C_{1,4,2,3} C_{3,4,4,5} C_{1,2,2,4}$$
$$- C_{1,4,3,4} C_{2,3,4,5} C_{1,2,2,4} - C_{2,3,2,4} C_{1,4,4,5} C_{1,2,3,4} - C_{1,2,2,4} C_{1,4,4,5} C_{2,3,3,4} + C_{2,4,3,4} C_{1,4,4,5} C_{1,2,2,3}$$

$$+ C_{1,2,1,4} C_{2,4,4,5} C_{2,3,3,4}) R_{1,3} R_{1,5} R_{2,5} R_{3,5} - (-C_{1,3,3,4} C_{3,5,4,5} C_{1,2,2,5} - C_{2,5,3,5} C_{3,4,4,5} C_{1,2,1,3}$$
$$+ C_{2,5,3,4} C_{1,2,4,5} C_{1,3,3,5} + C_{3,4,3,5} C_{1,3,4,5} C_{1,2,2,5} - C_{1,2,3,4} C_{2,5,4,5} C_{1,3,3,5} + C_{1,3,2,5} C_{3,4,4,5} C_{1,2,3,5}$$
$$+ C_{3,4,4,5} C_{1,2,2,5} C_{1,3,3,5}) R_{1,4} R_{1,5} R_{2,3} R_{2,4} - (C_{3,4,3,5} C_{1,3,4,5} C_{1,2,2,4} + C_{1,3,2,4} C_{3,4,4,5} C_{1,2,3,5}$$
$$- C_{2,4,3,5} C_{3,4,4,5} C_{1,2,1,3} - C_{1,3,3,4} C_{3,5,4,5} C_{1,2,2,4} + C_{1,2,4,5} C_{2,4,3,4} C_{1,3,3,5} - C_{1,2,3,4} C_{2,4,4,5} C_{1,3,3,5}$$
$$+ C_{1,2,2,4} C_{3,4,4,5} C_{1,3,3,5}) R_{1,4} R_{1,5} R_{2,3} R_{2,5} - (C_{1,2,2,5} C_{2,4,4,5} C_{1,3,3,5} - C_{1,3,2,4} C_{3,5,4,5} C_{1,2,2,5}$$
$$- C_{2,5,3,5} C_{2,4,4,5} C_{1,2,1,3} + C_{1,2,4,5} C_{2,4,2,5} C_{1,3,3,5} + C_{2,4,3,5} C_{1,3,4,5} C_{1,2,2,5} + C_{1,3,2,5} C_{2,4,4,5} C_{1,2,3,5}$$
$$- C_{2,5,4,5} C_{1,2,2,4} C_{1,3,3,5}) R_{1,4} R_{1,5} R_{2,3} R_{3,4} - (-C_{1,3,2,4} C_{3,4,4,5} C_{1,2,2,5} + C_{2,4,3,4} C_{1,3,4,5} C_{1,2,2,5}$$
$$- C_{2,5,4,5} C_{1,2,2,4} C_{1,3,3,4} + C_{1,2,2,5} C_{2,4,4,5} C_{1,3,3,4} - C_{2,5,3,4} C_{2,4,4,5} C_{1,2,1,3} + C_{1,2,4,5} C_{2,4,2,5} C_{1,3,3,4}$$
$$+ C_{1,3,2,5} C_{2,4,4,5} C_{1,2,3,4}) R_{1,4} R_{1,5} R_{2,3} R_{3,5} - (C_{3,4,4,5} C_{1,2,3,5} C_{1,3,2,3} - C_{3,4,4,5} C_{1,2,1,3} C_{2,3,3,5}$$
$$+ C_{2,3,3,4} C_{1,3,3,5} C_{1,2,4,5} + C_{3,4,3,5} C_{1,3,4,5} C_{1,2,2,3} + C_{3,4,4,5} C_{1,2,2,3} C_{1,3,3,5} - C_{2,3,4,5} C_{1,2,3,4} C_{1,3,3,5}$$
$$- C_{3,5,4,5} C_{1,3,3,4} C_{1,2,2,3}) R_{1,4} R_{1,5} R_{2,4} R_{2,5} - (C_{2,5,4,5} C_{2,3,3,5} C_{1,2,1,3} - C_{2,3,4,5} C_{1,3,3,5} C_{1,2,2,5}$$
$$+ C_{2,5,3,5} C_{1,3,4,5} C_{1,2,2,3} - C_{2,3,2,5} C_{1,3,4,5} C_{1,2,3,5} + C_{1,3,2,3} C_{3,5,4,5} C_{1,2,2,5} - C_{2,3,3,5} C_{1,3,4,5} C_{1,2,2,5}$$
$$- C_{1,3,2,5} C_{2,3,3,5} C_{1,2,4,5}) R_{1,4} R_{1,5} R_{2,4} R_{4,3} - (-C_{2,5,3,4} C_{2,3,3,5} C_{1,2,1,3} - C_{1,3,2,3} C_{3,4,3,5} C_{1,2,2,5}$$
$$+ C_{1,3,2,5} C_{2,3,3,5} C_{1,2,3,4} + C_{2,3,3,4} C_{1,3,3,5} C_{1,2,2,5} + C_{2,3,3,5} C_{1,3,3,4} C_{1,2,2,5} + C_{2,3,2,5} C_{1,3,3,4} C_{1,2,3,5}$$
$$- C_{2,5,3,5} C_{1,3,3,4} C_{1,2,2,3}) R_{1,4} R_{1,5} R_{2,4} R_{4,5} + (C_{1,3,3,4} C_{2,4,4,5} C_{1,2,3,5} + C_{1,2,3,4} C_{2,4,3,5} C_{1,3,4,5}$$
$$- C_{1,2,3,4} C_{1,3,2,4} C_{3,5,4,5} + C_{1,2,3,4} C_{2,4,4,5} C_{1,3,3,5} - C_{1,2,2,4} C_{3,4,4,5} C_{1,3,3,5} + C_{1,2,4,5} C_{2,4,3,4} C_{1,3,3,5}$$
$$- C_{3,4,3,5} C_{2,4,4,5} C_{1,2,1,3}) R_{1,4} R_{1,5} R_{2,5} R_{3,2} - (-C_{1,2,2,4} C_{2,3,4,5} C_{1,3,3,5} + C_{2,4,3,5} C_{1,3,4,5} C_{1,2,2,3}$$
$$+ C_{1,2,2,3} C_{2,4,4,5} C_{1,3,3,5} + C_{2,4,4,5} C_{1,2,3,5} C_{1,3,2,3} + C_{1,2,4,5} C_{2,3,2,4} C_{1,3,3,5} - C_{2,4,4,5} C_{2,3,3,5} C_{1,2,1,3}$$
$$- C_{1,3,2,4} C_{3,5,4,5} C_{1,2,2,3}) R_{1,4} R_{1,5} R_{2,5} R_{3,4} - (-C_{2,4,4,5} C_{1,2,1,3} C_{2,3,3,4} + C_{2,4,3,4} C_{1,3,4,5} C_{1,2,2,3}$$
$$- C_{1,3,2,4} C_{3,4,4,5} C_{1,2,2,3} - C_{1,2,2,4} C_{2,3,4,5} C_{1,3,3,4} + C_{1,2,2,3} C_{2,4,4,5} C_{1,3,3,4} + C_{2,4,4,5} C_{1,2,3,4} C_{1,3,2,3}$$
$$+ C_{1,2,4,5} C_{2,3,2,4} C_{1,3,3,4}) R_{1,4} R_{1,5} R_{2,5} R_{3,5} + (C_{2,3,4,5} C_{1,3,3,4} C_{1,2,3,5} + C_{1,3,4,5} C_{2,3,3,5} C_{1,2,3,4}$$
$$+ C_{2,3,3,5} C_{1,3,3,4} C_{1,2,4,5} + C_{2,3,3,4} C_{1,3,3,5} C_{1,2,4,5} - C_{1,3,2,3} C_{3,4,3,5} C_{1,2,4,5} - C_{3,5,4,5} C_{1,3,3,4} C_{1,2,2,3}$$
$$- C_{3,4,4,5} C_{1,2,1,3} C_{2,3,3,5}) R_{1,4} R_{1,5} R_{2,5} R_{4,2} - (-C_{2,3,3,5} C_{1,3,4,5} C_{1,2,2,4} + C_{2,4,4,5} C_{2,3,3,5} C_{1,2,1,3}$$
$$- C_{1,2,2,4} C_{2,3,4,5} C_{1,3,3,5} - C_{1,3,2,4} C_{2,3,3,5} C_{1,2,4,5} - C_{2,3,2,4} C_{1,3,4,5} C_{1,2,3,5} + C_{1,3,2,3} C_{3,5,4,5} C_{1,2,2,4}$$
$$+ C_{2,4,3,5} C_{1,3,4,5} C_{1,2,2,3}) R_{1,4} R_{1,5} R_{2,5} R_{4,3} - (C_{2,3,2,4} C_{1,3,3,4} C_{1,2,3,5} - C_{2,4,3,5} C_{1,3,3,4} C_{1,2,2,3}$$
$$+ C_{1,3,2,4} C_{2,3,3,5} C_{1,2,3,4} - C_{1,3,2,3} C_{3,4,3,5} C_{1,2,2,4} + C_{2,3,3,4} C_{1,3,3,5} C_{1,2,2,4} - C_{2,4,3,4} C_{2,3,3,5} C_{1,2,1,3}$$
$$+ C_{2,3,3,5} C_{1,3,3,4} C_{1,2,2,4}) R_{1,4} R_{1,5} R_{2,5} R_{4,5} - (-C_{2,5,3,5} C_{1,4,4,5} C_{1,2,1,3} - C_{1,3,1,4} C_{3,5,4,5} C_{1,2,2,5}$$
$$+ C_{1,3,2,5} C_{1,4,4,5} C_{1,2,3,5} + C_{1,4,3,5} C_{1,3,4,5} C_{1,2,2,5} + C_{1,4,4,5} C_{1,2,2,5} C_{1,3,3,5} + C_{1,2,4,5} C_{1,4,2,5} C_{1,3,3,5}$$
$$- C_{1,2,1,4} C_{2,5,4,5} C_{1,3,3,5}) R_{1,5} R_{2,3} R_{2,4} R_{3,4} - (-C_{2,5,3,4} C_{1,4,4,5} C_{1,2,1,3} + C_{1,3,2,5} C_{1,4,4,5} C_{1,2,3,4}$$
$$- C_{1,2,1,4} C_{2,5,4,5} C_{1,3,3,4} + C_{1,4,2,5} C_{1,2,4,5} C_{1,3,3,4} - C_{1,3,1,4} C_{3,4,4,5} C_{1,2,2,5} + C_{1,4,4,5} C_{1,2,2,5} C_{1,3,3,4}$$
$$+ C_{1,4,3,4} C_{1,3,4,5} C_{1,2,2,5}) R_{1,5} R_{2,3} R_{2,4} R_{3,5} - (-C_{1,2,1,4} C_{2,4,4,5} C_{1,3,3,5} + C_{1,4,3,5} C_{1,3,4,5} C_{1,2,2,4}$$
$$+ C_{1,3,2,4} C_{1,4,4,5} C_{1,2,3,5} - C_{2,4,3,5} C_{1,4,4,5} C_{1,2,1,3} - C_{1,3,1,4} C_{3,5,4,5} C_{1,2,2,4} + C_{1,2,2,4} C_{1,4,4,5} C_{1,3,3,5}$$
$$+ C_{1,2,4,5} C_{1,4,2,4} C_{1,3,3,5}) R_{1,5} R_{2,3} R_{2,5} R_{3,4} - (C_{1,4,3,4} C_{1,3,4,5} C_{1,2,2,4} + C_{1,2,2,4} C_{1,4,4,5} C_{1,3,3,4}$$
$$- C_{1,2,1,4} C_{2,4,4,5} C_{1,3,3,4} - C_{2,4,3,4} C_{1,4,4,5} C_{1,2,1,3} - C_{1,3,1,4} C_{3,4,4,5} C_{1,2,2,4} + C_{1,3,2,4} C_{1,4,4,5} C_{1,2,3,4}$$
$$+ C_{1,2,4,5} C_{1,4,2,4} C_{1,3,3,4}) R_{1,5} R_{2,3} R_{2,5} R_{3,5} - (-C_{1,3,3,4} C_{1,4,4,5} C_{1,2,3,5} - C_{1,4,3,4} C_{1,2,4,5} C_{1,3,3,5}$$
$$+ C_{1,2,1,4} C_{3,4,4,5} C_{1,3,3,5} + C_{1,2,3,4} C_{1,3,1,4} C_{3,5,4,5} - C_{1,4,4,5} C_{1,2,3,4} C_{1,3,3,5} - C_{1,2,3,4} C_{1,4,3,5} C_{1,3,4,5}$$
$$+ C_{3,4,3,5} C_{1,4,4,5} C_{1,2,1,3}) R_{1,5} R_{2,4} R_{2,5} R_{3,2} + (-C_{1,4,3,5} C_{1,3,4,5} C_{1,2,2,3} + C_{1,3,1,4} C_{3,5,4,5} C_{1,2,2,3}$$
$$- C_{1,4,4,5} C_{1,2,3,5} C_{1,3,2,3} - C_{1,2,4,5} C_{1,4,2,3} C_{1,3,3,5} + C_{1,4,4,5} C_{1,2,1,3} C_{2,3,3,5} + C_{1,2,1,4} C_{2,3,4,5} C_{1,3,3,5}$$
$$- C_{1,2,2,3} C_{1,4,4,5} C_{1,3,3,5}) R_{1,5} R_{2,4} R_{2,5} R_{3,4} + (-C_{1,2,4,5} C_{1,4,2,3} C_{1,3,3,4} + C_{1,4,4,5} C_{1,2,1,3} C_{2,3,3,4}$$
$$- C_{1,4,4,5} C_{1,2,3,4} C_{1,3,2,3} - C_{1,2,2,3} C_{1,4,4,5} C_{1,3,3,4} - C_{1,4,3,4} C_{1,3,4,5} C_{1,2,2,3} + C_{1,3,1,4} C_{3,4,4,5} C_{1,2,2,3}$$
$$+ C_{1,2,1,4} C_{2,3,4,5} C_{1,3,3,4}) R_{1,5} R_{2,4} R_{2,5} R_{3,5} + (C_{2,4,3,4} C_{1,3,4,5} C_{1,2,2,5} C_{2,3,3,5} - C_{1,2,2,4} C_{2,5,4,5} C_{2,3,3,4} C_{1,3,3,5}$$
$$+ C_{1,2,2,5} C_{2,4,4,5} C_{2,3,3,5} C_{1,3,3,4} + C_{1,2,2,4} C_{2,5,4,5} C_{1,3,2,3} C_{3,4,3,5} - C_{1,3,2,4} C_{3,4,4,5} C_{1,2,2,5} C_{2,3,3,5}$$
$$+ C_{2,4,3,5} C_{2,3,4,5} C_{1,2,2,5} C_{1,3,3,4} - C_{2,3,2,4} C_{3,5,4,5} C_{1,2,2,5} C_{1,3,3,4} - C_{2,5,3,5} C_{2,4,4,5} C_{1,3,3,4} C_{1,2,2,3}$$

$$+C_{2,3,2,5}\,C_{2,4,4,5}\,C_{1,3,3,4}\,C_{1,2,3,5}+C_{1,3,2,5}\,C_{2,4,4,5}\,C_{2,3,3,5}\,C_{1,2,3,4}-C_{1,2,2,4}\,C_{2,5,4,5}\,C_{2,3,3,5}\,C_{1,3,3,4}$$
$$-C_{1,2,4,5}\,C_{2,4,2,5}\,C_{1,3,2,3}\,C_{3,4,3,5}+C_{1,2,4,5}\,C_{2,4,2,5}\,C_{2,3,3,4}\,C_{1,3,3,5}+C_{1,2,4,5}\,C_{2,4,2,5}\,C_{2,3,3,5}\,C_{1,3,3,4}$$
$$+C_{1,2,2,5}\,C_{2,4,4,5}\,C_{2,3,3,4}\,C_{1,3,3,5}-C_{2,5,3,4}\,C_{2,4,4,5}\,C_{2,3,3,5}\,C_{1,2,1,3}-C_{1,2,2,5}\,C_{2,4,4,5}\,C_{1,3,2,3}\,C_{3,4,3,5}\big)\,R_{1,4}\,R_{1,5}$$
$$+\big(C_{1,4,2,5}\,C_{1,2,4,5}\,C_{2,3,3,5}\,C_{1,3,3,4}-C_{2,5,3,5}\,C_{1,4,4,5}\,C_{1,3,3,4}\,C_{1,2,2,3}+C_{2,3,2,5}\,C_{1,4,4,5}\,C_{1,3,3,4}\,C_{1,2,3,5}$$
$$-C_{2,5,3,4}\,C_{1,4,4,5}\,C_{2,3,3,5}\,C_{1,2,1,3}-C_{1,4,2,3}\,C_{3,5,4,5}\,C_{1,2,2,5}\,C_{1,3,3,4}+C_{1,3,2,5}\,C_{1,4,4,5}\,C_{2,3,3,5}\,C_{1,2,3,4}$$
$$+C_{1,4,4,5}\,C_{1,2,2,5}\,C_{2,3,3,5}\,C_{1,3,3,4}-C_{1,3,1,4}\,C_{3,4,4,5}\,C_{1,2,2,5}\,C_{2,3,3,5}-C_{1,2,1,4}\,C_{2,5,4,5}\,C_{2,3,3,5}\,C_{1,3,3,4}$$
$$+C_{1,4,3,4}\,C_{1,3,4,5}\,C_{1,2,2,5}\,C_{2,3,3,5}+C_{1,4,4,5}\,C_{1,2,2,5}\,C_{2,3,3,4}\,C_{1,3,3,5}-C_{1,2,1,4}\,C_{2,5,4,5}\,C_{2,3,3,4}\,C_{1,3,3,5}$$
$$-C_{1,4,2,5}\,C_{1,2,4,5}\,C_{1,3,2,3}\,C_{3,4,3,5}+C_{1,4,2,5}\,C_{1,2,4,5}\,C_{2,3,3,4}\,C_{1,3,3,5}-C_{1,4,4,5}\,C_{1,2,2,5}\,C_{1,3,2,3}\,C_{3,4,3,5}$$
$$+C_{1,2,1,4}\,C_{2,5,4,5}\,C_{1,3,2,3}\,C_{3,4,3,5}+C_{1,4,3,5}\,C_{2,3,4,5}\,C_{1,2,2,5}\,C_{1,3,3,4}\big)\,R_{1,5}\,R_{2,4}+\big(C_{1,2,2,4}\,C_{1,4,4,5}\,C_{2,3,3,4}\,C_{1,3,3,5}$$
$$+C_{1,2,2,4}\,C_{1,4,4,5}\,C_{2,3,3,5}\,C_{1,3,3,4}-C_{1,2,2,4}\,C_{1,4,4,5}\,C_{1,3,2,3}\,C_{3,4,3,5}+C_{1,2,1,4}\,C_{2,4,4,5}\,C_{1,3,2,3}\,C_{3,4,3,5}$$
$$-C_{1,2,1,4}\,C_{2,4,4,5}\,C_{2,3,3,5}\,C_{1,3,3,4}-C_{1,2,1,4}\,C_{2,4,4,5}\,C_{2,3,3,4}\,C_{1,3,3,5}-C_{1,2,1,3}\,C_{2,4,3,4}\,C_{1,4,4,5}\,C_{2,3,3,5}$$
$$+C_{1,4,3,5}\,C_{2,3,4,5}\,C_{1,2,2,4}\,C_{1,3,3,4}+C_{1,2,3,4}\,C_{1,3,2,4}\,C_{1,4,4,5}\,C_{2,3,3,5}-C_{1,4,2,3}\,C_{3,5,4,5}\,C_{1,2,2,4}\,C_{1,3,3,4}$$
$$+C_{1,2,3,5}\,C_{2,3,2,4}\,C_{1,4,4,5}\,C_{1,3,3,4}-C_{1,2,2,3}\,C_{2,4,3,5}\,C_{1,4,4,5}\,C_{1,3,3,4}+C_{1,4,3,4}\,C_{1,3,4,5}\,C_{1,2,2,4}\,C_{2,3,3,5}$$
$$-C_{1,2,4,5}\,C_{1,4,2,4}\,C_{1,3,2,3}\,C_{3,4,3,5}+C_{1,2,4,5}\,C_{1,4,2,4}\,C_{2,3,3,5}\,C_{1,3,3,4}+C_{1,2,4,5}\,C_{1,4,2,4}\,C_{2,3,3,4}\,C_{1,3,3,5}$$
$$-C_{1,3,1,4}\,C_{3,4,4,5}\,C_{1,2,2,4}\,C_{2,3,3,5}\big)\,R_{1,5}\,R_{2,5}:$$

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> *For planar case we* **use** *the following substitutions* :

$$> Su5 := seq\Big(seq\Big(seq\Big(seq\Big(C[i,j,k,l]=\frac{1}{2}\,(r[\min(i,l),\max(i,l)]^2+r[\min(j,k),\max(j,k)]^2-r[\min(i,\mathrm{k}),\max(i,\mathrm{k})]^2-r[\min(j,l),\max(j,1)]^2),\ l=k+1..5\Big),\ k=i..5\Big),\ j=i+1..5\Big),\ i=1..5\Big),$$
$$seq(seq(R[i,j]=r[\min(i,j),\max(i,j)],j=1..5),i=1..5)$$

$$Su5 := C_{1,2,1,2}=r_{1,2}^2,\ C_{1,2,1,3}=\frac{1}{2}r_{1,3}^2+\frac{1}{2}r_{1,2}^2-\frac{1}{2}r_{2,3}^2,\ C_{1,2,1,4}=\frac{1}{2}r_{1,4}^2+\frac{1}{2}r_{1,2}^2-\frac{1}{2}r_{2,4}^2, \quad (1)$$
$$C_{1,2,1,5}=\frac{1}{2}r_{1,5}^2+\frac{1}{2}r_{1,2}^2-\frac{1}{2}r_{2,5}^2,\ C_{1,2,2,3}=-\frac{1}{2}r_{1,2}^2+\frac{1}{2}r_{1,3}^2-\frac{1}{2}r_{2,3}^2,\ C_{1,2,2,4}=-\frac{1}{2}r_{1,2}^2$$
$$+\frac{1}{2}r_{1,4}^2-\frac{1}{2}r_{2,4}^2,\ C_{1,2,2,5}=-\frac{1}{2}r_{1,2}^2+\frac{1}{2}r_{1,5}^2-\frac{1}{2}r_{2,5}^2,\ C_{1,2,3,4}=-\frac{1}{2}r_{1,3}^2+\frac{1}{2}r_{1,4}^2+\frac{1}{2}r_{2,3}^2$$
$$-\frac{1}{2}r_{2,4}^2,\ C_{1,2,3,5}=-\frac{1}{2}r_{1,3}^2+\frac{1}{2}r_{1,5}^2+\frac{1}{2}r_{2,3}^2-\frac{1}{2}r_{2,5}^2,\ C_{1,2,4,5}=-\frac{1}{2}r_{1,4}^2+\frac{1}{2}r_{1,5}^2+\frac{1}{2}r_{2,4}^2$$
$$-\frac{1}{2}r_{2,5}^2,\ C_{1,3,1,2}=\frac{1}{2}r_{1,3}^2+\frac{1}{2}r_{1,2}^2-\frac{1}{2}r_{2,3}^2,\ C_{1,3,1,3}=r_{1,3}^2,\ C_{1,3,1,4}=\frac{1}{2}r_{1,4}^2+\frac{1}{2}r_{1,3}^2-\frac{1}{2}$$
$$r_{3,4}^2,\ C_{1,3,1,5}=\frac{1}{2}r_{1,5}^2+\frac{1}{2}r_{1,3}^2-\frac{1}{2}r_{3,5}^2,\ C_{1,3,2,3}=-\frac{1}{2}r_{1,2}^2+\frac{1}{2}r_{1,3}^2+\frac{1}{2}r_{2,3}^2,\ C_{1,3,2,4}=-\frac{1}{2}r_{1,2}^2$$
$$+\frac{1}{2}r_{1,4}^2+\frac{1}{2}r_{2,3}^2-\frac{1}{2}r_{3,4}^2,\ C_{1,3,2,5}=-\frac{1}{2}r_{1,2}^2+\frac{1}{2}r_{1,5}^2+\frac{1}{2}r_{2,3}^2-\frac{1}{2}r_{3,5}^2,\ C_{1,3,3,4}=-\frac{1}{2}r_{1,3}^2$$
$$+\frac{1}{2}r_{1,4}^2-\frac{1}{2}r_{3,4}^2,\ C_{1,3,3,5}=-\frac{1}{2}r_{1,3}^2+\frac{1}{2}r_{1,5}^2-\frac{1}{2}r_{3,5}^2,\ C_{1,3,4,5}=-\frac{1}{2}r_{1,4}^2+\frac{1}{2}r_{1,5}^2+\frac{1}{2}r_{3,4}^2$$
$$-\frac{1}{2}r_{3,5}^2,\ C_{1,4,1,2}=\frac{1}{2}r_{1,4}^2+\frac{1}{2}r_{1,2}^2-\frac{1}{2}r_{2,4}^2,\ C_{1,4,1,3}=\frac{1}{2}r_{1,4}^2+\frac{1}{2}r_{1,3}^2-\frac{1}{2}r_{3,4}^2,\ C_{1,4,1,4}=$$
$$r_{1,4}^2,\ C_{1,4,1,5}=\frac{1}{2}r_{1,5}^2+\frac{1}{2}r_{1,4}^2-\frac{1}{2}r_{4,5}^2,\ C_{1,4,2,3}=-\frac{1}{2}r_{1,2}^2+\frac{1}{2}r_{1,3}^2+\frac{1}{2}r_{2,4}^2-\frac{1}{2}r_{3,4}^2,\ C_{1,4,2,4}$$
$$=-\frac{1}{2}r_{1,2}^2+\frac{1}{2}r_{1,4}^2+\frac{1}{2}r_{2,4}^2,\ C_{1,4,2,5}=-\frac{1}{2}r_{1,2}^2+\frac{1}{2}r_{1,5}^2+\frac{1}{2}r_{2,4}^2-\frac{1}{2}r_{4,5}^2,\ C_{1,4,3,4}=-\frac{1}{2}r_{1,3}^2$$
$$+\frac{1}{2}r_{1,4}^2+\frac{1}{2}r_{3,4}^2,\ C_{1,4,3,5}=-\frac{1}{2}r_{1,3}^2+\frac{1}{2}r_{1,5}^2+\frac{1}{2}r_{3,4}^2-\frac{1}{2}r_{4,5}^2,\ C_{1,4,4,5}=-\frac{1}{2}r_{1,4}^2+\frac{1}{2}r_{1,5}^2$$
$$-\frac{1}{2}r_{4,5}^2,\ C_{1,5,1,2}=\frac{1}{2}r_{1,5}^2+\frac{1}{2}r_{1,2}^2-\frac{1}{2}r_{2,5}^2,\ C_{1,5,1,3}=\frac{1}{2}r_{1,5}^2+\frac{1}{2}r_{1,3}^2-\frac{1}{2}r_{3,5}^2,\ C_{1,5,1,4}=\frac{1}{2}$$
$$r_{1,5}^2+\frac{1}{2}r_{1,4}^2-\frac{1}{2}r_{4,5}^2,\ C_{1,5,1,5}=r_{1,5}^2,\ C_{1,5,2,3}=-\frac{1}{2}r_{1,2}^2+\frac{1}{2}r_{1,3}^2+\frac{1}{2}r_{2,5}^2-\frac{1}{2}r_{3,5}^2,\ C_{1,5,2,4}=$$

$-\frac{1}{2} r_{1,2}^2 + \frac{1}{2} r_{1,4}^2 + \frac{1}{2} r_{2,5}^2 - \frac{1}{2} r_{4,5}^2, C_{1,5,2,5} = -\frac{1}{2} r_{1,2}^2 + \frac{1}{2} r_{1,5}^2 + \frac{1}{2} r_{2,5}^2, C_{1,5,3,4} = -\frac{1}{2} r_{1,3}^2$
$+ \frac{1}{2} r_{1,4}^2 + \frac{1}{2} r_{3,5}^2 - \frac{1}{2} r_{4,5}^2, C_{1,5,3,5} = -\frac{1}{2} r_{1,3}^2 + \frac{1}{2} r_{1,5}^2 + \frac{1}{2} r_{3,5}^2, C_{1,5,4,5} = -\frac{1}{2} r_{1,4}^2 + \frac{1}{2} r_{1,5}^2$
$+ \frac{1}{2} r_{4,5}^2, C_{2,3,2,3} = r_{2,3}^2, C_{2,3,2,4} = \frac{1}{2} r_{2,4}^2 + \frac{1}{2} r_{2,3}^2 - \frac{1}{2} r_{3,4}^2, C_{2,3,2,5} = \frac{1}{2} r_{2,5}^2 + \frac{1}{2} r_{2,3}^2 - \frac{1}{2}$
$r_{3,5}^2, C_{2,3,3,4} = -\frac{1}{2} r_{2,3}^2 + \frac{1}{2} r_{2,4}^2 - \frac{1}{2} r_{3,4}^2, C_{2,3,3,5} = -\frac{1}{2} r_{2,3}^2 + \frac{1}{2} r_{2,5}^2 - \frac{1}{2} r_{3,5}^2, C_{2,3,4,5} = -\frac{1}{2}$
$r_{2,4}^2 + \frac{1}{2} r_{2,5}^2 + \frac{1}{2} r_{3,4}^2 - \frac{1}{2} r_{3,5}^2, C_{2,4,2,3} = \frac{1}{2} r_{2,4}^2 + \frac{1}{2} r_{2,3}^2 - \frac{1}{2} r_{3,4}^2, C_{2,4,2,4} = r_{2,4}^2, C_{2,4,2,5} = \frac{1}{2}$
$r_{2,5}^2 + \frac{1}{2} r_{2,4}^2 - \frac{1}{2} r_{4,5}^2, C_{2,4,3,4} = -\frac{1}{2} r_{2,3}^2 + \frac{1}{2} r_{2,4}^2 + \frac{1}{2} r_{3,4}^2, C_{2,4,3,5} = -\frac{1}{2} r_{2,3}^2 + \frac{1}{2} r_{2,5}^2 + \frac{1}{2} r_{3,4}^2$
$- \frac{1}{2} r_{4,5}^2, C_{2,4,4,5} = -\frac{1}{2} r_{2,4}^2 + \frac{1}{2} r_{2,5}^2 - \frac{1}{2} r_{4,5}^2, C_{2,5,2,3} = \frac{1}{2} r_{2,5}^2 + \frac{1}{2} r_{2,3}^2 - \frac{1}{2} r_{3,5}^2, C_{2,5,2,4}$
$= \frac{1}{2} r_{2,5}^2 + \frac{1}{2} r_{2,4}^2 - \frac{1}{2} r_{4,5}^2, C_{2,5,2,5} = r_{2,5}^2, C_{2,5,3,4} = -\frac{1}{2} r_{2,3}^2 + \frac{1}{2} r_{2,4}^2 + \frac{1}{2} r_{3,5}^2 - \frac{1}{2} r_{4,5}^2,$
$C_{2,5,3,5} = -\frac{1}{2} r_{2,3}^2 + \frac{1}{2} r_{2,5}^2 + \frac{1}{2} r_{3,5}^2, C_{2,5,4,5} = -\frac{1}{2} r_{2,4}^2 + \frac{1}{2} r_{2,5}^2 + \frac{1}{2} r_{4,5}^2, C_{3,4,3,4} = r_{3,4}^2,$
$C_{3,4,3,5} = \frac{1}{2} r_{3,5}^2 + \frac{1}{2} r_{3,4}^2 - \frac{1}{2} r_{4,5}^2, C_{3,4,4,5} = -\frac{1}{2} r_{3,4}^2 + \frac{1}{2} r_{3,5}^2 - \frac{1}{2} r_{4,5}^2, C_{3,5,3,4} = \frac{1}{2} r_{3,5}^2 + \frac{1}{2}$
$r_{3,4}^2 - \frac{1}{2} r_{4,5}^2, C_{3,5,3,5} = r_{3,5}^2, C_{3,5,4,5} = -\frac{1}{2} r_{3,4}^2 + \frac{1}{2} r_{3,5}^2 + \frac{1}{2} r_{4,5}^2, C_{4,5,4,5} = r_{4,5}^2, R_{1,1} = 0, R_{1,2}$
$= r_{1,2}, R_{1,3} = r_{1,3}, R_{1,4} = r_{1,4}, R_{1,5} = r_{1,5}, R_{2,1} = r_{1,2}, R_{2,2} = 0, R_{2,3} = r_{2,3}, R_{2,4} = r_{2,4}, R_{2,5} = r_{2,5},$
$R_{3,1} = r_{1,3}, R_{3,2} = r_{2,3}, R_{3,3} = 0, R_{3,4} = r_{3,4}, R_{3,5} = r_{3,5}, R_{4,1} = r_{1,4}, R_{4,2} = r_{2,4}, R_{4,3} = r_{3,4}, R_{4,4} = 0,$
$R_{4,5} = r_{4,5}, R_{5,1} = r_{1,5}, R_{5,2} = r_{2,5}, R_{5,3} = r_{3,5}, R_{5,4} = r_{4,5}, R_{5,5} = 0$

>

> $Sur := r_{1,2} = t1 + a12 + a23 + t2, r_{1,3} = t1 + a12 + a23 + b12 + b23 + t3, r_{1,4} = t1 + a12 + b12 + b23 + c23 + t4, r_{1,5} = t1 + b12 + c23 + t5, r_{2,3} = t2 + b12 + b23 + t3, r_{2,4} = t2 + b12 + b23 + a23 + c23 + t4, r_{2,5} = t2 + b12 + a23 + c23 + a12 + t5, r_{3,4} = t3 + a23 + c23 + t4, r_{3,5} = t3 + a23 + c23 + b23 + a12 + t5, r_{4,5} = t4 + b23 + a12 + t5$

$$Sur := r_{1,2} = t1 + a12 + a23 + t2, r_{1,3} = t1 + a12 + a23 + b12 + b23 + t3, r_{1,4} = t1 + a12 + b12 + b23 + c23 + t4, r_{1,5} = t1 + b12 + c23 + t5, r_{2,3} = t2 + b12 + b23 + t3, r_{2,4} = t2 + b12 + b23 + a23 + c23 + t4, r_{2,5} = t2 + b12 + a23 + c23 + a12 + t5, r_{3,4} = t3 + a23 + c23 + t4, r_{3,5} = t3 + a23 + c23 + b23 + a12 + t5, r_{4,5} = t4 + b23 + a12 + t5 \quad (2)$$

> $solve([Sur], \{t1, t2, t3, t4, t5, a12, a23, b12, b23, c23\})$

$$\left\{a12 = \frac{1}{2} r_{2,5} - \frac{1}{2} r_{1,5} - \frac{1}{2} r_{2,4} + \frac{1}{2} r_{1,4}, a23 = \frac{1}{2} r_{2,4} - \frac{1}{2} r_{2,3} - \frac{1}{2} r_{1,4} + \frac{1}{2} r_{1,3}, b12 = \frac{1}{2} r_{2,5} - \frac{1}{2} r_{1,2} - \frac{1}{2} r_{3,5} + \frac{1}{2} r_{1,3}, b23 = \frac{1}{2} r_{3,5} + \frac{1}{2} r_{2,4} - \frac{1}{2} r_{3,4} - \frac{1}{2} r_{2,5}, c23 = \frac{1}{2} r_{1,4} + \frac{1}{2} r_{3,5} - \frac{1}{2} r_{4,5} - \frac{1}{2} r_{1,3}, t1 = -\frac{1}{2} r_{2,5} + \frac{1}{2} r_{1,5} + \frac{1}{2} r_{1,2}, t2 = \frac{1}{2} r_{2,3} + \frac{1}{2} r_{1,2} - \frac{1}{2} r_{1,3}, t3 = -\frac{1}{2} r_{2,4} + \frac{1}{2} r_{3,4} + \frac{1}{2} r_{2,3}, t4 = -\frac{1}{2} r_{3,5} + \frac{1}{2} r_{3,4} + \frac{1}{2} r_{4,5}, t5 = \frac{1}{2} r_{4,5} + \frac{1}{2} r_{1,5} - \frac{1}{2} r_{1,4}\right\} \quad (3)$$

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> $AS5planr\_t = 1080\, m4321 + 2736\, m43111 + 1320\, m4222 + 9104\, m42211 + 1920\, m3331 + 304\, m3322 + 24096\, m33211 + 54864\, m32221 + 136800\, m22222$

>

> $$AS5planr\_t := 1080\, t1^{4}\, t2^{3}\, t3^{2}\, t4 + 1080\, t1^{4}\, t2^{3}\, t3^{2}\, t5 + 1080\, t1^{4}\, t2^{3}\, t3\, t4^{2}$$
$$+ 2736\, t1^{4}\, t2^{3}\, t3\, t4\, t5 + 1080\, t1^{4}\, t2^{3}\, t3\, t5^{2} + 1080\, t1^{4}\, t2^{3}\, t4^{2}\, t5 + 1080\, t1^{4}\, t2^{3}\, t4\, t5^{2}$$
$$+ 1080\, t1^{4}\, t2^{2}\, t3^{3}\, t4 + 1080\, t1^{4}\, t2^{2}\, t3^{3}\, t5 + 1320\, t1^{4}\, t2^{2}\, t3^{2}\, t4^{2} + 9104\, t1^{4}\, t2^{2}\, t3^{2}\, t4\, t5$$
$$+ 1320\, t1^{4}\, t2^{2}\, t3^{2}\, t5^{2} + 1080\, t1^{4}\, t2^{2}\, t3\, t4^{3} + 9104\, t1^{4}\, t2^{2}\, t3\, t4^{2}\, t5 + 9104\, t1^{4}\, t2^{2}\, t3\, t4\, t5^{2}$$
$$+ 1080\, t1^{4}\, t2^{2}\, t3\, t5^{3} + 1080\, t1^{4}\, t2^{2}\, t4^{3}\, t5 + 1320\, t1^{4}\, t2^{2}\, t4^{2}\, t5^{2} + 1080\, t1^{4}\, t2^{2}\, t4\, t5^{3}$$
$$+ 1080\, t1^{4}\, t2\, t3^{3}\, t4^{2} + 2736\, t1^{4}\, t2\, t3^{3}\, t4\, t5 + 1080\, t1^{4}\, t2\, t3^{3}\, t5^{2} + 1080\, t1^{4}\, t2\, t3^{2}\, t4^{3}$$
$$+ 9104\, t1^{4}\, t2\, t3^{2}\, t4^{2}\, t5 + 9104\, t1^{4}\, t2\, t3^{2}\, t4\, t5^{2} + 1080\, t1^{4}\, t2\, t3^{2}\, t5^{3} + 2736\, t1^{4}\, t2\, t3\, t4^{3}\, t5$$
$$+ 9104\, t1^{4}\, t2\, t3\, t4^{2}\, t5^{2} + 2736\, t1^{4}\, t2\, t3\, t4\, t5^{3} + 1080\, t1^{4}\, t2\, t4^{3}\, t5^{2} + 1080\, t1^{4}\, t2\, t4^{2}\, t5^{3}$$
$$+ 1080\, t1^{4}\, t3^{3}\, t4^{2}\, t5 + 1080\, t1^{4}\, t3^{3}\, t4\, t5^{2} + 1080\, t1^{4}\, t3^{2}\, t4^{3}\, t5 + 1320\, t1^{4}\, t3^{2}\, t4^{2}\, t5^{2}$$
$$+ 1080\, t1^{4}\, t3^{2}\, t4\, t5^{3} + 1080\, t1^{4}\, t3\, t4^{3}\, t5^{2} + 1080\, t1^{4}\, t3\, t4^{2}\, t5^{3} + 1080\, t1^{3}\, t2^{4}\, t3^{2}\, t4$$
$$+ 1080\, t1^{3}\, t2^{4}\, t3^{2}\, t5 + 1080\, t1^{3}\, t2^{4}\, t3\, t4^{2} + 2736\, t1^{3}\, t2^{4}\, t3\, t4\, t5 + 1080\, t1^{3}\, t2^{4}\, t3\, t5^{2}$$
$$+ 1080\, t1^{3}\, t2^{4}\, t4^{2}\, t5 + 1080\, t1^{3}\, t2^{4}\, t4\, t5^{2} + 1920\, t1^{3}\, t2^{3}\, t3^{3}\, t4 + 1920\, t1^{3}\, t2^{3}\, t3^{3}\, t5$$
$$+ 304\, t1^{3}\, t2^{3}\, t3^{2}\, t4^{2} + 24096\, t1^{3}\, t2^{3}\, t3^{2}\, t4\, t5 + 304\, t1^{3}\, t2^{3}\, t3^{2}\, t5^{2} + 1920\, t1^{3}\, t2^{3}\, t3\, t4^{3}$$
$$+ 24096\, t1^{3}\, t2^{3}\, t3\, t4^{2}\, t5 + 24096\, t1^{3}\, t2^{3}\, t3\, t4\, t5^{2} + 1920\, t1^{3}\, t2^{3}\, t3\, t5^{3} + 1920\, t1^{3}\, t2^{3}\, t4^{3}\, t5$$
$$+ 304\, t1^{3}\, t2^{3}\, t4^{2}\, t5^{2} + 1920\, t1^{3}\, t2^{3}\, t4\, t5^{3} + 1080\, t1^{3}\, t2^{2}\, t3^{4}\, t4 + 1080\, t1^{3}\, t2^{2}\, t3^{4}\, t5$$
$$+ 304\, t1^{3}\, t2^{2}\, t3^{3}\, t4^{2} + 24096\, t1^{3}\, t2^{2}\, t3^{3}\, t4\, t5 + 304\, t1^{3}\, t2^{2}\, t3^{3}\, t5^{2} + 304\, t1^{3}\, t2^{2}\, t3^{2}\, t4^{3}$$
$$+ 54864\, t1^{3}\, t2^{2}\, t3^{2}\, t4^{2}\, t5 + 54864\, t1^{3}\, t2^{2}\, t3^{2}\, t4\, t5^{2} + 304\, t1^{3}\, t2^{2}\, t3^{2}\, t5^{3}$$
$$+ 1080\, t1^{3}\, t2^{2}\, t3\, t4^{4} + 24096\, t1^{3}\, t2^{2}\, t3\, t4^{3}\, t5 + 54864\, t1^{3}\, t2^{2}\, t3\, t4^{2}\, t5^{2}$$
$$+ 24096\, t1^{3}\, t2^{2}\, t3\, t4\, t5^{3} + 1080\, t1^{3}\, t2^{2}\, t3\, t5^{4} + 1080\, t1^{3}\, t2^{2}\, t4^{4}\, t5 + 304\, t1^{3}\, t2^{2}\, t4^{3}\, t5^{2}$$
$$+ 304\, t1^{3}\, t2^{2}\, t4^{2}\, t5^{3} + 1080\, t1^{3}\, t2^{2}\, t4\, t5^{4} + 1080\, t1^{3}\, t2\, t3^{4}\, t4^{2} + 2736\, t1^{3}\, t2\, t3^{4}\, t4\, t5$$
$$+ 1080\, t1^{3}\, t2\, t3^{4}\, t5^{2} + 1920\, t1^{3}\, t2\, t3^{3}\, t4^{3} + 24096\, t1^{3}\, t2\, t3^{3}\, t4^{2}\, t5 + 24096\, t1^{3}\, t2\, t3^{3}\, t4\, t5^{2}$$
$$+ 1920\, t1^{3}\, t2\, t3^{3}\, t5^{3} + 1080\, t1^{3}\, t2\, t3^{2}\, t4^{4} + 24096\, t1^{3}\, t2\, t3^{2}\, t4^{3}\, t5$$
$$+ 54864\, t1^{3}\, t2\, t3^{2}\, t4^{2}\, t5^{2} + 24096\, t1^{3}\, t2\, t3^{2}\, t4\, t5^{3} + 1080\, t1^{3}\, t2\, t3^{2}\, t5^{4}$$
$$+ 2736\, t1^{3}\, t2\, t3\, t4^{4}\, t5 + 24096\, t1^{3}\, t2\, t3\, t4^{3}\, t5^{2} + 24096\, t1^{3}\, t2\, t3\, t4^{2}\, t5^{3}$$
$$+ 2736\, t1^{3}\, t2\, t3\, t4\, t5^{4} + 1080\, t1^{3}\, t2\, t4^{4}\, t5^{2} + 1920\, t1^{3}\, t2\, t4^{3}\, t5^{3} + 1080\, t1^{3}\, t2\, t4^{2}\, t5^{4}$$
$$+ 1080\, t1^{3}\, t3^{4}\, t4^{2}\, t5 + 1080\, t1^{3}\, t3^{4}\, t4\, t5^{2} + 1920\, t1^{3}\, t3^{3}\, t4^{3}\, t5 + 304\, t1^{3}\, t3^{3}\, t4^{2}\, t5^{2}$$
$$+ 1920\, t1^{3}\, t3^{3}\, t4\, t5^{3} + 1080\, t1^{3}\, t3^{2}\, t4^{4}\, t5 + 304\, t1^{3}\, t3^{2}\, t4^{3}\, t5^{2} + 304\, t1^{3}\, t3^{2}\, t4^{2}\, t5^{3}$$
$$+ 1080\, t1^{3}\, t3^{2}\, t4\, t5^{4} + 1080\, t1^{3}\, t3\, t4^{4}\, t5^{2} + 1920\, t1^{3}\, t3\, t4^{3}\, t5^{3} + 1080\, t1^{3}\, t3\, t4^{2}\, t5^{4}$$
$$+ 1080\, t1^{2}\, t2^{4}\, t3^{3}\, t4 + 1080\, t1^{2}\, t2^{4}\, t3^{3}\, t5 + 1320\, t1^{2}\, t2^{4}\, t3^{2}\, t4^{2} + 9104\, t1^{2}\, t2^{4}\, t3^{2}\, t4\, t5$$
$$+ 1320\, t1^{2}\, t2^{4}\, t3^{2}\, t5^{2} + 1080\, t1^{2}\, t2^{4}\, t3\, t4^{3} + 9104\, t1^{2}\, t2^{4}\, t3\, t4^{2}\, t5 + 9104\, t1^{2}\, t2^{4}\, t3\, t4\, t5^{2}$$
$$+ 1080\, t1^{2}\, t2^{4}\, t3\, t5^{3} + 1080\, t1^{2}\, t2^{4}\, t4^{3}\, t5 + 1320\, t1^{2}\, t2^{4}\, t4^{2}\, t5^{2} + 1080\, t1^{2}\, t2^{4}\, t4\, t5^{3}$$
$$+ 1080\, t1^{2}\, t2^{3}\, t3^{4}\, t4 + 1080\, t1^{2}\, t2^{3}\, t3^{4}\, t5 + 304\, t1^{2}\, t2^{3}\, t3^{3}\, t4^{2} + 24096\, t1^{2}\, t2^{3}\, t3^{3}\, t4\, t5$$
$$+ 304\, t1^{2}\, t2^{3}\, t3^{3}\, t5^{2} + 304\, t1^{2}\, t2^{3}\, t3^{2}\, t4^{3} + 54864\, t1^{2}\, t2^{3}\, t3^{2}\, t4^{2}\, t5$$
$$+ 54864\, t1^{2}\, t2^{3}\, t3^{2}\, t4\, t5^{2} + 304\, t1^{2}\, t2^{3}\, t3^{2}\, t5^{3} + 1080\, t1^{2}\, t2^{3}\, t3\, t4^{4} + 24096\, t1^{2}\, t2^{3}\, t3\, t4^{3}\, t5$$
$$+ 54864\, t1^{2}\, t2^{3}\, t3\, t4^{2}\, t5^{2} + 24096\, t1^{2}\, t2^{3}\, t3\, t4\, t5^{3} + 1080\, t1^{2}\, t2^{3}\, t3\, t5^{4}$$
$$+ 1080\, t1^{2}\, t2^{3}\, t4^{4}\, t5 + 304\, t1^{2}\, t2^{3}\, t4^{3}\, t5^{2} + 304\, t1^{2}\, t2^{3}\, t4^{2}\, t5^{3} + 1080\, t1^{2}\, t2^{3}\, t4\, t5^{4}$$
$$+ 1320\, t1^{2}\, t2^{2}\, t3^{4}\, t4^{2} + 9104\, t1^{2}\, t2^{2}\, t3^{4}\, t4\, t5 + 1320\, t1^{2}\, t2^{2}\, t3^{4}\, t5^{2} + 304\, t1^{2}\, t2^{2}\, t3^{3}\, t4^{3}$$
$$+ 54864\, t1^{2}\, t2^{2}\, t3^{3}\, t4^{2}\, t5 + 54864\, t1^{2}\, t2^{2}\, t3^{3}\, t4\, t5^{2} + 304\, t1^{2}\, t2^{2}\, t3^{3}\, t5^{3}$$
$$+ 1320\, t1^{2}\, t2^{2}\, t3^{2}\, t4^{4} + 54864\, t1^{2}\, t2^{2}\, t3^{2}\, t4^{3}\, t5 + 136800\, t1^{2}\, t2^{2}\, t3^{2}\, t4^{2}\, t5^{2}$$
$$+ 54864\, t1^{2}\, t2^{2}\, t3^{2}\, t4\, t5^{3} + 1320\, t1^{2}\, t2^{2}\, t3^{2}\, t5^{4} + 9104\, t1^{2}\, t2^{2}\, t3\, t4^{4}\, t5$$
$$+ 54864\, t1^{2}\, t2^{2}\, t3\, t4^{3}\, t5^{2} + 54864\, t1^{2}\, t2^{2}\, t3\, t4^{2}\, t5^{3} + 9104\, t1^{2}\, t2^{2}\, t3\, t4\, t5^{4}$$
$$+ 1320\, t1^{2}\, t2^{2}\, t4^{4}\, t5^{2} + 304\, t1^{2}\, t2^{2}\, t4^{3}\, t5^{3} + 1320\, t1^{2}\, t2^{2}\, t4^{2}\, t5^{4} + 1080\, t1^{2}\, t2\, t3^{4}\, t4^{3}$$
$$+ 9104\, t1^{2}\, t2\, t3^{4}\, t4^{2}\, t5 + 9104\, t1^{2}\, t2\, t3^{4}\, t4\, t5^{2} + 1080\, t1^{2}\, t2\, t3^{4}\, t5^{3} + 1080\, t1^{2}\, t2\, t3^{3}\, t4^{4}$$
$$+ 24096\, t1^{2}\, t2\, t3^{3}\, t4^{3}\, t5 + 54864\, t1^{2}\, t2\, t3^{3}\, t4^{2}\, t5^{2} + 24096\, t1^{2}\, t2\, t3^{3}\, t4\, t5^{3}$$

$$+ 1080\, t1^{2}\, t2\, t3^{3}\, t5^{4} + 9104\, t1^{2}\, t2\, t3^{2}\, t4^{4}\, t5 + 54864\, t1^{2}\, t2\, t3^{2}\, t4^{3}\, t5^{2}$$
$$+ 54864\, t1^{2}\, t2\, t3^{2}\, t4^{2}\, t5^{3} + 9104\, t1^{2}\, t2\, t3^{2}\, t4\, t5^{4} + 9104\, t1^{2}\, t2\, t3\, t4^{4}\, t5^{2}$$
$$+ 24096\, t1^{2}\, t2\, t3\, t4^{3}\, t5^{3} + 9104\, t1^{2}\, t2\, t3\, t4^{2}\, t5^{4} + 1080\, t1^{2}\, t2\, t4^{4}\, t5^{3} + 1080\, t1^{2}\, t2\, t4^{3}\, t5^{4}$$
$$+ 1080\, t1^{2}\, t3^{4}\, t4^{3}\, t5 + 1320\, t1^{2}\, t3^{4}\, t4^{2}\, t5^{2} + 1080\, t1^{2}\, t3^{4}\, t4\, t5^{3} + 1080\, t1^{2}\, t3^{3}\, t4^{4}\, t5$$
$$+ 304\, t1^{2}\, t3^{3}\, t4^{3}\, t5^{2} + 304\, t1^{2}\, t3^{3}\, t4^{2}\, t5^{3} + 1080\, t1^{2}\, t3^{3}\, t4\, t5^{4} + 1320\, t1^{2}\, t3^{2}\, t4^{4}\, t5^{2}$$
$$+ 304\, t1^{2}\, t3^{2}\, t4^{3}\, t5^{3} + 1320\, t1^{2}\, t3^{2}\, t4^{2}\, t5^{4} + 1080\, t1^{2}\, t3\, t4^{4}\, t5^{3} + 1080\, t1^{2}\, t3\, t4^{3}\, t5^{4}$$
$$+ 1080\, t1\, t2^{4}\, t3^{3}\, t4^{2} + 2736\, t1\, t2^{4}\, t3^{3}\, t4\, t5 + 1080\, t1\, t2^{4}\, t3^{3}\, t5^{2} + 1080\, t1\, t2^{4}\, t3^{2}\, t4^{3}$$
$$+ 9104\, t1\, t2^{4}\, t3^{2}\, t4^{2}\, t5 + 9104\, t1\, t2^{4}\, t3^{2}\, t4\, t5^{2} + 1080\, t1\, t2^{4}\, t3^{2}\, t5^{3} + 2736\, t1\, t2^{4}\, t3\, t4^{3}\, t5$$
$$+ 9104\, t1\, t2^{4}\, t3\, t4^{2}\, t5^{2} + 2736\, t1\, t2^{4}\, t3\, t4\, t5^{3} + 1080\, t1\, t2^{4}\, t4^{3}\, t5^{2} + 1080\, t1\, t2^{4}\, t4^{2}\, t5^{3}$$
$$+ 1080\, t1\, t2^{3}\, t3^{4}\, t4^{2} + 2736\, t1\, t2^{3}\, t3^{4}\, t4\, t5 + 1080\, t1\, t2^{3}\, t3^{4}\, t5^{2} + 1920\, t1\, t2^{3}\, t3^{3}\, t4^{3}$$
$$+ 24096\, t1\, t2^{3}\, t3^{3}\, t4^{2}\, t5 + 24096\, t1\, t2^{3}\, t3^{3}\, t4\, t5^{2} + 1920\, t1\, t2^{3}\, t3^{3}\, t5^{3} + 1080\, t1\, t2^{3}\, t3^{2}\, t4^{4}$$
$$+ 24096\, t1\, t2^{3}\, t3^{2}\, t4^{3}\, t5 + 54864\, t1\, t2^{3}\, t3^{2}\, t4^{2}\, t5^{2} + 24096\, t1\, t2^{3}\, t3^{2}\, t4\, t5^{3}$$
$$+ 1080\, t1\, t2^{3}\, t3^{2}\, t5^{4} + 2736\, t1\, t2^{3}\, t3\, t4^{4}\, t5 + 24096\, t1\, t2^{3}\, t3\, t4^{3}\, t5^{2}$$
$$+ 24096\, t1\, t2^{3}\, t3\, t4^{2}\, t5^{3} + 2736\, t1\, t2^{3}\, t3\, t4\, t5^{4} + 1080\, t1\, t2^{3}\, t4^{4}\, t5^{2} + 1920\, t1\, t2^{3}\, t4^{3}\, t5^{3}$$
$$+ 1080\, t1\, t2^{3}\, t4^{2}\, t5^{4} + 1080\, t1\, t2^{2}\, t3^{4}\, t4^{3} + 9104\, t1\, t2^{2}\, t3^{4}\, t4^{2}\, t5 + 9104\, t1\, t2^{2}\, t3^{4}\, t4\, t5^{2}$$
$$+ 1080\, t1\, t2^{2}\, t3^{4}\, t5^{3} + 1080\, t1\, t2^{2}\, t3^{3}\, t4^{4} + 24096\, t1\, t2^{2}\, t3^{3}\, t4^{3}\, t5$$
$$+ 54864\, t1\, t2^{2}\, t3^{3}\, t4^{2}\, t5^{2} + 24096\, t1\, t2^{2}\, t3^{3}\, t4\, t5^{3} + 1080\, t1\, t2^{2}\, t3^{3}\, t5^{4}$$
$$+ 9104\, t1\, t2^{2}\, t3^{2}\, t4^{4}\, t5 + 54864\, t1\, t2^{2}\, t3^{2}\, t4^{3}\, t5^{2} + 54864\, t1\, t2^{2}\, t3^{2}\, t4^{2}\, t5^{3}$$
$$+ 9104\, t1\, t2^{2}\, t3^{2}\, t4\, t5^{4} + 9104\, t1\, t2^{2}\, t3\, t4^{4}\, t5^{2} + 24096\, t1\, t2^{2}\, t3\, t4^{3}\, t5^{3}$$
$$+ 9104\, t1\, t2^{2}\, t3\, t4^{2}\, t5^{4} + 1080\, t1\, t2^{2}\, t4^{4}\, t5^{3} + 1080\, t1\, t2^{2}\, t4^{3}\, t5^{4} + 2736\, t1\, t2\, t3^{4}\, t4^{3}\, t5$$
$$+ 9104\, t1\, t2\, t3^{4}\, t4^{2}\, t5^{2} + 2736\, t1\, t2\, t3^{4}\, t4\, t5^{3} + 2736\, t1\, t2\, t3^{3}\, t4^{4}\, t5$$
$$+ 24096\, t1\, t2\, t3^{3}\, t4^{3}\, t5^{2} + 24096\, t1\, t2\, t3^{3}\, t4^{2}\, t5^{3} + 2736\, t1\, t2\, t3^{3}\, t4\, t5^{4}$$
$$+ 9104\, t1\, t2\, t3^{2}\, t4^{4}\, t5^{2} + 24096\, t1\, t2\, t3^{2}\, t4^{3}\, t5^{3} + 9104\, t1\, t2\, t3^{2}\, t4^{2}\, t5^{4}$$
$$+ 2736\, t1\, t2\, t3\, t4^{4}\, t5^{3} + 2736\, t1\, t2\, t3\, t4^{3}\, t5^{4} + 1080\, t1\, t3^{4}\, t4^{3}\, t5^{2} + 1080\, t1\, t3^{4}\, t4^{2}\, t5^{3}$$
$$+ 1080\, t1\, t3^{3}\, t4^{4}\, t5^{2} + 1920\, t1\, t3^{3}\, t4^{3}\, t5^{3} + 1080\, t1\, t3^{3}\, t4^{2}\, t5^{4} + 1080\, t1\, t3^{2}\, t4^{4}\, t5^{3}$$
$$+ 1080\, t1\, t3^{2}\, t4^{3}\, t5^{4} + 1080\, t2^{4}\, t3^{3}\, t4^{2}\, t5 + 1080\, t2^{4}\, t3^{3}\, t4\, t5^{2} + 1080\, t2^{4}\, t3^{2}\, t4^{3}\, t5$$
$$+ 1320\, t2^{4}\, t3^{2}\, t4^{2}\, t5^{2} + 1080\, t2^{4}\, t3^{2}\, t4\, t5^{3} + 1080\, t2^{4}\, t3\, t4^{3}\, t5^{2} + 1080\, t2^{4}\, t3\, t4^{2}\, t5^{3}$$
$$+ 1080\, t2^{3}\, t3^{4}\, t4^{2}\, t5 + 1080\, t2^{3}\, t3^{4}\, t4\, t5^{2} + 1920\, t2^{3}\, t3^{3}\, t4^{3}\, t5 + 304\, t2^{3}\, t3^{3}\, t4^{2}\, t5^{2}$$
$$+ 1920\, t2^{3}\, t3^{3}\, t4\, t5^{3} + 1080\, t2^{3}\, t3^{2}\, t4^{4}\, t5 + 304\, t2^{3}\, t3^{2}\, t4^{3}\, t5^{2} + 304\, t2^{3}\, t3^{2}\, t4^{2}\, t5^{3}$$
$$+ 1080\, t2^{3}\, t3^{2}\, t4\, t5^{4} + 1080\, t2^{3}\, t3\, t4^{4}\, t5^{2} + 1920\, t2^{3}\, t3\, t4^{3}\, t5^{3} + 1080\, t2^{3}\, t3\, t4^{2}\, t5^{4}$$
$$+ 1080\, t2^{2}\, t3^{4}\, t4^{3}\, t5 + 1320\, t2^{2}\, t3^{4}\, t4^{2}\, t5^{2} + 1080\, t2^{2}\, t3^{4}\, t4\, t5^{3} + 1080\, t2^{2}\, t3^{3}\, t4^{4}\, t5$$
$$+ 304\, t2^{2}\, t3^{3}\, t4^{3}\, t5^{2} + 304\, t2^{2}\, t3^{3}\, t4^{2}\, t5^{3} + 1080\, t2^{2}\, t3^{3}\, t4\, t5^{4} + 1320\, t2^{2}\, t3^{2}\, t4^{4}\, t5^{2}$$
$$+ 304\, t2^{2}\, t3^{2}\, t4^{3}\, t5^{3} + 1320\, t2^{2}\, t3^{2}\, t4^{2}\, t5^{4} + 1080\, t2^{2}\, t3\, t4^{4}\, t5^{3} + 1080\, t2^{2}\, t3\, t4^{3}\, t5^{4}$$
$$+ 1080\, t2\, t3^{4}\, t4^{3}\, t5^{2} + 1080\, t2\, t3^{4}\, t4^{2}\, t5^{3} + 1080\, t2\, t3^{3}\, t4^{4}\, t5^{2} + 1920\, t2\, t3^{3}\, t4^{3}\, t5^{3}$$
$$+ 1080\, t2\, t3^{3}\, t4^{2}\, t5^{4} + 1080\, t2\, t3^{2}\, t4^{4}\, t5^{3} + 1080\, t2\, t3^{2}\, t4^{3}\, t5^{4} :$$

```
> Y := sort( map(factor, collect( AS5planr_2 − AS5planr_t, {t1, t2, t3, t4, t5}, distributed) ), [t1, t2, t3, t4, t5]) : length(Y);
  nops(Y)
```

1684654

1992 **(4)**

```
> Ya := subs(Sub5, Y) : length(Ya), nops(Ya)
```

1299484, 1992 **(5)**

>

>

> *The wanted Atiyah's determinant has two parts :*

(1) AS5planr_t (depending on tangential segments t1, t2, t3, t4, t5 only)

and

> (2) $Ya1$
(depending on at least one of intermediate variables $a12, a23, b12, b23, c23$ abreviated via substitution $(Sub5 := a12 = U, a23 = u, b12 = V, b23 = v, c23 = w)$

> $Ya1 := sort(map(factor, collect(Ya, \{U, V, u, v, w\}, distributed)), [u, v, w, U, V]) : length(Ya1), nops(Ya1)$

$$1485144, 2552 \quad (6)$$

>

$$\{U, V, t1, t2, t3, t4, t5, u, v, w\} \quad (7)$$

> The 2552 terms of Ya1, divided **by** 8, are listed **in** the following order:

> The first 10 terms are:

> **for** k **to** 10 **do** $k, op\left(k, \frac{Ya1}{8}\right)$ **od**

$$1, 795\ (t3 + t4)\ U^4 V^5$$

$$2, (2286\ t1\ t3 + 2286\ t1\ t4 + 2787\ t2\ t3 + 2787\ t2\ t4 + 1780\ t3^2 + 6216\ t3\ t4 + 2787\ t3\ t5 + 1780\ t4^2 + 2787\ t4\ t5)\ U^4 V^4$$

$$3, 2\ (64\ t1^2 + 128\ t1\ t2 + 128\ t1\ t3 + 215\ t1\ t4 + 459\ t1\ t5 + 128\ t2\ t4 + 209\ t2\ t5 + 128\ t3\ t4 + 209\ t3\ t5 + 64\ t4^2 + 459\ t4\ t5 + 215\ t5^2)\ u^6\ v\ V$$

$$4, (126139\ t1 + 126139\ t2 + 107719\ t3 + 121872\ t4 + 107719\ t5)\ u\ v^2\ w^2\ U\ V^3$$

$$5, (61424\ t1^2 + 190074\ t1\ t2 + 186147\ t1\ t3 + 155050\ t1\ t4 + 153955\ t1\ t5 + 61424\ t2^2 + 153955\ t2\ t3 + 155050\ t2\ t4 + 186147\ t2\ t5 + 53381\ t3^2 + 126174\ t3\ t4 + 171180\ t3\ t5 + 40750\ t4^2 + 126174\ t4\ t5 + 53381\ t5^2)\ u^2\ v^2\ w^2\ U^2$$

$$6, (8533\ t1 + 10471\ t2 + 4096\ t3 + 5760\ t4 + 5248\ t5)\ v\ w^4\ U\ V^3$$

$$7, (13587\ t1^2 + 42057\ t1\ t2 + 41130\ t1\ t3 + 45978\ t1\ t4 + 36909\ t1\ t5 + 16459\ t2^2 + 44006\ t2\ t3 + 51876\ t2\ t4 + 45597\ t2\ t5 + 6912\ t3^2 + 26096\ t3\ t4 + 27376\ t3\ t5 + 9728\ t4^2 + 25328\ t4\ t5 + 8448\ t5^2)\ v\ w^3\ U\ V^3$$

$$8, (15126\ t1^3 + 96954\ t1^2\ t2 + 104032\ t1^2\ t3 + 90558\ t1^2\ t4 + 82348\ t1^2\ t5 + 83779\ t1\ t2^2 + 232140\ t1\ t2\ t3 + 253815\ t1\ t2\ t4 + 269415\ t1\ t2\ t5 + 87790\ t1\ t3^2 + 227111\ t1\ t3\ t4 + 282427\ t1\ t3\ t5 + 77869\ t1\ t4^2 + 228334\ t1\ t4\ t5 + 80611\ t1\ t5^2 + 7294\ t2^3 + 42549\ t2^2\ t3 + 55605\ t2^2\ t4 + 87975\ t2^2\ t5 + 43209\ t2\ t3^2 + 135900\ t2\ t3\ t4 + 236198\ t2\ t3\ t5 + 55605\ t2\ t4^2 + 231411\ t2\ t4\ t5 + 98229\ t2\ t5^2 + 7614\ t3^3 + 43209\ t3^2\ t4 + 84608\ t3^2\ t5 + 42549\ t3\ t4^2 + 197931\ t3\ t4\ t5 + 98195\ t3\ t5^2 + 7294\ t4^3 + 62859\ t4^2\ t5 + 73974\ t4\ t5^2 + 14007\ t5^3)\ u^2\ v^2\ w\ U^2$$

$$9, (4917\ t1^3 + 29600\ t1^2\ t2 + 43361\ t1^2\ t3 + 48898\ t1^2\ t4 + 34670\ t1^2\ t5 + 31038\ t1\ t2^2 + 122402\ t1\ t2\ t3 + 143542\ t1\ t2\ t4 + 108982\ t1\ t2\ t5 + 44661\ t1\ t3^2 + 140811\ t1\ t3\ t4 + 123669\ t1\ t3\ t5 + 51356\ t1\ t4^2 + 114967\ t1\ t4\ t5 + 34227\ t1\ t5^2 + 5779\ t2^3 + 45116\ t2^2\ t3 + 54527\ t2^2\ t4 + 44328\ t2^2\ t5 + 44023\ t2\ t3^2 + 141687\ t2\ t3\ t4 + 140441\ t2\ t3\ t5 + 55546\ t2\ t4^2 + 135327\ t2\ t4\ t5 + 43363\ t2\ t5^2 + 3584\ t3^3 + 27068\ t3^2\ t4 + 31036\ t3^2\ t5 + 29740\ t3\ t4^2 + 87464\ t3\ t4\ t5 + 31724\ t3\ t5^2 + 5248\ t4^3 + 27692\ t4^2\ t5 + 25964\ t4\ t5^2 + 4224\ t5^3)\ v\ w^2\ U\ V^3$$

(8)

$$10, 2\,(5003\,t1 + 5867\,t2 + 5003\,t3 + 8161\,t4 + 7869\,t5)\,u^2\,w^4\,U\,V^2 \qquad \textbf{(8)}$$

>

> *The last* 10 *terms are* :

>

$$2538, (770\,t1^4\,t2\,t3 + 1170\,t1^4\,t2\,t4 + 770\,t1^4\,t2\,t5 + 640\,t1^4\,t3^2 + 2717\,t1^4\,t3\,t4$$
$$+ 1647\,t1^4\,t3\,t5 + 826\,t1^4\,t4^2 + 1663\,t1^4\,t4\,t5 + 384\,t1^4\,t5^2 + 3820\,t1^3\,t2^2\,t3$$
$$+ 5762\,t1^3\,t2^2\,t4 + 3820\,t1^3\,t2^2\,t5 + 7424\,t1^3\,t2\,t3^2 + 31732\,t1^3\,t2\,t3\,t4$$
$$+ 21448\,t1^3\,t2\,t3\,t5 + 11299\,t1^3\,t2\,t4^2 + 25702\,t1^3\,t2\,t4\,t5 + 5888\,t1^3\,t2\,t5^2$$
$$+ 1920\,t1^3\,t3^3 + 18410\,t1^3\,t3^2\,t4 + 11990\,t1^3\,t3^2\,t5 + 19406\,t1^3\,t4^2\,t3$$
$$+ 44386\,t1^3\,t3\,t4\,t5 + 10191\,t1^3\,t3\,t5^2 + 2793\,t1^3\,t4^3 + 12318\,t1^3\,t4^2\,t5 + 9845\,t1^3\,t4\,t5^2$$
$$+ 896\,t1^3\,t5^3 + 3820\,t1^2\,t2^3\,t3 + 5762\,t1^2\,t2^3\,t4 + 3820\,t1^2\,t2^3\,t5 + 13751\,t1^2\,t2^2\,t3^2$$
$$+ 60428\,t1^2\,t2^2\,t3\,t4 + 45256\,t1^2\,t2^2\,t3\,t5 + 23654\,t1^2\,t2^2\,t4^2 + 60428\,t1^2\,t2^2\,t4\,t5$$
$$+ 13751\,t1^2\,t2^2\,t5^2 + 8922\,t1^2\,t2\,t3^3 + 86816\,t1^2\,t2\,t3^2\,t4 + 63192\,t1^2\,t2\,t3^2\,t5$$
$$+ 95825\,t1^2\,t2\,t3\,t4^2 + 257892\,t1^2\,t2\,t3\,t4\,t5 + 60444\,t1^2\,t2\,t3\,t5^2 + 15769\,t1^2\,t2\,t4^3$$
$$+ 83927\,t1^2\,t2\,t4^2\,t5 + 72542\,t1^2\,t2\,t4\,t5^2 + 7386\,t1^2\,t2\,t5^3 + 896\,t1^2\,t3^4$$
$$+ 19509\,t1^2\,t3^3\,t4 + 15343\,t1^2\,t3^3\,t5 + 44620\,t1^2\,t3^2\,t4^2 + 121461\,t1^2\,t3^2\,t4\,t5$$
$$+ 29152\,t1^2\,t3^2\,t5^2 + 21790\,t1^2\,t3\,t4^3 + 122743\,t1^2\,t3\,t4^2\,t5 + 102364\,t1^2\,t3\,t4\,t5^2$$
$$+ 11715\,t1^2\,t3\,t5^3 + 1415\,t1^2\,t4^4 + 16414\,t1^2\,t4^3\,t5 + 27770\,t1^2\,t4^2\,t5^2$$
$$+ 10559\,t1^2\,t4\,t5^3 + 384\,t1^2\,t5^4 + 770\,t1\,t2^4\,t3 + 1170\,t1\,t2^4\,t4 + 770\,t1\,t2^4\,t5$$
$$+ 5888\,t1\,t2^3\,t3^2 + 25702\,t1\,t2^3\,t3\,t4 + 21448\,t1\,t2^3\,t3\,t5 + 11299\,t1\,t2^3\,t4^2$$
$$+ 31732\,t1\,t2^3\,t4\,t5 + 7424\,t1\,t2^3\,t5^2 + 7386\,t1\,t2^2\,t3^3 + 72542\,t1\,t2^2\,t3^2\,t4$$
$$+ 60444\,t1\,t2^2\,t3^2\,t5 + 83927\,t1\,t2^2\,t3\,t4^2 + 257892\,t1\,t2^2\,t3\,t4\,t5 + 63192\,t1\,t2^2\,t3\,t5^2$$
$$+ 15769\,t1\,t2^2\,t4^3 + 95825\,t1\,t2^2\,t4^2\,t5 + 86816\,t1\,t2^2\,t4\,t5^2 + 8922\,t1\,t2^2\,t5^3$$
$$+ 1756\,t1\,t2\,t3^4 + 40609\,t1\,t2\,t3^3\,t4 + 36023\,t1\,t2\,t3^3\,t5 + 98818\,t1\,t2\,t3^2\,t4^2$$
$$+ 310729\,t1\,t2\,t3^2\,t4\,t5 + 77270\,t1\,t2\,t3^2\,t5^2 + 52280\,t1\,t2\,t3\,t4^3 + 341454\,t1\,t2\,t3\,t4^2\,t5$$
$$+ 310729\,t1\,t2\,t3\,t4\,t5^2 + 36023\,t1\,t2\,t3\,t5^3 + 3972\,t1\,t2\,t4^4 + 52280\,t1\,t2\,t4^3\,t5$$
$$+ 98818\,t1\,t2\,t4^2\,t5^2 + 40609\,t1\,t2\,t4\,t5^3 + 1756\,t1\,t2\,t5^4 + 3458\,t1\,t3^4\,t4$$
$$+ 3438\,t1\,t3^4\,t5 + 20063\,t1\,t3^3\,t4^2 + 59579\,t1\,t3^3\,t4\,t5 + 17067\,t1\,t3^3\,t5^2$$
$$+ 21316\,t1\,t3^2\,t4^3 + 134545\,t1\,t3^2\,t4^2\,t5 + 125288\,t1\,t3^2\,t4\,t5^2 + 15753\,t1\,t3^2\,t5^3$$
$$+ 4556\,t1\,t3\,t4^4 + 61500\,t1\,t3\,t4^3\,t5 + 123349\,t1\,t3\,t4^2\,t5^2 + 49077\,t1\,t3\,t4\,t5^3$$
$$+ 2643\,t1\,t3\,t5^4 + 3866\,t1\,t4^4\,t5 + 16436\,t1\,t4^3\,t5^2 + 14015\,t1\,t4^2\,t5^3 + 2137\,t1\,t4\,t5^4$$
$$+ 384\,t2^4\,t3^2 + 1663\,t2^4\,t3\,t4 + 1647\,t2^4\,t3\,t5 + 826\,t2^4\,t4^2 + 2717\,t2^4\,t4\,t5$$
$$+ 640\,t2^4\,t5^2 + 896\,t2^3\,t3^3 + 9845\,t2^3\,t3^2\,t4 + 10191\,t2^3\,t3^2\,t5 + 12318\,t2^3\,t3\,t4^2$$
$$+ 44386\,t2^3\,t3\,t4\,t5 + 11990\,t2^3\,t3\,t5^2 + 2793\,t2^3\,t4^3 + 19406\,t2^3\,t4^2\,t5$$
$$+ 18410\,t2^3\,t4\,t5^2 + 1920\,t2^3\,t5^3 + 384\,t2^2\,t3^4 + 10559\,t2^2\,t3^3\,t4 + 11715\,t2^2\,t3^3\,t5$$
$$+ 27770\,t2^2\,t3^2\,t4^2 + 102364\,t2^2\,t3^2\,t4\,t5 + 29152\,t2^2\,t3^2\,t5^2 + 16414\,t2^2\,t3\,t4^3$$
$$+ 122743\,t2^2\,t3\,t4^2\,t5 + 121461\,t2^2\,t3\,t4\,t5^2 + 15343\,t2^2\,t3\,t5^3 + 1415\,t2^2\,t4^4$$
$$+ 21790\,t2^2\,t4^3\,t5 + 44620\,t2^2\,t4^2\,t5^2 + 19509\,t2^2\,t4\,t5^3 + 896\,t2^2\,t5^4 + 2137\,t2\,t3^4\,t4$$
$$+ 2643\,t2\,t3^4\,t5 + 14015\,t2\,t3^3\,t4^2 + 49077\,t2\,t3^3\,t4\,t5 + 15753\,t2\,t3^3\,t5^2$$

$+ 16436\, t2\, t3^2\, t4^3 + 123349\, t2\, t3^2\, t4^2\, t5 + 125288\, t2\, t3^2\, t4\, t5^2 + 17067\, t2\, t3^2\, t5^3$
$+ 3866\, t2\, t3\, t4^4 + 61500\, t2\, t3\, t4^3\, t5 + 134545\, t2\, t3\, t4^2\, t5^2 + 59579\, t2\, t3\, t4\, t5^3$
$+ 3438\, t2\, t3\, t5^4 + 4556\, t2\, t4^4\, t5 + 21316\, t2\, t4^3\, t5^2 + 20063\, t2\, t4^2\, t5^3 + 3458\, t2\, t4\, t5^4$
$+ 1081\, t3^4\, t4^2 + 3437\, t3^4\, t4\, t5 + 1351\, t3^4\, t5^2 + 2957\, t3^3\, t4^3 + 19747\, t3^3\, t4^2\, t5$
$+ 20624\, t3^3\, t4\, t5^2 + 3098\, t3^3\, t5^3 + 1397\, t3^2\, t4^4 + 20529\, t3^2\, t4^3\, t5 + 44452\, t3^2\, t4^2\, t5^2$
$+ 20624\, t3^2\, t4\, t5^3 + 1351\, t3^2\, t5^4 + 3848\, t3\, t4^4\, t5 + 20529\, t3\, t4^3\, t5^2 + 19747\, t3\, t4^2\, t5^3$
$+ 3437\, t3\, t4\, t5^4 + 1397\, t4^4\, t5^2 + 2957\, t4^3\, t5^3 + 1081\, t4^2\, t5^4)\, u\, U\, V^2$

$2539, 8\, (32\, t1 + 73\, t2 + 122\, t3 + 32\, t4 + 32\, t5)\, u\, w\, U^6\, V$

$2540, 8\, (32\, t1 + 32\, t2 + 122\, t3 + 73\, t4 + 32\, t5)\, v\, w\, U^6\, V$

$2541, (2176\, t1 + 3927\, t2 + 5317\, t3 + 1792\, t4 + 2304\, t5)\, u^2\, w\, U^5\, V$

$2542, (4651\, t1 + 7158\, t2 + 11492\, t3 + 7676\, t4 + 4977\, t5)\, u\, w\, U^5\, V^2$

$2543, 2\, (1551\, t1 + 1980\, t2 + 5039\, t3 + 3706\, t4 + 1615\, t5)\, v\, w\, U^5\, V^2$

$2544, 128\, (3\, t3 + 2\, t4 + 2\, t1 + 2\, t2 + 2\, t5)\, u\, v\, w\, U^6$

$2545, 128\, (t3 + t4)\, (t1 + t5)\, v^2\, U^6$

$2546, 128\, (2\, t2\, t3 + 2\, t2\, t4 + t3^2 + 3\, t3\, t4 + t4^2)\, (t1 + t5)\, v\, U^6$

$2547, 128\, (2\, t1\, t4 + 2\, t1\, t5 + 2\, t2\, t4 + 2\, t2\, t5 + 2\, t3\, t4 + 2\, t3\, t5 + t4^2 + 3\, t4\, t5$
$+ t5^2)\, u^6\, w\, V$

$2548, 512\, (t4 + t5)\, u\, w^3\, V^5$

$2549, 128\, (17\, t1\, t4 + 17\, t1\, t5 + 18\, t2\, t4 + 18\, t2\, t5 + 14\, t3\, t4 + 14\, t3\, t5 + 6\, t4^2 + 18\, t4\, t5$
$+ 6\, t5^2)\, u\, w^2\, V^5$

$2550, 2\, (384\, t1^2\, t4 + 384\, t1^2\, t5 + 1146\, t1\, t2\, t4 + 1146\, t1\, t2\, t5 + 2234\, t1\, t3\, t4$
$+ 2234\, t1\, t3\, t5 + 1056\, t1\, t4^2 + 3253\, t1\, t4\, t5 + 1056\, t1\, t5^2 + 384\, t2^2\, t4 + 384\, t2^2\, t5$
$+ 1978\, t2\, t3\, t4 + 1978\, t2\, t3\, t5 + 1120\, t2\, t4^2 + 3427\, t2\, t4\, t5 + 1120\, t2\, t5^2 + 640\, t3^2\, t4$
$+ 640\, t3^2\, t5 + 832\, t3\, t4^2 + 2660\, t3\, t4\, t5 + 832\, t3\, t5^2 + 128\, t4^3 + 832\, t4^2\, t5 + 832\, t4\, t5^2$
$+ 128\, t5^3)\, u\, w\, V^5$

$2551, 512\, (t3 + t4)\, (t1 + t2)\, v^3\, V^5$

$2552, 128\, (3\, t1^2\, t3 + 3\, t1^2\, t4 + 9\, t1\, t2\, t3 + 9\, t1\, t2\, t4 + 6\, t1\, t3^2 + 18\, t1\, t3\, t4 + 14\, t1\, t3\, t5$ **(9)**
$+ 6\, t1\, t4^2 + 14\, t1\, t4\, t5 + 3\, t2^2\, t3 + 3\, t2^2\, t4 + 6\, t2\, t3^2 + 18\, t2\, t3\, t4 + 14\, t2\, t3\, t5$
$+ 6\, t2\, t4^2 + 14\, t2\, t4\, t5)\, v^2\, V^5$

>

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> *COMPLETE FORMULA is* 103 $-$ *pages long positive polynomial* **for** *ATIYAH SUTCLIFFE DETERMINANT AS5* **in** *t1,..., t5 , a12 = U, a23 = u, b12 = V, b23 = v, c23 = w* (*tangential* **and** *middle variables*) *. THIS COMPLETES THE PROOF OF planar* 5 $-$ *point ATIYAH*

's LINEAR INDEPENDENCE CONJECTURE.

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Our trigonometric (euclidean) Eastwood–Norbury formula (where $c_{i\_jk} := \cos(ij, ik)$ and $c_{ij\_kl} := \cos(ij, kl)$):

$$
\begin{aligned}
16Re(D_4) = \; & (1 + c_{3\_12} + c_{2\_34})(1 + c_{1\_24} + c_{4\_13}) + \\
& (1 + c_{2\_13} + c_{3\_24})(1 + c_{4\_12} + c_{1\_34}) + \\
& (1 + c_{3\_12} + c_{1\_34})(1 + c_{2\_14} + c_{4\_23}) + \\
& (1 + c_{1\_23} + c_{3\_14})(1 + c_{2\_34} + c_{4\_12}) + \\
& (1 + c_{2\_13} + c_{1\_24})(1 + c_{3\_14} + c_{4\_23}) + \\
& (1 + c_{1\_23} + c_{2\_14})(1 + c_{3\_24} + c_{4\_13}) + \\
& 2(c_{14\_23}c_{13\_24} - c_{14\_23}c_{12\_34} + c_{13\_24}c_{12\_34}) + \\
& 72(\text{normalized volume})^2.
\end{aligned}
$$

Open problems:

Hyperbolic (euclidean) version for $n \geq 4$ ($n \geq 5$) points in terms of distances, or in terms of angles.

NEW TRIGONOMETRIC FORMULA FOR NORMALIZED VOLUME OF A TETRAHEDRON

A law of sines for tetrahedra and the space of all shapes of tetrahedra

A corollary of the usual law of sines is that in a tetrahedron with vertices O, A, B, C, we have

$$\sin\angle OAB \cdot \sin\angle OBC \cdot \sin\angle OCA = \sin\angle OAC \cdot \sin\angle OCB \cdot \sin\angle OBA.$$

Putting any of the four vertices in the role of O yields four such identities, but at most three of them are independent: If the "clockwise" sides of three of them are multiplied and the product is inferred to be equal to the product of the "counterclockwise" sides of the same three identities, and then common factors are canceled from both sides, the result is the fourth identity. Three angles are the angles of some triangle if and only if their sum is 180° (Pi radians). What condition on 12 angles is necessary and sufficient for them to be the 12 angles of some tetrahedron? Clearly the sum of the angles of any side of the tetrahedron must be 180°. Since there are four such triangles, there are four such constraints on sums of angles, and the number of degrees of freedom is thereby reduced from 12 to 8. The four relations given by this sine law further reduce the number of degrees of freedom, from 8 down to not 4 but 5, since the fourth constraint is not independent of the first three. Thus the space of all shapes of tetrahedra is 5-dimensional.

Recall the Cayley-Menger matrix, the squared volume of a tetrahedron:

$$M_4 = \begin{bmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & r_{12}^2 & r_{13}^2 & r_{14}^2 \\ 1 & r_{12}^2 & 0 & r_{23}^2 & r_{24}^2 \\ 1 & r_{13}^2 & r_{23}^2 & 0 & r_{34}^2 \\ 1 & r_{14}^2 & r_{24}^2 & r_{34}^2 & 0 \end{bmatrix}$$

$$Vsq := \frac{Determinant\,(M_4)}{2^3\,(3!)^2}$$

$$
\begin{aligned}
Vsq := & -\frac{r_{12}^4 r_{34}^2}{144} - \frac{r_{12}^2 r_{13}^2 r_{23}^2}{144} + \frac{r_{12}^2 r_{13}^2 r_{24}^2}{144} + \frac{r_{12}^2 r_{13}^2 r_{34}^2}{144} \\
& + \frac{r_{12}^2 r_{14}^2 r_{23}^2}{144} - \frac{r_{12}^2 r_{14}^2 r_{24}^2}{144} + \frac{r_{12}^2 r_{14}^2 r_{34}^2}{144} + \frac{r_{12}^2 r_{23}^2 r_{34}^2}{144} \\
& + \frac{r_{12}^2 r_{24}^2 r_{34}^2}{144} - \frac{r_{12}^2 r_{34}^4}{144} - \frac{r_{13}^4 r_{24}^2}{144} + \frac{r_{13}^2 r_{14}^2 r_{23}^2}{144} \\
& + \frac{r_{13}^2 r_{14}^2 r_{24}^2}{144} - \frac{r_{13}^2 r_{14}^2 r_{34}^2}{144} + \frac{r_{13}^2 r_{23}^2 r_{24}^2}{144} - \frac{r_{13}^2 r_{24}^4}{144} \\
& + \frac{r_{13}^2 r_{24}^2 r_{34}^2}{144} - \frac{r_{14}^4 r_{23}^2}{144} - \frac{r_{14}^2 r_{23}^4}{144} + \frac{r_{14}^2 r_{23}^2 r_{24}^2}{144} \\
& + \frac{r_{14}^2 r_{23}^2 r_{34}^2}{144} - \frac{r_{23}^2 r_{24}^2 r_{34}^2}{144}
\end{aligned}
$$

$$vols1 := 144 V sq$$

$$\begin{aligned} vols1 := & - r_{12}^4 r_{34}^2 - r_{12}^2 r_{13}^2 r_{23}^2 + r_{12}^2 r_{13}^2 r_{24}^2 + r_{12}^2 r_{13}^2 r_{34}^2 \\ & + r_{12}^2 r_{14}^2 r_{23}^2 - r_{12}^2 r_{14}^2 r_{24}^2 + r_{12}^2 r_{14}^2 r_{34}^2 + r_{12}^2 r_{23}^2 r_{34}^2 \\ & + r_{12}^2 r_{24}^2 r_{34}^2 - r_{12}^2 r_{34}^4 - r_{13}^4 r_{24}^2 + r_{13}^2 r_{14}^2 r_{23}^2 \\ & + r_{13}^2 r_{14}^2 r_{24}^2 - r_{13}^2 r_{14}^2 r_{34}^2 + r_{13}^2 r_{23}^2 r_{24}^2 - r_{13}^2 r_{24}^4 \\ & + r_{13}^2 r_{24}^2 r_{34}^2 - r_{14}^4 r_{23}^2 - r_{14}^2 r_{23}^4 + r_{14}^2 r_{23}^2 r_{24}^2 \\ & + r_{14}^2 r_{23}^2 r_{34}^2 - r_{23}^2 r_{24}^2 r_{34}^2 \end{aligned}$$

To each vertex $i = 1..4$ and a cyclic orientation of its complement we associate the following quantities:

$$b1 := (r_{12}^2 - r_{13}^2 + r_{23}^2)(r_{13}^2 - r_{14}^2 + r_{34}^2)(-r_{12}^2 + r_{14}^2 + r_{24}^2) + \\ + (-r_{12}^2 + r_{13}^2 + r_{23}^2)(-r_{13}^2 + r_{14}^2 + r_{34}^2)(r_{12}^2 - r_{14}^2 + r_{24}^2)$$

$$b2 := (r_{12}^2 + r_{13}^2 - r_{23}^2)(r_{23}^2 - r_{24}^2 + r_{34}^2)(-r_{12}^2 + r_{14}^2 + r_{24}^2) + \\ + (-r_{12}^2 + r_{13}^2 + r_{23}^2)(-r_{23}^2 + r_{24}^2 + r_{34}^2)(r_{12}^2 + r_{14}^2 - r_{24}^2)$$

$$b3 := (r_{12}^2 + r_{13}^2 - r_{23}^2)(r_{23}^2 + r_{24}^2 - r_{34}^2)(-r_{13}^2 + r_{14}^2 + r_{34}^2) + \\ + (r_{12}^2 - r_{13}^2 + r_{23}^2)(-r_{23}^2 + r_{24}^2 + r_{34}^2)(r_{13}^2 + r_{14}^2 - r_{34}^2)$$

$$b4 := (r_{12}^2 + r_{14}^2 - r_{24}^2)(r_{23}^2 + r_{24}^2 - r_{34}^2)(r_{13}^2 - r_{14}^2 + r_{34}^2) + \\ + (r_{12}^2 - r_{14}^2 + r_{24}^2)(r_{23}^2 - r_{24}^2 + r_{34}^2)(r_{13}^2 + r_{14}^2 - r_{34}^2)$$

Then it follows that

$$4\, vols1 = b_1 + b_2 + b_3 + b_4\,.$$

Recall the notation for cosine of angle at vertex i in a triangle with vertices i, j, k:

$$C_{i\_jk} = \frac{r_{ij}^2 + r_{ik}^2 - r_{jk}^2}{2r_{ij}r_{ik}}.$$

Then the last computation reads as the following NEW FORMULA: (for the normalized squared volume of a tetrahedron)

$$\frac{288V^2}{64\, r_{12}\, r_{13}\, r_{14}\, r_{23}\, r_{24}\, r_{34}} = \frac{1}{16} \sum C_{i\_jl}\, C_{j\_kl}\, C_{k\_il}$$

where sumation of triple products of cosines is over all 8 oriented three $-$ cycles $< ijk >$ of vertices of our tetrahedron.

Corollary For a semiregular tetrahedron ($r_{12} = r_{34} = c$, $r_{13} = r_{24} = b$, and $r_{14} = r_{23} = a$) we obtain the well known formula:

$$72\, V^2 = \left(-a^2 + b^2 + c^2\right) \left(a^2 - b^2 + c^2\right) \left(a^2 + b^2 - c^2\right).$$

INTRINSIC FORMULA FOR THE HYPERBOLIC 4-POINT ATIYAH DETERMINANT

$$\begin{aligned} p_1 &:= (z-t_{12})(z-t_{13})(z-t_{14}) \\ p_2 &:= (z-t_{21})(z-t_{23})(z-t_{24}) \\ p_3 &:= (z-t_{31})(z-t_{32})(z-t_{34}) \\ p_4 &:= (z-t_{41})(z-t_{42})(z-t_{43}) \end{aligned}$$

$$M_4 := \begin{bmatrix} 1 & -t_{12}-t_{13}-t_{14} & t_{12}t_{13}+t_{14}t_{12}+t_{14}t_{13} & -t_{12}t_{13}t_{14} \\ 1 & -t_{21}-t_{23}-t_{24} & t_{21}t_{23}+t_{24}t_{21}+t_{24}t_{23} & -t_{21}t_{23}t_{24} \\ 1 & -t_{31}-t_{32}-t_{34} & t_{31}t_{32}+t_{34}t_{31}+t_{34}t_{32} & -t_{31}t_{32}t_{34} \\ 1 & -t_{41}-t_{42}-t_{43} & t_{41}t_{42}+t_{43}t_{41}+t_{43}t_{42} & -t_{41}t_{42}t_{43} \end{bmatrix}$$

$\Delta_4 := Determinant(M_4)$

$prod := (t_{12}-t_{21})(t_{13}-t_{31})(t_{23}-t_{32})(t_{14}-t_{41})(t_{24}-t_{42})(t_{34}-t_{43})$

$$D_4 := \frac{\Delta_4}{prod} = \frac{t_{21}t_{41}t_{42}t_{43}t_{31}t_{32} + \cdots \text{214 more similar terms} \cdots + t_{34}t_{12}t_{13}t_{14}t_{24}t_{23}}{(t_{12}-t_{21})(t_{13}-t_{31})(t_{23}-t_{32})(t_{14}-t_{41})(t_{24}-t_{42})(t_{34}-t_{43})}$$

Let D_{4a} be equal to D_4 after substitutions

$$t_{ij} = (s\,A_{ij} + B_{ij})/2, i = 1..j-1, j = 1..4,$$
$$t_{ji} = (s\,A_{ij} - B_{ij})/2, i = 1..j-1, j = 1..4.$$

$$cc(i,j,k,l) := \frac{(t_{kl}-t_{ji})(t_{ij}-t_{lk})}{(t_{ij}-t_{ji})(t_{kl}-t_{lk})}$$

$$C(i,j,k,l) := 2\frac{(t_{kl}-t_{ji})(t_{ij}-t_{lk})}{(t_{ij}-t_{ji})(t_{kl}-t_{lk})} - 1$$

$$AMPLITUDE(12, 13, 24)$$

$$Q121324{:=}{-}1{+}C(1{,}2{,}1{,}3)^2{+}C(1{,}2{,}2{,}4)^2{+}C(1{,}3{,}2{,}4)^2{+}2C(1{,}2{,}1{,}3)C(1{,}2{,}4{,}2)C(1{,}3{,}2{,}4)$$

$$Q121324{:=}\frac{4(t_{12}t_{13}t_{21} - + \cdots \text{10 more similar terms} \cdots + -t_{24}t_{31}t_{42})^2}{(t_{12}-t_{21})^2(t_{13}-t_{31})^2(t_{24}-t_{42})^2}$$

$Q121324$ is symmetric in $A_{ij} = t_{ij} + t_{ji}$, and antisymmetric in $B_{ij} = t_{ij} - t_{ji}$ coordinates:

$$Q121324 = \frac{1}{4}\frac{(A_{13}-A_{24})(A_{12}-A_{24})(A_{12}-A_{13})s^3}{B_{12}B_{13}B_{24}} - \frac{1}{4}\frac{(A_{13}-A_{24})B_{12}s}{B_{13}B_{24}} + \frac{1}{4}\frac{(A_{12}-A_{24})B_{13}s}{B_{24}B_{12}} - \frac{1}{4}\frac{(A_{12}-A_{13})B_{24}s}{B_{12}B_{13}}$$

$$F123 = \frac{1}{2}\left(\cos\left(\frac{A}{2}\right)^2 + \cos\left(\frac{B}{2}\right)^2 + \cos\left(\frac{C}{2}\right)^2\right)$$

$$G123 = (-1 + \cos(A)^2 + \cos(B)^2 + \cos(C)^2 + 2\cos(A)\cos(B)\cos(C))^{\frac{1}{2}}$$

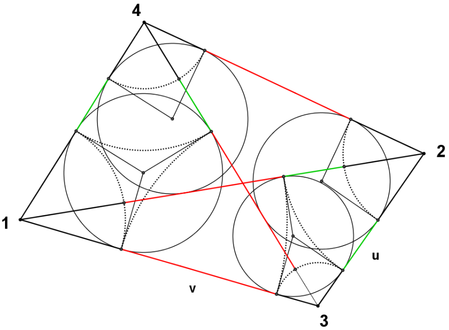
Positive parametrization of distances between 4 points

Key Lemma. (Shear coordinates of a tetrahedron)

In any tetrahedron (degenerate or not) one has the following type of nonnegative splitting of edge lengths:

$$r_{12} = t_1 + u + v + t_2, r_{13} = t_1 + v + t_3,$$
$$r_{23} = t_2 + u + t_3, r_{14} = t_1 + u + t_4,$$
$$r_{24} = t_2 + v + t_4, r_{34} = t_3 + u + v + t_4$$

if and only if $r_{12} + r_{34} = \max\{r_{12} + r_{34}, r_{13} + r_{24}, r_{14} + r_{23}\}$.



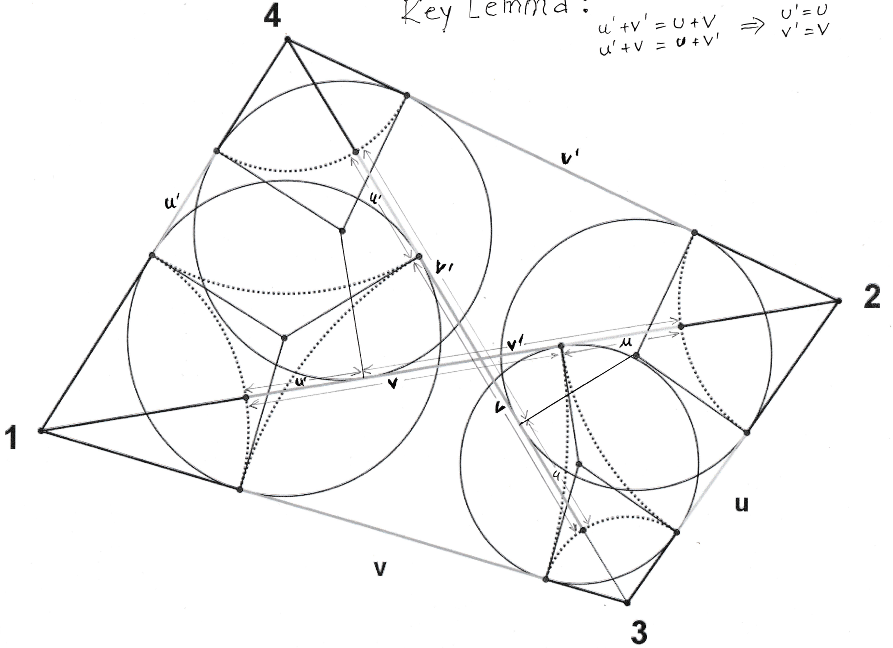
Proof.

The form of the solution:

$$t_1 = \frac{r_{13} + r_{14} - r_{34}}{2}, t_2 = \frac{r_{23} + r_{24} - r_{34}}{2}, t_3 = \frac{r_{13} + r_{23} - r_{12}}{2},$$
$$t_4 = \frac{r_{14} + r_{24} - r_{12}}{2}, u = \frac{r_{12} + r_{34} - (r_{13} + r_{24})}{2}, v = \frac{r_{12} + r_{34} - (r_{14} + r_{23})}{2}$$

proves the Lemma immediately.

Key Lemma: $\begin{array}{l} u'+v' = u+v \\ u'+v = u+v' \end{array} \Rightarrow \begin{array}{l} u'=u \\ v'=v \end{array}$



Verification of the Atiyah–Sutcliffe four–point conjectures

Let us recall the original Eastwood–Norbury formula for the real part of the Atiyah's determinat D_4 of a tetrahedron:

$$Re(D_4) := prod - 4d_3(r_{12}r_{34}, r_{13}r_{24}, r_{23}r_{14}) + A_4 + vols;$$

where $d_3(a,b,c) := (-a+b+c)(a-b+c)(a+b-c);$

$$A_4 =$$
$$(r_{14}((r_{24}+r_{34})^2 - r_{23}^2) + r_{24}((r_{14}+r_{34})^2 - r_{13}^2) + r_{34}((r_{24}+r_{14})^2 - r_{12}^2))d_3(r_{12}, r_{13}, r_{23}) +$$
$$+(r_{13}((r_{23}+r_{34})^2 - r_{24}^2) + r_{23}((r_{13}+r_{34})^2 - r_{14}^2) + r_{34}((r_{23}+r_{13})^2 - r_{12}^2))d_3(r_{12}, r_{14}, r_{24}) +$$
$$+(r_{12}((r_{23}+r_{24})^2 - r_{34}^2) + r_{23}((r_{12}+r_{24})^2 - r_{14}^2) + r_{24}((r_{23}+r_{12})^2 - r_{13}^2))d_3(r_{13}, r_{14}, r_{34}) +$$
$$+(r_{12}((r_{13}+r_{14})^2 - r_{34}^2) + r_{13}((r_{12}+r_{14})^2 - r_{24}^2) + r_{14}((r_{13}+r_{12})^2 - r_{23}^2))d_3(r_{23}, r_{24}, r_{34});$$

$$prod := 64r_{12}r_{13}r_{23}r_{14}r_{24}r_{34};$$

$$vols := 2(r_{12}^2r_{34}^2(r_{13}^2+r_{14}^2+r_{23}^2+r_{24}^2-r_{12}^2-r_{34}^2)+r_{13}^2r_{24}^2(-r_{13}^2+r_{14}^2+r_{23}^2-r_{24}^2+r_{12}^2+r_{34}^2)+$$
$$r_{14}^2r_{23}^2(r_{13}^2-r_{14}^2-r_{23}^2+r_{24}^2+r_{12}^2+r_{34}^2) - r_{12}^2r_{13}^2r_{23}^2 - r_{12}^2r_{14}^2r_{24}^2 - r_{13}^2r_{14}^2r_{34}^2 - r_{23}^2r_{24}^2r_{34}^2);$$

$(vols = 288volume^2)$ and normalized Atiyah determinant of face triangles:

$$\delta_1 := 1 + \frac{1}{8}\frac{d_3(r_{23}, r_{24}, r_{34})}{r_{23}r_{24}r_{34}}, \delta_2 := 1 + \frac{1}{8}\frac{d_3(r_{13}, r_{14}, r_{34})}{r_{13}r_{14}r_{34}},$$
$$\delta_3 := 1 + \frac{1}{8}\frac{d_3(r_{12}, r_{14}, r_{24})}{r_{12}r_{14}r_{24}}, \delta_4 := 1 + \frac{1}{8}\frac{d_3(r_{12}, r_{13}, r_{23})}{r_{12}r_{13}r_{23}}.$$

We first prove a stronger four–point conjecture of Svrtan – Urbiha (arXiv:math0609174v1 (Conjecture 2.1 (weak version)) which implies (c.f. Proposition 2.2 in loc.cit) all three four–point conjectures C_1, C_2, C_3 of Atiyah – Sutcliffe).

The substitution from the Key Lemma

$Sub := \{r_{12} = t_1 + u + v + t_2, r_{13} = t_1 + v + t_3, r_{23} = t_2 + u + t_3,$
$r_{14} = t_1 + u + t_4, r_{24} = t_2 + v + t_4, r_{34} = t_3 + u + v + t_4\};$

in the Maple code DifferSU :=

$$\left\{ \text{coeffs}\left(\text{expand}\left(\text{subs}\left(\text{Sub}, \frac{1}{64}\text{numer}\left(\frac{Re(D_4) - 4vols}{prod} - \frac{\delta_1^2 + \delta_2^2 + \delta_3^2 + \delta_4^2}{4} \right)\right)\right)\right)\right\};$$

gives the output DifferSU $= \{2, 3, 4, \ldots, 5328, 5564, 6036\}$ which proves the conjecture coefficientwise.

The Maple code for the strongest Atiyah – Sutcliffe conjecture DifferAS :=

$$\left\{ \text{coeffs}\left(\text{expand}\left(\text{subs}\left(\text{Sub}, \frac{1}{64}\text{numer}\left(\left(\frac{Re(D_4) - 4vols}{prod}\right)^2 - \delta_1\delta_2\delta_3\delta_4 \right)\right)\right)\right)\right\};$$

gives the output DifferAS $= \{64, 128, 192, \ldots, 233472, 246720, 261888\}$

(coefficients of a 4512 terms inequality of degree 12 in 6 distances).

Remark 1. Similarly to DifferSU one can check the upper estimate with the additional coefficient equal to $37/27$.

Remark 2. Recently we also proved Atiyah – Sutcliffe conjecture C_2 directly from the following new formula:

$$Re(D_4) = 64\prod_{1\leq i<j\leq 4} r_{ij} + 8d_3(r_{12}r_{34}, r_{13}r_{24}, r_{14}r_{23}) + 4vols + 32R_4\ ,$$

where

$$\begin{aligned} R_4 = {} & 4m_{2211} + (s_{13}p_{24}^2 + s_{24}p_{13}^2)u + (s_{14}p_{23}^2 + s_{23}p_{14}^2)v + (m_{221} + 8m_{2111})w+ \\ & + 2(\tau_{13}^2 + \tau_{14}^2 + \tau_{13}\tau_{14})uv + (2m_{211} + 8m_{1111})(2u^2 + uv + 2v^2) \\ & + 4m_{111}(u^3 + v^3) + (3m_{21} + 14m_{111} + 3m_{11}w)uvw + \left[(s_{14}p_{14} + s_{23}p_{23})(u + w)+\right. \\ & \left. +(s_{13}p_{13} + s_{24}p_{24})(v + w)\right] uv + \left[(\tau_{13} + \tau_{14})(u^2 + uv + v^2)+\right. \\ & \left. +\tau_{14}u^2 + \tau_{13}v^2\right] uv + 2(m_1 + w)(4m_1 + 3w)u^2v^2 \end{aligned}$$

and where

$$u = \frac{r_{12}+r_{34}-r_{13}-r_{24}}{2},\ v = \frac{r_{12}+r_{34}-r_{14}-r_{23}}{2},\ w = u + v, \tau_{13} = t_1t_3 + t_2t_4,$$

$$\tau_{14} = t_1t_4 + t_2t_3,\ t_1 = \frac{r_{13}+r_{14}-r_{34}}{2},\ t_2 = \frac{r_{23}+r_{24}-r_{34}}{2},\ t_3 = \frac{r_{13}+r_{23}-r_{12}}{2},$$

$$t_4 = \frac{r_{14}+r_{24}-r_{12}}{2},\ s_{ij} = t_i + t_j,\ p_{ij} = t_it_j, m_1 = t_1 + t_2 + t_3 + t_4,$$

$$m_{11} = t_1t_2 + \cdots,\ m_{21} = t_1^2t_2 + \cdots,\ m_{111} = t_1t_2t_3 + \cdots, m_{1111} = t_1t_2t_3t_4,$$

$$m_{2111} = t_1^2t_2t_3t_4 + \cdots,\ m_{221} = t_1^2t_2^2t_3 + \cdots,\ m_{2211} = t_1^2t_2^2t_3t_4 + \cdots$$

Mixed Atiyah determinants

We further generalize Atiyah normalized determinant $D(x_1,\ldots,x_n)$ to $D^\Gamma(x_1,\ldots,x_n)$, where Γ is any (simple) graph with the vertex set $\{x_1,\ldots,x_n\}$.

Definition.

We start with the normalized Atiyah determinant D viewed as a function of all directions u_{ij} $(1 \le i \ne j \le n)$. Then we define D^Γ by simultaneously switching the roles of directions (i.e. replacing u_{ij} by u_{ji} and also replacing u_{ji} by u_{ij}) for each pair ij such that x_ix_j is an edge of Γ.

For $n=3$ we obtain eight mixed Atiyah's determinants (mixed energies) which we can label by binary sequences $D_3 = D_3^{000}, D_3^{001}, \ldots, D_3^{111}$ for which we also have simple explicit trigonometric formulas, which can be obtained from the original Atiyah determinant by suitable sign changes of the lengths of the sides of a triangle.

Observe that

$$D_3 = D_3^{000}, D_3^{111} = 1 + e^p \prod \sinh(p_a)/\sinh(a)$$

are both ≥ 1. All other mixed determinants, eg.

$$D_3^{110} = 1 - e^{p_c} \sinh(p) \sinh(p_a) \sinh(p_b)/\prod \sinh(a),$$

are between 0 and 1.

Main Theorem

Now we state our

Main Theorem.

We have $\sum_\Gamma D^\Gamma = n!$, where the summation extends over all simple graphs on n vertices.

The proof is obtained by our method of computing Atiyah's determinants.

Corollary.

For any configuration of points in a hyperbolic 3–space at least one of the mixed Atiyah determinants is nonzero.

Proof of the main Theorem

Proof of the Main Theorem.

In coordinates $B_{ij} = u_{ij} - u_{ji}$ (antisymmetric) and $A_{ij} = u_{ij} + u_{ji}$ (symmetric) $1 \leq i \neq j \leq n$, D^Γ differs from D in changing signs of B_{ij}'s for each edge $ij \in \Gamma$. Let us first observe that each nonconstant term in D (and in each D^Γ) is a square free Laurent monomial w.r.t. all variables B_{ij}'s, hence in the sum over Γ its contribution is zero.

Therefore, we have to compute the constant term (C.T.) of D (which is the same in all D^Γ). Since D is a symmetrization over S_n of its main diagonal term, we have $C.T.(D) = n!C.T.(diagonal\ term)$. But diagonal term of D is equal to

$$\frac{1 \cdot (-u_{21} + \cdots)((-u_{31})(-u_{32}) + \cdots)\cdots[(-u_{n,1})(-u_{n,1})\cdots(-u_{n,n-1})]}{(u_{12} - u_{21})(u_{13} - u_{31})(u_{23} - u_{32})\cdots(u_{1,n} - u_{n,1})\cdots(u_{n-1,n} - u_{n,n-1})}$$

so $C.T.(diag.term) = C.T.\dfrac{\frac{B_{12}}{2}\frac{B_{13}}{2}\frac{B_{23}}{2}\cdots}{B_{12}B_{13}B_{23}\cdots} = \dfrac{1}{2^{\binom{n}{2}}}$ and $C.T.(D) = \dfrac{n!}{2^{\binom{n}{2}}}$ and

$$C.T.\left(\sum_\Gamma D^\Gamma\right) = n!.$$

□

New developments (1/3)

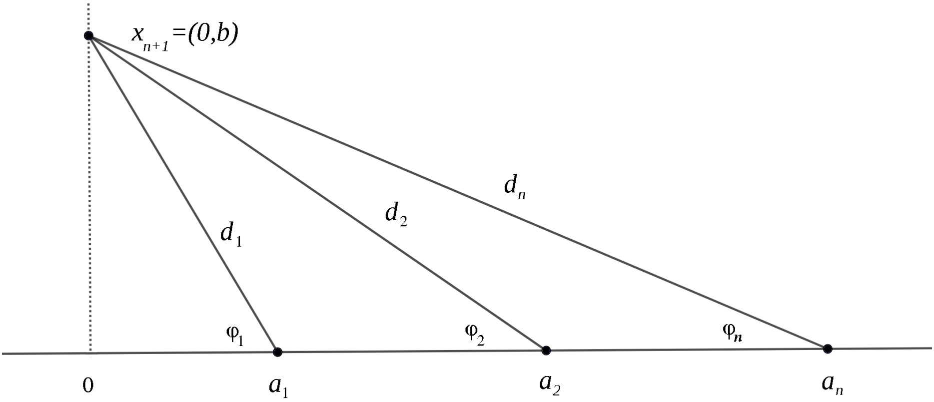
- In 2011. M.Mazur and B.V.Petrenko restated the original Eastwood Norbury formula in trigonometric form which besides face angles of a tetrahedron uses also angles of so called Crelle triangle (associated to the tetrahedron). Our formula in [5] does not involve Crelle's angles, but uses "skew" angles.
- C_2 proved for convex (planar) quadrilaterals
- C_3 proved for cyclic quadrilaterals (we have it proved already in [5])
- Three conjectures stated which are consequences of some of our conjectures in [5]. (Hence we have a proof of all three.)

New developments (2/3)

- In a recent paper M.B.Khuzam and M.J.Johnson (arXiv:1401.2787v1) gave a verification (by linear programming) of both C_2 and C_3 four–point conjectures of Atiyah and Sutcliffe, by using symmetric functions of degree 12 in 12 variables $t_{il} = r_{ij} + r_{ik} - r_{jk}$, $\{i, j, k, l\} = \{1, 2, 3, 4\}$ (which are linearly dependent), so for C_2 (resp. C_3) they use 64 (resp. 114) huge monomial symmetric functions.
- In a recent paper J. Malkoun defined a symplectic version of Atiyah conjecture and proved it for $n = 2$ (which also follows from [5], 2.6 Atiyah - Sutcliffe conjectures for parallelograms). We observe that symplectic Atiyah determinants are special case of ordinary Atiyah determinants for centrally symmetric configurations!

New developments (3/3)
Tropical Version of Atiyah–Sutcliffe Conjecture for almost collinear configurations

Almost collinear configurations. Đoković's approach and generalizations.
Type A configuration $(n+1)$ (n points collinear)



$$
\begin{aligned}
\lambda_1 &= a_1 + \sqrt{b^2 + a_1^2} = a_1 + d_1 = d_1(1 + \cos(\varphi_1)) \,, \\
&\dots \\
\lambda_2 &= a_2 + \sqrt{b^2 + a_2^2} = a_2 + d_2 = d_2(1 + \cos(\varphi_2)) \,, \\
\lambda_n &= a_n + \sqrt{b^2 + a_n^2} = a_n + d_n = d_n(1 + \cos(\varphi_n)) \,.
\end{aligned}
$$

$$
M_{n,1} = \begin{vmatrix} 1 & \lambda_1 & & & \\ & 1 & \lambda_2 & & \\ & & & & \\ (-1)^n e_n & & & -e_1 & 1 \end{vmatrix}, \quad \begin{aligned} e_1 &= \lambda_1 + \lambda_2 + \cdots \lambda_n \\ e_2 &= \lambda_1\lambda_2 + \cdots + \lambda_{n-1}\lambda_n \\ &\dots \\ e_n &= \lambda_1 \cdots \lambda_n \end{aligned}
$$

$$
\begin{aligned}
D_{n,1} = det(M_{n,1}) &= 1 + \lambda_n e_1 + \lambda_n \lambda_{n-1} e_2 + \cdots + \lambda_n \cdots \lambda_1 e_n \\
&\geq 1 + e_1(\lambda_1^2, \dots, \lambda_n^2) + \cdots + e_n(\lambda_1^2, \dots, \lambda_n^2) \\
&= (1 + \lambda_1^2)(1 + \lambda_2^2) \cdots (1 + \lambda_n^2) \Rightarrow proof\ of\ C_2
\end{aligned}
$$

Atiyah–Sutcliffe conjectures:

$C_2 : \ \mathcal{D}_{n,1} = D_{n,1} / \prod (1 + \lambda_k^2) \geq 1$

$C_3 : \ (D_{n,1}(\lambda_1, \cdots, \lambda_n))^{n-1} \geq \prod D_{n-1,1}(\lambda_1, \cdots, \widehat{\lambda_k}, \cdots, \lambda_n)$

Already in 2004., we generalized asymmetric Đoković's formula for

$$D_{n,1} = 1 + \lambda_n e_1(\lambda_1, \ldots, \lambda_n) + \lambda_n \lambda_{n-1} e_2(\lambda_1, \ldots, \lambda_n) + \cdots + \lambda_n \cdots \lambda_1 e_n(\lambda_1, \ldots, \lambda_n)$$

by introducing new parameters $A_1 \geq A_2 \geq \cdots \geq A_n > 0$ for non-symmetrically appearing $\lambda_1, \ldots, \lambda_n$ and commutative variables $a_1, a_2, \ldots, a_n$ for symmetrically appearing $\lambda_1, \ldots, \lambda_n$.

$$\Psi^{A_1, A_2, \ldots, A_n}_{a_1, a_2, \ldots, a_n} := 1 + A_1 e_1(a_1, \ldots, a_n) + A_1 A_2 e_2(a_1, \ldots, a_n) + \cdots + A_1 A_2 \cdots A_n e_n(a_1, \ldots, a_n)$$

which we abbreviate as (with $A_{1\ldots k} := A_1 A_2 \cdots A_k$)

$$\Psi^{12\cdots n}_{12\cdots n} = 1 + A_1 e_1 + A_{12} e_2 + \cdots + A_{1\ldots n} e_n$$

and proposed a conjecture (in [4], c.f. Conj. 1.5. in [9])

$$\boxed{\left(\Psi^{12\ldots n}_{12\ldots n}\right)^{n-1} \geq \prod_{k=1}^{n} \Psi^{1\ldots\widehat{k}\ldots n}_{1\ldots\widehat{k}\ldots n}} \qquad (*)$$

hold coefficientwise in the ring $\mathbf{R}[a_1, \ldots, a_n]$ of polynomials in (commuting) indeterminants $a_1, a_2, \ldots, a_n$ (and verified it with a number of refinements for $n \leq 9$).

It is trivial for $n = 2$:

$$\Psi_{12}^{12} = a + A_1(a_1 + a_2) + A_1A_2a_1a_2 \geq (1 + A_2a_2)(1 + A_1a_1) = \Psi_2^2\Psi_1^1$$

because

$$\Psi_{12}^{12} - \Psi_2^2\Psi_1^1 = (A_1 - A_2)a_2$$

(and we assumed $A_1 \geq A_2 > 0$).

Note that the r.h.s. of the Conjecture is not symmetric in variables $a_1, \ldots, a_n$, but by studying the derivatives of

$$\frac{\Psi_{1\ldots n}^{1\ldots n}}{\Psi_{1\ldots\widehat{k}\ldots n}^{1\ldots\widehat{k}\ldots n}}$$

in [4] we stated a strengthened symmetric version.

Conjecture

Let $A_1 \geq \cdots \geq A_n \geq, a_1, \ldots a_n \geq 0$. Then the following inequality for symmetric functions in $a_1, \ldots, a_n$

$$\Psi^{112\ldots n-1}_{123\ldots n} \Psi^{1223\ldots n-1}_{1234\ldots n} \cdots \Psi^{12\ldots n-2\ n-1\ n-1}_{12\ldots n-2\ n-1\ n} \geq \Psi^{1\ 2\ldots n-1}_{1\ 2\ldots n-1} \Psi^{1\ 2\ldots n-1}_{1\ 2\ldots n-2\ n} \cdots \Psi^{1\ 2\ldots n-1}_{2\ 3\ldots n-1}$$

i.e.

$$\boxed{\prod_{k=1}^{n-1} \Psi^{1\ 2\ldots k\ \ k\ \ldots n}_{1\ 2\ldots k\ k+1\ldots n} \geq \prod_{k=1}^{n} \Psi^{1\ 2\ \ldots\ n-1}_{1\ 2\ldots \widehat{k}\ldots n}} \qquad (**)$$

holds true coefficientwise (m–positivity).

Now by the following Lemma we interpreted the Conjecture $(**)$ as (polynomial wrt Schur functions) a Hadamard type inequality for certain non symmetric matrices.

Lemma

For any k, $(1 \leq k \leq n)$, we have

$$\Psi^{1\ldots k\ldots n-1}_{1\ldots \widehat{k}\ldots n} = \sum_{j=0}^{n-1} c_j a_k^{n-1-j}$$

where

$$c_{n-1-j} = (-1)^j \sum_{i=j}^{n-1} X_1 \cdots X_i e_{i-j}, \quad j = 0, \ldots, n-1\,.$$

By the Lemma, the right hand side of $(**)$ can be written as $R_n = \prod_{k=1}^{n}\left(\sum_{j=0}^{n-1} c_j \xi_k^{n-1-j}\right)$ and can be written as

$$R_n = \left|\begin{array}{ccccccccc} 1 & -e_1 & e_2 & -e_3 & \cdots & (-1)^n e_n & & & \\ & 1 & -e_1 & e_2 & -e_3 & \cdots & & & \\ & & \ddots & & & & & & \\ & & & 1 & -e_1 & \cdots & & & \\ c_0 & c_1 & c_2 & \cdots & c_n & & & & \\ & c_0 & c_1 & c_2 & \cdots & c_n & & & \\ & & \ddots & & & & & & \\ & & & c_0 & c_1 & c_2 & & \cdots & c_n \end{array}\right| \quad \left(=: \left|\begin{array}{cc} A & B \\ C & D \end{array}\right|\right)$$

can be simplified as $= |A| \cdot |D - CA^{-1}B| = |D - CA^{-1}B|.$

The entries of the $n \times n$ matrix $\Delta := D - CA^{-1}B$ are given by

$$\delta_{ij} = \begin{cases} (-1)^{j-i-1} \displaystyle\sum_{k=j+1}^{n} A_1 \cdots A_{k+i-j} e_k, & 0 \le i < j \le n-1 \\ (-1)^{j-i} \displaystyle\sum_{k=0}^{j} A_1 \cdots A_{k+i-j} e_k, & 0 \le j \le i \le n-1 \end{cases}$$

For example, for $n = 3$

$$\Delta_3 = \left|\begin{array}{ccc} 1 & A_1e_2 + A_1A_2e_3 & -A_1e_3 \\ -A_1 & 1 + A_1e_1 & A_1A_2e_3 \\ A_1A_2 & -A_1 - A_1A_2e_1 & 1 + A_1e_1 + A_1A_2e_2 \end{array}\right|$$

By elementary operations we get

$$\Delta_3 = \begin{vmatrix} 1 & * & * \\ 0 & \Psi^{112}_{123} & A_1(A_2 - A_1)e_3 \\ 0 & A_2 - A_1 & \Psi^{122}_{123} \end{vmatrix} = \begin{vmatrix} \Psi^{112}_{123} & A_1(A_2 - A_1)e_3 \\ A_2 - A_1 & \Psi^{122}_{123} \end{vmatrix}$$

Similarly, for $n = 4$ we obtain

$$\Delta_4 = \begin{vmatrix} \Psi^{1123}_{1234} & -A_1(A_1 - A_2)e_3 - A_1A_2(A_1 - A_3)e_4 & A_1(A_1 - A_2)e_4 \\ -(A_1 - A_2) & \Psi^{1223}_{1234} & -A_1A_2(A_2 - A_3)e_4 \\ A_1(A_2 - A_3) & -(A_1 - A_3) - A_1(A_2 - A_3)e_1 & \Psi^{1233}_{1234} \end{vmatrix}$$

In general

$$\Delta_n = \det(\delta'_{ij})_{1 \le i,j \le n-1}$$

where

$$\delta'_{ij} = \begin{cases} (-1)^{j-i} \sum_{k=j+1}^{n} A_1 \cdots A_{k+i-j-1}(A_i - A_{k+i-j})e_k & , 1 \le i < j \le n-1 \\ \Psi^{1 \ \dots \ i \ i \ \dots \ n}_{1 \ 2 \ \dots \ n} & , i = j \\ (-1)^{j-i} \sum_{k=0}^{j} A_1 \cdots A_{k+i-j-1}(A_{k+i-j} - A_i)e_k & , 1 \le j < i \le n-1 \end{cases}$$

Then we conjecture that the following Hadamard type inequality

$$\prod_{i=1}^{n-1} \delta'_{ii} \ge \det(\delta'_{ij}), \ \ (\delta'_{ij})_{1 \le i,j \le n-1}$$

should hold coefficientwise w.r.t. Schur functions in $a_1, \dots, a_n$.

Let $a_1, \ldots, a_n, A_1, \ldots, A_n, n \geq 1$ be two sets of commuting indeterminates. For any $l, 1 \leq l \leq n$ and any sequences $1 \leq i_1 \leq \cdots \leq i_l \leq n, 1 \leq j_1, \ldots, j_l \leq n$ we define polynomials $\Psi^I_J = \Psi^{i_1 \cdots i_l}_{j_1 \cdots j_l} \in \mathcal{Q}[a_1, \ldots, a_n, A_1, \ldots, A_n]$ as follows:

$$\Psi^I_J := \sum_{k=0}^{l} e_k(a_{j_1}, a_{j_2}, \ldots, a_{j_l}) A_{i_1} A_{i_2} \cdots A_{i_k}, \ (l \geq 1), \ \Psi^{\emptyset}_{\emptyset} := 1 \ (j = 0)$$

where e_k is the k-th elementary symmetric function.

The polynomials Ψ^I_J are symmetric w.r.t. $a_{j_1}, a_{j_2}, \ldots, a_{j_l}$, but nonsymmetric w.r.t. $A_{i_1}, A_{i_2}, \ldots, A_{i_l}$. By specializing A_i's to assume real values such that $A_{i_1} \geq A_{i_2} \geq \ldots \geq A_{i_l} \geq 0$ then we obtain polynomials in a_j's satisfying the following simple but important property.

Proposition (Partition property)
Let $(I_1, \ldots, I_s)$ and $(J_1, \ldots, J_s)$ be ordered set partitions of respective sets $I = \bigcup_{p=1}^{s} I_p$ and $J = \bigcup_{p=1}^{s} J_p$ such that $|I_p| = |J_p|, 1 \leq p \leq s$. Then the inequality

$$\Psi^I_J \geq \prod_{p=1}^{s} \Psi^{I_p}_{J_p}$$

holds coefficientwise w.r.t. a_j's.

Proof. Proof is evident from the definition of Ψ^I_J and the monotonicity of A_i's. □

For the powers $\left(\Psi^I_J\right)^m$ we made the following conjecture in ([4]):

Conjecture(Weighted Multiset Partition Conjecture)
For given natural number m and sets I and J, $|I| = |J|$, of natural numbers let $(I_1, \ldots, I_s)$ and $(J_1, \ldots, J_s)$ be the partitions of the multiset I^m consisting of m copies of all elements of I and similarly for J^m.

(i) Then the inequality $\left(\Psi^I_J\right)^m \geq \prod_{p=1}^{s} \Psi^{I_p}_{J_p}$ holds coefficientwise w.r.t. a_j's.

(ii) The difference $\left(\Psi^I_J\right)^m - \prod_{p=1}^{s} \Psi^{I_p}_{J_p}$ is multi–Schur positive with respect to partial alphabets corresponding to the atoms of the intersection lattice of the set system $\{J_1, \ldots, J_s\}$.

Tropical Version of Atiyah–Sutcliffe Conjecture for almost collinear configurations ($n = 3$)

In case $n = 3$ we illustrate the tropical version of Atiyah–Sutcliffe conjecture.
Let $n = 3,\ A_1 \geq A_2 \geq A_3 > 0$.

$$
\begin{aligned}
E(x) &= (1 + a_1x)(1 + a_2x)(1 + a_3x) \\
&= 1 + e_1x + e_2x^2 + e_3x^3 \\
E^{(1)}(x) &= (1 + a_2x)(1 + a_3x) \\
&= 1 + (a_2 + a_3)x + a_2a_3x^2 = 1 + e_1^{(1)}x + e_2^{(1)}x^2 \\
E^{(2)}(x) &= (1 + a_1x)(1 + a_3x) \\
&= 1 + (a_1 + a_3)x + a_1a_3x^2 = 1 + e_1^{(2)}x + e_2^{(2)}x^2 \\
E^{(3)}(x) &= (1 + a_1x)(1 + a_2x) \\
&= 1 + (a_1 + a_2)x + a_1a_2x^2 = 1 + e_1^{(3)}x + e_2^{(3)}x^2
\end{aligned}
$$

$$
\begin{aligned}
AS_3 &= \Psi^{123}_{123}\Psi^{123}_{123} - \Psi^{12}_{12}\Psi^{13}_{13}\Psi^{23}_{23}\\
&= (1 + A_1e_1 + A_1A_2e_2 + A_1A_2A_3e_3)^2 - \left(1 + A_2e_1^{(1)} + A_2A_3e_2^{(1)}\right)\cdot\\
&\qquad \cdot\left(1 + A_1e_1^{(2)} + A_1A_3e_2^{(2)}\right)\cdot\\
&\qquad \cdot\left(1 + A_1e_1^{(3)} + A_1A_2e_2^{(3)}\right)
\end{aligned}
$$

$$
\begin{aligned}
AS_3^{trop} &= \left[t^1\right]\ subs\left(A_1 = e^{\alpha_1 t}, A_2 = e^{\alpha_2 t}, A_3 = e^{\alpha_3 t}, AS_3\right)\\
&= \left[t^1\right]\ subs\left(A_1 = 1 + \alpha_1 t, A_2 = 1 + \alpha_2 t, A_3 = 1 + \alpha_3 t, AS_3\right)\\
&= \left[t^1\right]\Big[\ (1 + (1 + \alpha_1 t)e_1 + (1 + (\alpha_1 + \alpha_2)t)e_2 + (1 + (\alpha_1 + \alpha_2 + \alpha_3)t)e_3)^2 -\\
&\qquad - \left(1 + (1 + \alpha_2 t)e_1^{(1)} + (1 + (\alpha_2 + \alpha_3)t)e_2^{(1)}\right)\cdot\\
&\qquad \cdot\left(1 + (1 + \alpha_1 t)e_1^{(2)} + (1 + (\alpha_1 + \alpha_3)t)e_2^{(2)}\right)\cdot\\
&\qquad \cdot\left(1 + (1 + \alpha_1 t)e_1^{(3)} + (1 + (\alpha_1 + \alpha_2)t)e_2^{(3)}\right)\ \Big]\\
&= \left[t^1\right]\Big[\ (1 + e_1 + e_2 + e_3 + (\alpha_1e_1 + (\alpha_1 + \alpha_2)e_2 + (\alpha_1 + \alpha_2 + \alpha_3)e_3)t)^2\\
&\qquad - \left(1 + e_1^{(1)} + e_2^{(1)} + \left(\alpha_2e_1^{(1)} + (\alpha_2 + \alpha_3)e_2^{(1)}\right)t\right)\cdot\\
&\qquad \cdot\left(1 + e_1^{(2)} + e_2^{(2)} + \left(\alpha_1e_1^{(2)} + (\alpha_1 + \alpha_3)e_2^{(2)}\right)t\right)\cdot\\
&\qquad \cdot\left(1 + e_1^{(3)} + e_2^{(3)} + \left(\alpha_1e_1^{(3)} + (\alpha_1 + \alpha_2)e_2^{(3)}\right)t\right)\ \Big] =
\end{aligned}
$$

$$= \left[t^1\right] \Big[\left(E(1) + (\alpha_1 e_1 + (\alpha_1+\alpha_2)e_2 + (\alpha_1+\alpha_2+\alpha_3)e_3 t)^2\right)$$
$$- \left(E^{(1)}(1) + \left(\alpha_2 e_1^{(1)} + (\alpha_2+\alpha_3)e_2^{(1)}\right) t\right) \cdot$$
$$\cdot \left(E^{(2)}(1) + \left(\alpha_1 e_1^{(2)} + (\alpha_1+\alpha_3)e_2^{(2)}\right) t\right) \cdot$$
$$\cdot \left(E^{(3)}(1) + \left(\alpha_1 e_1^{(3)} + (\alpha_1+\alpha_2)e_2^{(3)}\right) t\right) \Big] =$$

$$= 2E(1)(\alpha_1 e_1 + (\alpha_1+\alpha_2)e_2 + (\alpha_1+\alpha_2+\alpha_3)e_3 t)^2$$
$$- E^{(2)}(1)E^{(2)}(1) \left(\alpha_2 e_1^{(1)} + (\alpha_2+\alpha_3)e_2^{(1)}\right)$$
$$- E^{(1)}(1)E^{(3)}(1) \left(\alpha_1 e_1^{(2)} + (\alpha_1+\alpha_3)e_2^{(2)}\right)$$
$$- E^{(1)}(1)E^{(2)}(1) \left(\alpha_1 e_1^{(3)} + (\alpha_1+\alpha_2)e_2^{(3)}\right) =$$

Now we use $E^{(2)}(1)E^{(3)}(1) = (1+a_1)(1+a_3)(1+a_1)(1+a_2) = E(1)(1+a_1)$ etc.

$$= E(1) \Big[2\alpha_1 e_1 + (2\alpha_1 + 2\alpha_2)e_2 + (2\alpha_1 + 2\alpha_2 + 2\alpha_3)e_3$$
$$- (1+a_1) \left(\qquad \alpha_2 e_1^{(1)} + (\qquad \alpha_2 + \alpha_3)e_2^{(1)}\right)$$
$$- (1+a_2) \left(\alpha_1 e_1^{(2)} \qquad + (\alpha_1 + \qquad \alpha_3)e_2^{(2)}\right)$$
$$- (1+a_3) \left(\alpha_1 e_1^{(3)} \qquad + (\alpha_1 + \alpha_2 \qquad)e_2^{(3)}\right) \Big] =$$

By using basic relation between elementary symmetric polynomials

$$e_k(a_1,\ldots,a_n) = a_j e_{k-1}(a_1,\ldots,\widehat{a_j}\ldots,a_n) + e_k(a_1,\ldots,\widehat{a_j}\ldots,a_n)$$

$$\begin{aligned}
= E(1)\Big\{ & 2\alpha_1 e_1 + (2\alpha_1 + 2\alpha_2)e_2 + (2\alpha_1 + 2\alpha_2 + 2\alpha_3)e_3 \\
& - \Big[\alpha_2 e_1^{(1)} + \alpha_2 e_2 - \cancel{\alpha_2 e_2^{(1)}} + (\underline{\alpha_2} + \alpha_3)\underline{e_2^{(1)}} + (\alpha_2 + \alpha_3)e_3 \Big] \\
& - \Big[\alpha_1 e_1^{(2)} + \alpha_1 e_2 - \cancel{\alpha_1 e_2^{(2)}} + (\underline{\alpha_1} + \alpha_3)\underline{e_2^{(2)}} + (\alpha_1 + \alpha_3)e_3 \Big] \\
& - \Big[\alpha_1 e_1^{(3)} + \alpha_1 e_2 - \cancel{\alpha_1 e_2^{(3)}} + (\underline{\alpha_1} + \alpha_2)\underline{e_2^{(3)}} + (\alpha_1 + \alpha_2)e_3 \Big] \Big\} =
\end{aligned}$$

By using $2e_1 = e_1^{(1)} + e_1^{(2)} + e_1^{(3)}$, $e_2 = e_2^{(1)} + e_2^{(2)} + e_2^{(3)}$

$$= E(1)\left[(\alpha_1 - \alpha_{\mathbf{2}})e_1^{(1)} + \alpha_2 e_2 - \left(\alpha_3 e_2^{(1)} + \alpha_3 e_2^{(2)} + \alpha_2 e_2^{(3)}\right)\right]$$

$$\boxed{AS_3^{trop} = E(1)\left[(\alpha_1 - \alpha_2)e_1^{(1)} + (\alpha_2 - \alpha_3)\left(e_2^{(1)} + e_2^{(2)}\right)\right]}$$

$(E(1) = 1 + e_1 + e_2 + e_3)$

In general it would be

$$AS_n^{trop} = (E(1))^{n-2}\left[(\alpha_1 - \alpha_2)e_1^{(1)} + \cdots + (\alpha_{n-1} - \alpha_n)\left(e_{n-1}^{(1)} + \cdots + e_{n-1}^{(n-1)}\right)\right]$$

Example Verification supporting our Multiset Partition Conjecture Using Maple (1/4)

Let $a1, a2, \ldots, a6$ be commuting variables and let $A1 \geq A2 \geq \cdots \geq A6 > 0$ be nonnegative real numbers.

Let $f1 = a2 + a4 + a5$, $f2 = a2 \cdot a4 + a2 \cdot a5 + a4 \cdot a5$, $f3 = a2 \cdot a4 \cdot a5$, $h1 = a2 + a5$, $h2 = a2 \cdot a5$, and $e1 = a1 + a3 + a6$,$e2 = a1 \cdot a3 + a1 \cdot a6 + a3 \cdot a6$, $e3 = a1 \cdot a3 \cdot a6$ be the elementary symmetric functions of the alphabets $a2, a4, a5$ and $a2, a5$ and $a1, a3, a6$.

The $psi123456$ function of the original alphabet $a1, a2, a3, a4, a5, a6$, is

$$psi123456 =$$
$$1 + A[1](e1 + f1) + A[1]A[2] \cdot (e2 + e1 \cdot f1 + f2) + A[1]A[2]A[3](e3 + e2 \cdot f1 + e1 \cdot f2 + f3) + A[1]A[2]A[3]A$$
$$+A[1]A[2]A[3]A[4]A[5]A[6]e3 \cdot f3 (= psi6a).$$

The $psi245$ function of the alphabet $a2, a4, a5$ is

$$psi245 = 1 + A[2] \cdot f1 + A[2] \cdot A[4] \cdot f2 + A[2] \cdot A[4] \cdot A[5] \cdot f3 (= D1).$$

The $psi136$ function of the alphabet $a1, a3, a6$ is

$$psi136 = 1 + A[1] \cdot e1 + A[1] \cdot A[2] \cdot e2 + A[1] \cdot A[3] \cdot A[6] \cdot e3.$$

The $psi1346$ function of the alphabet $a1, a3, a4, a6$ is

$$psi1346 = 1 + A[1](e1 + a[4]) + A[1]A[3](e2 + e1a[4]) + A[1]A[3]A[4](e3 + e2a[4]) + A[1]A[3]A[4]A[6]e3a[4] (= d1).$$

The $psi12356$ function of the alphabet $a1, a2, a3, a5, a6$ is

$$psi12356 =$$
$$(1 + A[1] \cdot (e1 + h1) + A[1] \cdot A[2] \cdot (e2 + e1 \cdot h1 + h2) + A[1] \cdot A[2] \cdot A[3] \cdot (e3 + e2 \cdot h1 + e1 \cdot h2) +$$
$$+A[1] \cdot A[2] \cdot A[3] \cdot A[5] \cdot (e3 \cdot h1 + e2 \cdot h2) + A[1] \cdot A[2] \cdot A[3] \cdot A[5] \cdot A[6] \cdot e3 \cdot h2) (= d2).$$

Then our Weighted Multiset Partition Conjecture (Conjecture 3.2 in arxiv.org.math.0609174.pdf) reads as the following coefficientwise inequality:

$$psi123456^2 >= psi245 \cdot psi1346 \cdot psi12356.$$

Example Verification supporting our Multiset Partition Conjecture Using Maple (2/4)

The final formula for

$$(psi123456)^2 - psi245 \cdot psi1346 \cdot psi12356$$

in terms of Schur functions $t1, t11, t111, t2, t21, t211, t22, t221, t222$ of $a[1], a[3], a[6]$ and Schur functions $s[1], s[2], s[1,1], s[2,1], s[2,2], s[1,1,1], s[2,1,1], s[2,2,1], s[2,2,2]$ of $a[2], a[4], a[5]$ is $Z6c$ bellow (and its coefficients are positive).

$Z6c :=$
$sort(map(factor, collect(Z6b, \{t1, t11, t111, t2, t21, t211, t22, t221, t222, s[1], s[2], s[1,1], s[2,1], s[2,2],$
$s[1,1,1], s[2,1,1], s[2,2,1], s[2,2,2]\}, distributed)), \{A[1], A[2], A[3], A[4], A[5], A[6]\});$

Example Verification supporting our Multiset Partition Conjecture Using Maple (3/4)

$Z6c := (A[2]-A[3])\cdot s[1,1]\cdot t22\cdot A[1]^2\cdot A[2]\cdot A[3]\cdot A[4]+(A[2]\cdot A[3]-A[2]\cdot A[4]+A[2]\cdot A[5]$
$-A[4]\cdot A[6])\cdot s[2,2]\cdot t111\cdot A[1]^2\cdot A[2]\cdot A[3]\cdot A[4]+(A[2]-A[3])\cdot t22\cdot A[1]^2\cdot A[2]+(2\cdot A[2]\cdot A[3]$
$-A[2]\cdot A[4]-A[4]\cdot A[5])\cdot s[2,2,1]\cdot A[1]^2\cdot A[2]+(A[2]-A[4])\cdot s[2,2]\cdot A[1]^2\cdot A[2]$
$+(A[1]-A[4])\cdot s[2,1]\cdot A[1]\cdot A[2]+(2\cdot A[1]\cdot A[3]-A[1]\cdot A[4]-A[4]\cdot A[5])\cdot s[2,1,1]\cdot A[1]\cdot A[2]$
$+(2\cdot A[1]\cdot A[2]-A[1]\cdot A[3]-A[2]^2+2\cdot A[2]\cdot A[3]-2\cdot A[2]\cdot A[4])\cdot s[1,1]\cdot t1\cdot A[1]$
$+(A[1]-A[2])\cdot s[2]\cdot A[1]+(2\cdot A[1]\cdot A[2]^2+A[1]\cdot A[2]\cdot A[3]-A[1]\cdot A[2]\cdot A[4]-A[1]\cdot A[3]\cdot A[4]$
$-A[2]^2\cdot A[3]-A[2]^2\cdot A[4]+A[2]\cdot A[3]\cdot A[4])\cdot s[1,1]\cdot t11\cdot A[1]+(A[1]-A[2])\cdot s[1]\cdot t1\cdot A[1]$
$+(2\cdot A[1]\cdot A[2]-A[1]\cdot A[3]-A[2]^2)\cdot s[1]\cdot t11\cdot A[1]+(4\cdot A[1]\cdot A[2]^2\cdot A[3]-A[1]\cdot A[2]^2\cdot A[4]$
$-A[1]\cdot A[2]\cdot A[3]^2+A[1]\cdot A[2]\cdot A[3]\cdot A[4]-A[1]\cdot A[2]\cdot A[3]\cdot A[5]-A[1]\cdot A[3]\cdot A[4]\cdot A[6]$
$-A[2]^2\cdot A[3]\cdot A[4]-A[2]^2\cdot A[3]\cdot A[5]-A[2]\cdot A[3]\cdot A[4]^2+2\cdot A[2]\cdot A[3]\cdot A[4]\cdot A[5])\cdot s[1,1]$
$\cdot t111\cdot A[1]+(2\cdot A[1]\cdot A[3]-A[1]\cdot A[4]-A[4]\cdot A[5])\cdot s[1,1,1]\cdot A[2]+(A[1]\cdot A[2]^2$
$+A[1]\cdot A[2]\cdot A[3]-A[1]\cdot A[3]\cdot A[4]-A[2]^2\cdot A[3]+A[2]\cdot A[3]\cdot A[4]-A[2]\cdot A[3]\cdot A[5])\cdot s[1]\cdot t111$
$\cdot A[1]+(A[5]-A[6])\cdot s[2,2]\cdot t222\cdot A[1]^2\cdot A[2]^2\cdot A[3]^2\cdot A[4]^2\cdot A[5]+(A[5]-A[6])\cdot s[2,2,1]$
$\cdot t221\cdot A[1]^2\cdot A[2]^2\cdot A[3]^2\cdot A[4]^2\cdot A[5]+(A[5]-A[6])\cdot s[2,2,1]\cdot t222\cdot A[1]^2\cdot A[2]^2\cdot A[3]^2$
$\cdot A[4]^2\cdot A[5]\cdot A[6]+(A[4]-A[5])\cdot s[2,2,2]\cdot t211\cdot A[1]^2\cdot A[2]^2\cdot A[3]^2\cdot A[4]\cdot A[5]\cdot A[6]$

$\cdots$

$+(A[2]-A[5])\cdot s[1]\cdot t222\cdot A[1]^2\cdot A[2]\cdot A[3]^2\cdot A[4]+(2\cdot(A[3]-A[5]))\cdot s[2,2,2]\cdot t1\cdot A[1]^2$
$\cdot A[2]^2\cdot A[3]\cdot A[4]+(A[3]\cdot A[5]-A[4]\cdot A[6])\cdot s[2,2]\cdot t211\cdot A[1]^2\cdot A[2]^2\cdot A[3]\cdot A[4]$
$+(2\cdot A[3]\cdot A[4]\cdot A[5]-A[3]\cdot A[4]\cdot A[6]-A[3]\cdot A[5]^2+A[3]\cdot A[5]\cdot A[6]-A[4]\cdot A[5]\cdot A[6])\cdot s[2,2,1]$
$\cdot t211\cdot A[1]^2\cdot A[2]^2\cdot A[3]\cdot A[4]+(A[3]-A[5])\cdot s[2,2,1]\cdot t2\cdot A[1]^2\cdot A[2]^2\cdot A[3]\cdot A[4]$
$+(A[3]-A[4])\cdot s[2,2]\cdot t21\cdot A[1]^2\cdot A[2]^2\cdot A[3]\cdot A[4]+(A[3]-A[5])\cdot s[2,2,1]\cdot t21\cdot A[1]^2\cdot A[2]^2$
$\cdot A[3]\cdot A[4]^2+(2\cdot A[4]\cdot A[5]-A[4]\cdot A[6]-A[5]^2)\cdot s[2,1,1]\cdot t221\cdot A[1]^2\cdot A[2]^2\cdot A[3]^2\cdot A[4]$
$+(A[4]-A[6])\cdot s[2,1]\cdot t221\cdot A[1]^2\cdot A[2]^2\cdot A[3]^2\cdot A[4]$

$>nops(Z6c);$
86

Example Verification supporting our Multiset Partition Conjecture Using Maple (4/4)

for k to 86 do k, op(k, Z6c) end do

$$1, (A[5]-A[6])s[2,2,1]t222A[1]^2A[2]^2A[3]^2A[4]^2A[5]A[6]$$
$$2, (A[5]-A[6])s[2,2,1]t221A[1]^2A[2]^2A[3]^2A[4]^2A[5]$$
$$3, (A[5]-A[6])s[2,2]t222A[1]^2A[2]^2A[3]^2A[4]^2A[5]$$
$$4, (A[4]-A[5])s[2,2,2]t211A[1]^2A[2]^2A[3]^2A[4]A[5]A[6]$$
$$5, (A[4]-A[6])s[2,1,1]t222A[1]^2A[2]^2A[3]^2A[4]A[5]A[6]$$
$$6, (A[5]-A[6])s[2,2]t221A[1]^2A[2]^2A[3]^2A[4]^2$$
$$7, (A[4]-A[6])s[2,1]t222A[1]^2A[2]^2A[3]^2A[4]A[5]$$
$$8, (A[4]-A[5])s[2,2,2]t21A[1]^2A[2]^2A[3]^2A[4]A[5]$$
$$9, (A[2]-A[6])s[1,1,1]t222A[1]^2A[2]A[3]^2A[4]A[5]A[6]$$
$$10, (A[4]-A[5])s[2,2,2]t2A[1]^2A[2]^2A[3]^2A[4]$$
$$11, (A[4]-A[5])s[2]t222A[1]^2A[2]^2A[3]^2A[4]$$
$$12, (A[4]-A[6])s[2,1]t221A[1]^2A[2]^2A[3]^2A[4]$$
$$13, (A[4]A[5]-A[4]A[6]-A[5]^2)^2s[2,1,1]t221A[1]^2A[2]^2A[3]^2A[4]$$
$$\cdots$$
$$83, (A[1]A[2]^2+A[1]A[2]A[3]-A[1]A[3]A[4]-A[2]^2A[3]+A[2]A[3]A[4]-A[2]A[3]A[5])s[1]t111A[1]$$
$$84, (2A[1]A[3]-A[1]A[4]-A[4]A[5])s[1,1,1]A[2]$$
$$85, (A[1]-A[4])s[1,1]A[2]$$
$$86, (A[1]-A[2])s[1]$$

Tropical Version of Atiyah–Sutcliffe Conjecture for almost collinear configurations ($n = 4$)

Tropical version for $n = 4$ is reduced analogously to the following expression

$$
\begin{aligned}
AS_4^{trop} = E(1)^2[&3\alpha_1 e_1 + 3(\alpha_1+\alpha_2)e_2 + 3(\alpha_1+\alpha_2+\alpha_3)e_3 + 3(\alpha_1+\alpha_2+\alpha_3+\alpha_4)e_4 \\
&-(\alpha_2 e_1^{(1)} + \alpha_2 e_2 + \alpha_3 e_2^{(1)} + (\alpha_2+\alpha_3)e_3 + \alpha_4 e_3^{(1)} + (\alpha_2+\alpha_3+\alpha_4)e_4) \\
&-(\alpha_1 e_1^{(2)} + \alpha_1 e_2 + \alpha_3 e_2^{(2)} + (\alpha_1+\alpha_3)e_3 + \alpha_4 e_3^{(2)} + (\alpha_1+\alpha_3+\alpha_4)e_4) \\
&-(\alpha_1 e_1^{(3)} + \alpha_1 e_2 + \alpha_2 e_2^{(3)} + (\alpha_1+\alpha_2)e_3 + \alpha_4 e_3^{(3)} + (\alpha_1+\alpha_2+\alpha_4)e_4) \\
&-(\alpha_1 e_1^{(4)} + \alpha_1 e_2 + \alpha_2 e_2^{(4)} + (\alpha_1+\alpha_2)e_3 + \alpha_3 e_3^{(4)} + (\alpha_1+\alpha_2+\alpha_3)e_4)]
\end{aligned}
$$

$$
\begin{aligned}
= E(1)^2[&3\alpha_1 e_1 + 3(\alpha_1+\alpha_2)e_2 + 3(\alpha_1+\alpha_2+\alpha_3)e_3 + 3(\alpha_1+\alpha_2+\alpha_3+\alpha_4)e_4 \\
&-\Big(\alpha_1 \underbrace{\left(e_1^{(1)}+e_1^{(2)}+e_1^{(3)}+e_1^{(4)}\right)}_{=3e_1} + (\alpha_2-\alpha_1)e_1^{(1)} + 3(\alpha_1+\alpha_2)e_2 + (\alpha_3-\alpha_2)\left(e_2^{(1)}+e_2^{(2)}\right) + \\
&+(3(\alpha_1+\alpha_2)+2\alpha_3+\alpha_3)e_3 + (\alpha_4-\alpha_3)\left(e_3^{(1)}+e_3^{(2)}+e_3^{(3)}\right)\Big)
\end{aligned}
$$

$$
\boxed{AS_4^{trop} = E(1)^2\left[(\alpha_1-\alpha_2)e_1^{(1)} + (\alpha_2-\alpha_3)\left(e_2^{(1)}+e_2^{(2)}\right) + (\alpha_3-\alpha_4)\left(e_3^{(1)}+e_3^{(2)}+e_3^{(3)}\right)\right]}
$$

Here we used the elementary formulas:

$$
e_1^{(1)}+e_1^{(2)}+e_1^{(3)}+e_1^{(4)} = 3e_1,\; e_2^{(1)}+e_2^{(2)}+e_2^{(3)}+e_2^{(4)} = 2e_2,\; e_3^{(1)}+e_3^{(2)}+e_3^{(3)}+e_3^{(4)} = e_3.
$$

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THANK YOU