



Combinatorics and artificial intelligence

5th Croatian Combinatorial Days

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Introduction

- ▶ Artificial intelligence and its rapid development:
 - today: Narrow AI (less than human intelligence)
 - 2040: General AI (equal to human intelligence)
 - 2060: Super AI (greater than human intelligence)
 - source: https://www.researchgate.net/figure/Future-evolution-of-Artificial-Intelligence-56_fig2_353712184
- ▶ Impact of AI on education and science (improvements and concerns)

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AI i.e. ChatGPT and combinatorics in schools

1. **Example:** There are 12 blue and 9 red points on the plane, no three of them belong to the same straight line. How many triangles can be formed with vertices among these points such that not all three vertices are the same color?
2. **Example:** How many four-digit numbers are there that can be written with at most two characters?
3. **Example:** How many six-digit numbers can be formed from the digits 0,1,2,3,4,5 with the condition that each digit appears exactly once and that even digits are next to each other. (Note: 0 is an even number)

► Literature: Ž. Ivanović, S. Ognjanović, "Matematika-Zbirka zadataka za 1.razred gimnazija i tehničkih škola", Krug, Beograd, 2008. (1. Example: Problem 93. – Page 16, 2. Example: Problem 129. – Page 20, 3. Example: Problem 104. – Page 17)

1. Example:

1. Example-solution given by ChatGPT

- ▶ To solve this, we need to find how many triangles can be formed using the 12 blue and 9 red points on the plane, with the condition that not all three vertices of the triangle are of the same color.
- ▶ Here's a step-by-step breakdown:

Step 1:

- ▶ Total number of triangles The total number of triangles that can be formed from 21 points (12 blue + 9 red) is given by choosing 3 points out of the 21, which is:

$$\binom{21}{3} = \frac{21 \cdot 20 \cdot 19}{3 \cdot 2 \cdot 1} = 1330$$

Step 2: Subtracting monochromatic triangles We need to subtract the triangles where all three points are of the same color:

1. **All blue points**: The number of triangles formed using only the 12 blue points is:

$$\binom{12}{3} = \frac{12 \cdot 11 \cdot 10}{3 \cdot 2 \cdot 1} = 220$$

2. **All red points**: The number of triangles formed using only the 9 red points is:

$$\binom{9}{3} = \frac{9 \cdot 8 \cdot 7}{3 \cdot 2 \cdot 1} = 84$$

So, the total number of monochromatic triangles is:

$$220 + 84 = 304$$

Step 3: Triangles with vertices of different colors Finally, we subtract the number of monochromatic triangles from the total number of triangles:

$$1330 - 304 = 1026$$

Thus, the number of triangles where not all vertices are the same color is $** 1026 **$.



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2. Example:

2. Example – solution given by ChatGPT (1.iteration)

- ▶ To find how many four-digit numbers can be written with at most two distinct characters, we need to count how many such numbers can be formed where the digits are restricted to only two possible values.

Case 1: All four digits are the same

- In this case, the four-digit number consists of a repeated single character. Since the number must be a four-digit number, the first digit cannot be zero. The choices for the first digit are:

$$\{1, 2, 3, 4, 5, 6, 7, 8, 9\}$$

- So, there are 9 possible numbers in this case, corresponding to 1111, 2222, 3333, etc.

Case 2: Two distinct characters In this case, the number must be written using exactly two distinct characters, and the first digit cannot be zero.

1. ****Choosing the two distinct characters**:**

The first character must be a non-zero digit, and the second character can be any digit (including zero). So, the choices for the two characters are:

- Choose the first digit (non-zero) from

$$\{1, 2, 3, 4, 5, 6, 7, 8, 9\} : 9 \text{ choices.}$$

- Choose the second digit from

$$\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\} \setminus \{\text{first digit}\} : 9 \text{ choices.}$$

Therefore, there are $9 \cdot 9 = 81$ ways to select the two distinct characters.

2. ****Arranging the characters****:

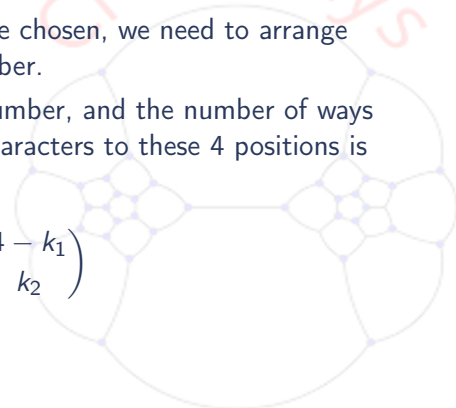
- ▶ Once the two distinct digits are chosen, we need to arrange them to form a four-digit number.
- ▶ There are 4 positions in the number, and the number of ways to assign exactly 2 different characters to these 4 positions is given by:

$$\binom{4}{k_1} \cdot \binom{4 - k_1}{k_2}$$

Where k_1 is the number of

- ▶ **STOP**

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2. Example-solution is given by ChatGPT (2. iteration)

- ▶ To solve this problem, we need to count how many four-digit numbers can be formed using at most two distinct digits. The number must be a four-digit number, meaning the first digit cannot be zero.

Case 1: All four digits are the same

- ▶ In this case, the number consists of a single repeated digit. Since the first digit cannot be zero, the choices for the repeated digit are:

$$\{1, 2, 3, 4, 5, 6, 7, 8, 9\}$$

- ▶ Thus, there are 9 possible four-digit numbers in this case, corresponding to numbers like 1111, 2222, ..., 9999.

Case 2: Two distinct digits

- ▶ In this case, the number must be written using exactly two distinct digits, with the first digit non-zero.

Step 1: Choosing the two distinct digits

- The first digit must be a non-zero digit, so there are 9 choices for the first digit (1 to 9).
- The second digit can be any digit from 0 to 9, except for the first digit. Therefore, there are 9 choices for the second digit.

Thus, there are $9 \cdot 9 = 81$ ways to choose the two distinct digits.

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Step 2: Arranging the digits

- ▶ Once the two distinct digits are chosen, we need to arrange them in 4 positions. We can have the first digit appear in 1, 2, or 3 positions, and the second digit will fill the remaining positions.
- ▶ The number of ways to do this is the number of ways to distribute one of the digits in the four positions.
- ▶ The number of distinct ways to distribute two digits across four positions is given by choosing the number of positions for one of the digits.
- ▶ There are three nontrivial cases here (since all digits being the same is covered in Case 1):

- ▶ ****1st digit appears in 3 positions**:**

There are

$$\binom{4}{3} = 4$$

ways to place the first digit in 3 positions, and the second digit fills the remaining position.

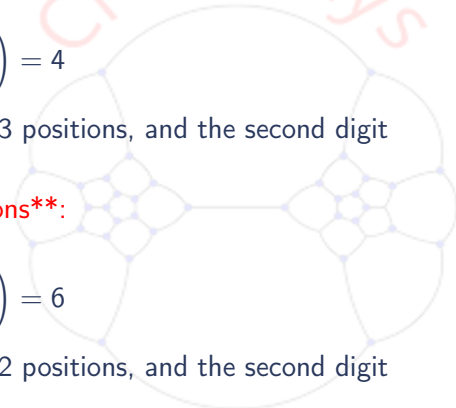
- ▶ ****1st digit appears in 2 positions**:**

There are

$$\binom{4}{2} = 6$$

ways to place the first digit in 2 positions, and the second digit fills the other 2 positions.

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****1st digit appears in 1 position**:**

- ▶ There are

$$\binom{4}{1} = 4$$

ways to place the first digit in 1 position, and the second digit fills the remaining 3 positions.

- ▶ For each pair of two distinct digits, we have:

$$\begin{aligned} &4(\text{for 3 positions}) + 6(\text{for 2 positions}) + 4(\text{for 1 position}) \\ &= 14 \text{ ways to arrange the digits.} \end{aligned}$$

Step 3: Total for two distinct digits

For each of the 81 possible pairs of two distinct digits, we have 14 ways to arrange them, so the total number of four-digit numbers that can be written using exactly two distinct digits is:

$$81 \cdot 14 = 1134$$

- ▶ Total number of four-digit numbers with at most two distinct digits
- ▶ Finally, to find the total number of four-digit numbers that can be written with at most two distinct digits, we add the numbers from Case 1 and Case 2:

$$9(\text{all digits the same}) + 1134(\text{two distinct digits}) = 1143$$

- ▶ Thus, the total number of four-digit numbers that can be written with at most two distinct digits is ****1,143****.



3. Example

3. Example-solution given by ChatGPT (solution number 1)

- ▶ To find how many six-digit numbers can be formed from the digits 0, 1, 2, 3, 4, 5 with the condition that each digit appears exactly once and that the even digits are next to each other, let's break the problem into steps.

Step 1: Identify the even and odd digits

- **Even digits**: 0, 2, 4
 - **Odd digits**: 1, 3, 5
- ▶ We need to arrange the digits such that the even digits are grouped together (i.e., they must form a block).

Step 2: Treat the group of even digits as a block Since the even digits 0, 2, 4 must stay next to each other, we can treat this group of even digits as a single block or "super-digit." Now, we are arranging 4 objects:

- ▶ The "block" of even digits
- ▶ The digits 1, 3, 5 (which are odd digits and can be placed anywhere)
- ▶ The number of ways to arrange these 4 objects (the block and the 3 odd digits) is:

$$4! = 4 \cdot 3 \cdot 2 \cdot 1 = 24$$

Step 3: Arrange the even digits within the block Within the block, we can arrange the three even digits 0, 2, 4 in different ways. However, the first digit of the six-digit number cannot be 0 (since the number cannot start with zero). Therefore, we need to make sure that 0 is not the first digit within the block.

- ▶ If the even block is at the beginning of the number, then 0 cannot be the first digit within the block. The number of valid ways to arrange 2 and 4 at the beginning of the block and 0 in the second or third position is:

$$2! = 2 \cdot 1 = 2$$

- ▶ This accounts for the restriction that 0 cannot be at the start of the block.
- ▶ If the even block is not at the beginning of the number (i.e., it is placed in the second, third, or fourth position), there are no restrictions, and all 3 even digits can be arranged in:

$$3! = 6 \text{ ways}$$

Step 4: Total number of valid arrangements

- ▶ There are 24 ways to arrange the block and the odd digits. For each arrangement:
- ▶ In 1 case (when the even block is at the beginning), there are 2 valid arrangements of the even digits.
- ▶ In the other 3 cases (when the even block is in any other position), there are 6 valid arrangements of the even digits.
- ▶ Thus, the total number of valid six-digit numbers is:

$$(1 \cdot 2) + (3 \cdot 6) = 2 + 18 = 20$$

- ▶ Therefore, the total number of six-digit numbers that can be formed with the given conditions is ****20****.

3. Example-solution given by ChatGPT (solution number 2)

- ▶ We need to determine how many six-digit numbers can be formed from the digits $\{0, 1, 2, 3, 4, 5\}$ such that each digit appears exactly once and all even digits are adjacent to each other. The even digits are $\{0, 2, 4\}$, and the odd digits are $\{1, 3, 5\}$.

Step 1: Treat the even digits as a single "block"

- ▶ Since the even digits $\{0, 2, 4\}$ must be adjacent, we can treat this group of three digits as one "block."
- ▶ Therefore, instead of arranging 6 digits, we now need to arrange 4 elements: the "even digit block" and the 3 odd digits $\{1, 3, 5\}$.
- ▶ The number of ways to arrange these 4 elements is:

$$4! = 4 \cdot 3 \cdot 2 \cdot 1 = 24$$

Step 2: Arranging the even digits within the block

- ▶ The even digits $\{0, 2, 4\}$ can be arranged in $3!$ ways within the block. However, we must ensure that 0 is not in the leading position (since it would make the number not a valid six-digit number). Let's count how many valid arrangements of the even digits there are:
 - First, there are $3! = 6$ ways to arrange the three even digits without any restrictions.
 - Out of these 6 arrangements, 2 will have 0 in the leading position (since the other two digits $\{2, 4\}$ can be arranged in $2! = 2$ ways when 0 is first).
 - Therefore, the number of valid arrangements of the even digits (with 0 not in the leading position) is:

$$3! - 2 = 6 - 2 = 4$$

Step 3: Total number of valid six-digit numbers

- ▶ Now, multiply the number of ways to arrange the four elements (Step 1) by the number of valid ways to arrange the even digits within the block (Step 2):

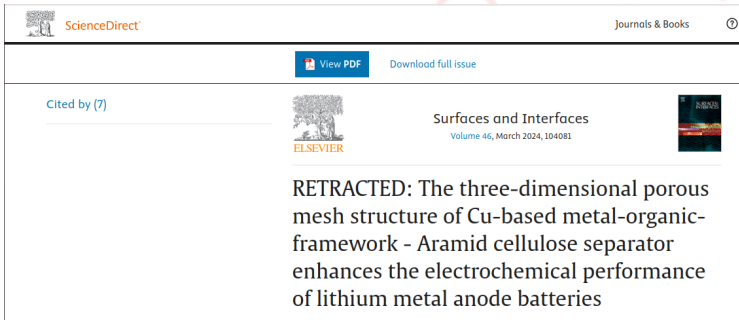
$$24 \cdot 4 = 96$$

- ▶ Thus, the total number of six-digit numbers that can be formed is ****96****.

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AI and science



The screenshot shows a ScienceDirect article page. At the top left is the ScienceDirect logo. To the right, it says 'Journals & Books'. Below this are two buttons: 'View PDF' and 'Download full issue'. On the left side, it says 'Cited by (7)'. The article title is 'Surfaces and Interfaces', Volume 46, March 2024, 104081. The article is marked as 'RETRACTED'. The title of the retracted article is 'RETRACTED: The three-dimensional porous mesh structure of Cu-based metal-organic-framework - Aramid cellulose separator enhances the electrochemical performance of lithium metal anode batteries'. The Elsevier logo is visible on the left side of the article information.

- ▶ print screen of source: <https://www.sciencedirect.com/science/article/pii/S2468023024002402>
- ▶ According to source https://www.researchgate.net/figure/Dialogue-from-a-command-prompt-given-by-AI-based-on-a-large-language-model-eg_fig3_381697010, the Introduction sentence of this paper started with words "Certainly, here is a possible introduction for your topic: ..."

History of Tverberg theorem—research paper

1. Radon's lemma (1921)

- Radon used it in his proof of Helly's theorem.
- It says that any set X of $d + 2$ points in \mathbb{R}^d can be split into two sets whose convex hulls intersect. So it is the case $r = 2$ of Tverberg's theorem.

2. Another result linked to this theorem is Radon's centerpoint theorem.

- This states that for any set X of n points in \mathbb{R}^d , there is a point p such that any closed half-space that contains p also contains at least

$$\left\lceil \frac{n}{d+1} \right\rceil$$

points of X .

- The standard proof of this result uses Helly's theorem.

► Tverberg's theorem implies it in few lines:

- setting $r = \left\lceil \frac{n}{d+1} \right\rceil$
- there is a partition of X into r parts X_1, \dots, X_r and a point $p \in \mathbb{R}^d$ such that

$$p \in \bigcap_{j=1}^r \text{conv}X_j.$$

- Then p is a centerpoint of X : every closed half-space containing p contains at least one point from each X_j .

3. Birch "On $3N$ points in a plane",

- proves that any $3N$ points in the plane determine N triangles that have a point in common.
- His motivation was the (planar) centerpoint theorem.
- He proves more, namely the case $d = 2$ of Tverberg's theorem, and he states the general case as a conjecture.

4. Tverberg's original motivation was also the centerpoint theorem, and he learned about Birch's result and conjecture only later.

5. He proved it first for $d = 3$ in 1963 and in full generality in 1964.

History of Tverberg theorem–AI

Give me a brief history of Tveberg's theorem.

Answer:

****Tverberg's theorem****

is a result in combinatorial geometry, named after the Norwegian mathematician Helge Tverberg, who first formulated it in 1966. The theorem is an important result in discrete geometry and concerns the partitioning of points in Euclidean space.

****Statement of the Theorem:****

Tverberg's theorem states that for any set of $(r - 1)(d + 1) + 1$ points in \mathbb{R}^d , it is possible to partition the points into r subsets whose convex hulls all intersect in at least one common point.

****History:****

▶ ****1966:****

Helge Tverberg published the theorem, which became a fundamental result in the field of combinatorial geometry. His work built on earlier results, including Radon's theorem, which states that any set of $d + 2$ points in \mathbb{R}^d can be partitioned into two subsets with intersecting convex hulls.

▶ ****1970s-1980s:****

The theorem gained recognition for its applications in various areas, including topological combinatorics and optimization. Several mathematicians worked on generalizing the theorem, leading to what is known as "colored Tverberg theorems," where points are colored, and the partitions must satisfy additional constraints.

► ****1990s-Present:****

Research continued on extensions and variations of Tverberg's theorem, including applications in areas like data analysis and computational geometry. The theorem also inspired a significant amount of research into Tverberg-type problems in higher dimensions and with different combinatorial structures.

► ****1990s-Present:****

Research continued on extensions and variations of Tverberg's theorem, including applications in areas like data analysis and computational geometry. The theorem also inspired a significant amount of research into Tverberg-type problems in higher dimensions and with different combinatorial structures.

Tverberg theorem and AI Tverberg theorem

Theorem (Tverberg theorem)

Given $(r - 1)(d + 1) + 1$ points in \mathbb{R}^d , there is a partition of them into r parts whose convex hulls intersect.

AI: State Tveberg's theorem.

Answer: Tverberg's Theorem can be stated as follows:

For any integer $r \geq 2$ and any set of $(r - 1)(d + 1) + 1$ points in \mathbb{R}^d , it is possible to partition the set into r subsets such that the convex hulls of these subsets all have a non-empty intersection. In other words, there exists a point that lies in the convex hull of each of the r subsets.

****Mathematical Formulation:****

- ▶ Given a set S of $(r - 1)(d + 1) + 1$ points in \mathbb{R}^d , there exists a partition S_1, S_2, \dots, S_r of S such that:

$$\bigcap_{i=1}^r \text{conv}(S_i) \neq \emptyset,$$

where $\text{conv}(S_i)$ denotes the convex hull of the subset S_i . This means that you can always find a way to divide the set of points into r groups, such that their convex hulls share at least one common point.

Special cases of Tverberg theorem and AI

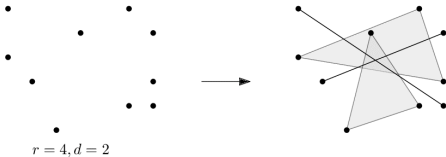


Figura: Math literature: $r = 4, d = 2$

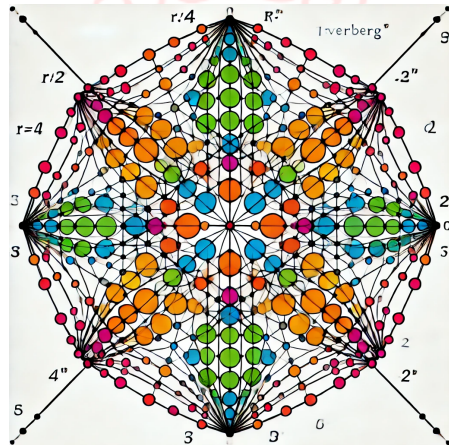


Figura: AI: $r = 4, d = 2$

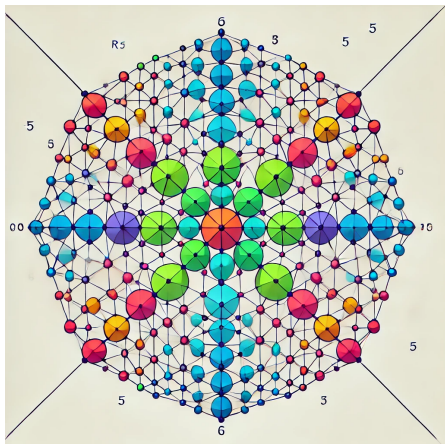


Figura: Al: $r = 5, d = 2$

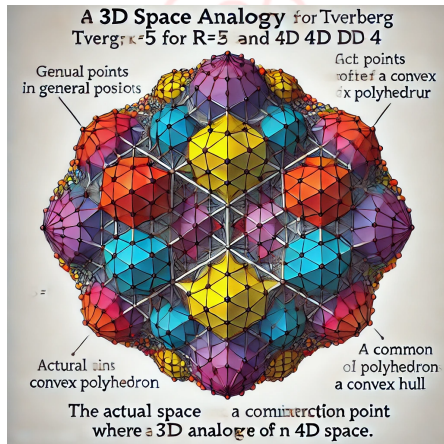


Figura: Al: $r = 5, d = 4$



Figura: AI: $r = 5$, $d = 3$

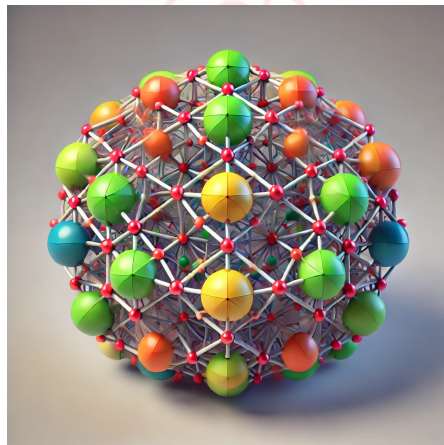


Figura: AI: $r = 4$, $d = 3$

Conclusion

- ▶ AI - like a powerful professional tool with a limitations
- ▶ AI - useful or misleading
- ▶ AI - misconceptions or incomprehensibility



Literature



https://www.researchgate.net/figure/Future-evolution-of-Artificial-Intelligence-56_fig2_353712184



Ž. Ivanović, S. Ognjanović, "*Matematika-Zbirka zadataka za 1.razred gimnazija i tehničkih škola*", Krug, Beograd, 2008.



I. Bárány, P. Soberón, *Tveberg's theorem is 50 years old: a survey*, Bulletin (New Series) of the AMS, Vol. 55, Number 4, October 2018, Pages 459–492.

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Available: <https://www.ams.org/journals/bull/2018-55-04/S0273-0979-2018-01634-0/S0273-0979-2018-01634-0.pdf>



<https://www.sciencedirect.com/science/article/pii/S2468023024002402>



https://www.researchgate.net/figure/Dialogue-from-a-command-prompt-given-by-AI-based-on-a-large_fig3_381697010



<https://chatgpt.com/>

Feedback

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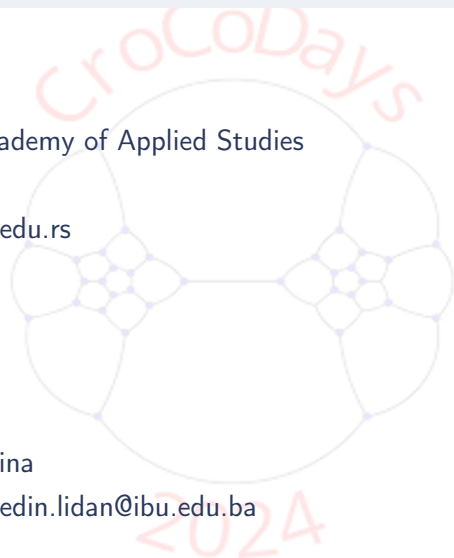
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Thank you for your attention.
Questions?



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