On the Petersen Coloring Conjecture

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(joint work with Riste Škrekovski)

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Introduction

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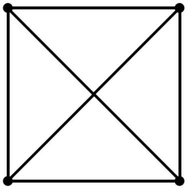

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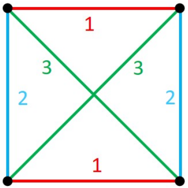

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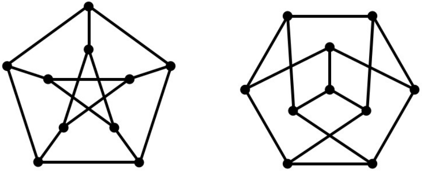
The **Petersen graph** P_{10} is the smallest snark.



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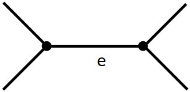

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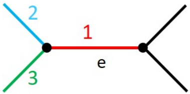

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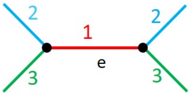

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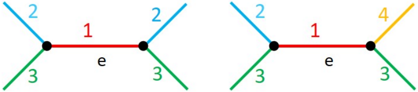

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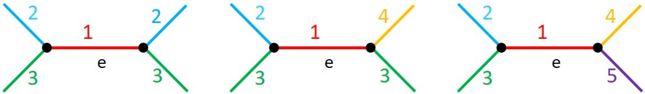

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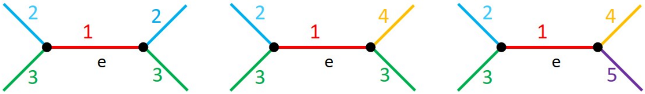

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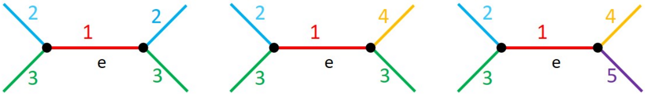

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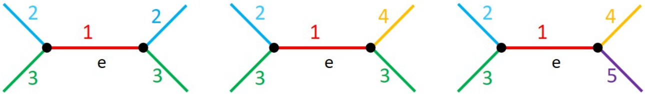

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Petersen Coloring Conjecture (restatement). If G is a bridgeless cubic graph, then G has a normal 5-edge-coloring.

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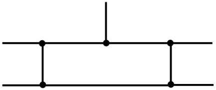

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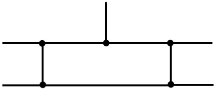

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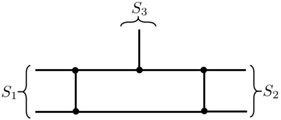

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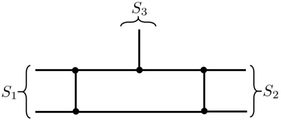

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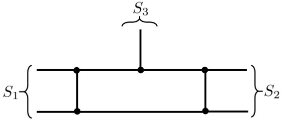

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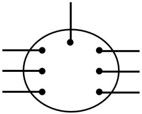

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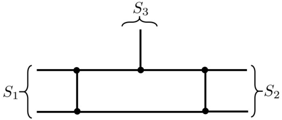

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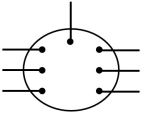

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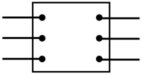

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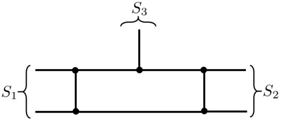

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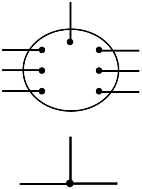
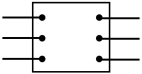
supervertex 

superedge 

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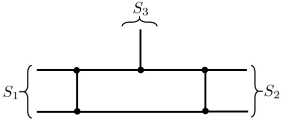

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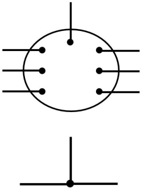

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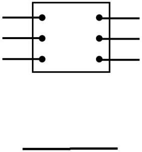

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A **superposition** $G(\mathcal{V}, \mathcal{E})$ is obtained from a snark G by replacing:

- each vertex v with a supervertex $\mathcal{V}(v)$;
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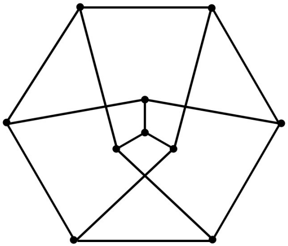

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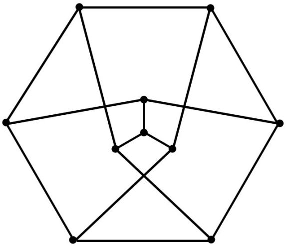

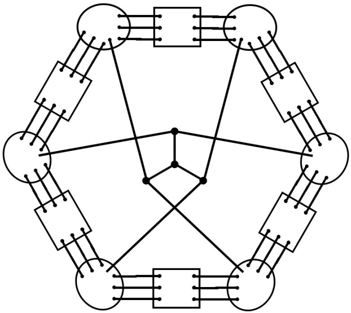

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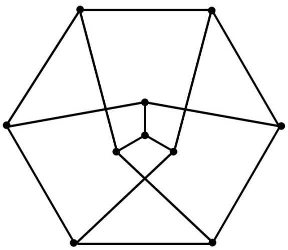

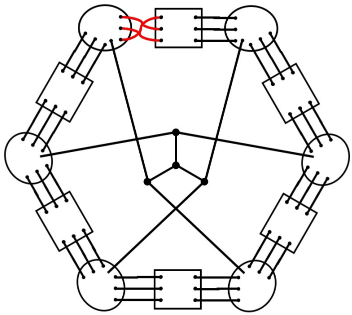

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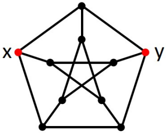

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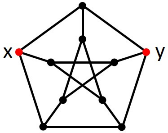

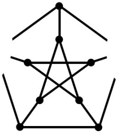

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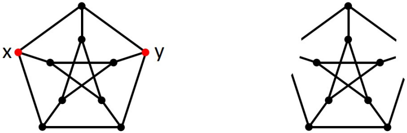

Theorem. If G is a snark and $\mathcal{E}(e)$ a proper superedge for every edge e, then $G(\mathcal{V}, \mathcal{E})$ is also a snark.

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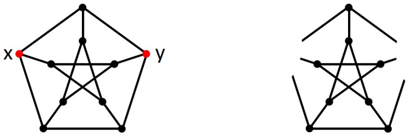

Theorem. If G is a snark and $\mathcal{E}(e)$ a proper superedge for every edge e, then $G(\mathcal{V}, \mathcal{E})$ is also a snark.

We will consider superpositions by proper superedges $H_{x,y}$ where H is:

- a hypohamiltonian snark;
- a Flower snark.

Superposition and normal colorings

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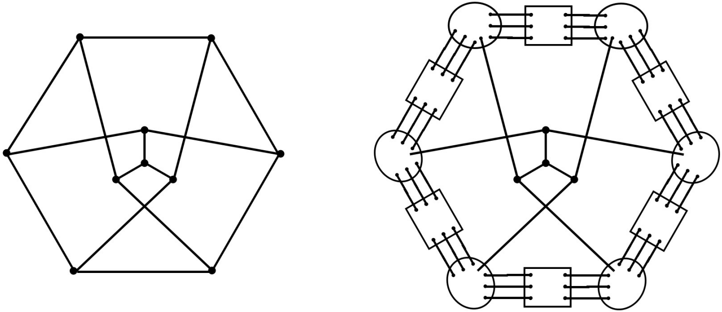

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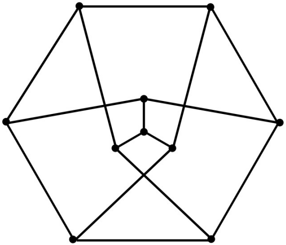

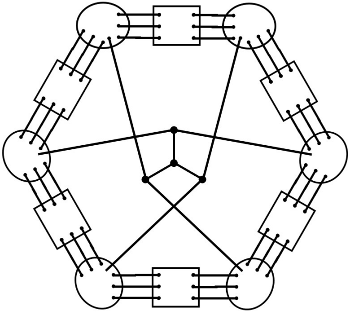

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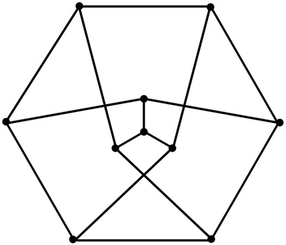

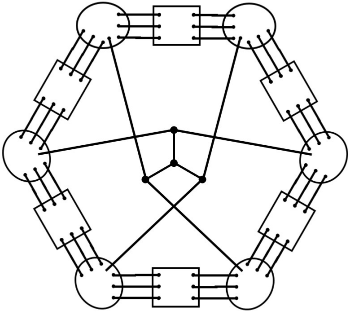

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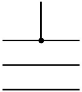

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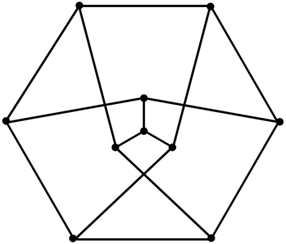

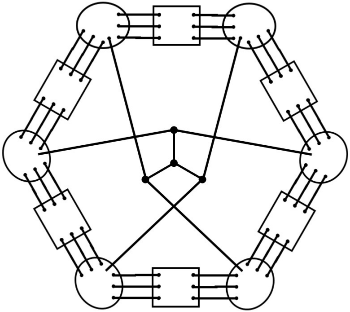

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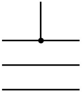
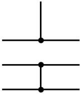
A 

A' 

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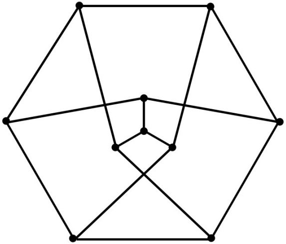

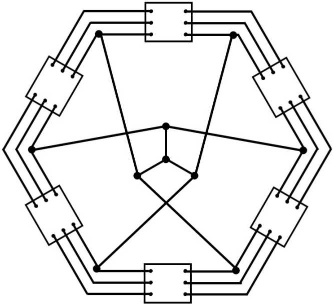

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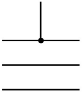

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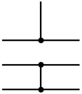

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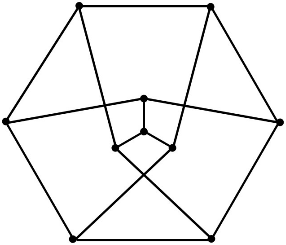

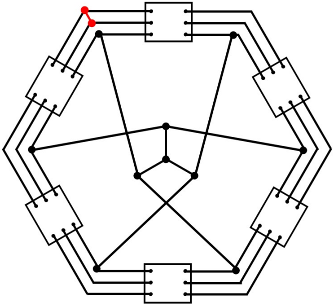

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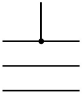
A



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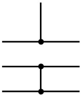

Superposition and normal colorings

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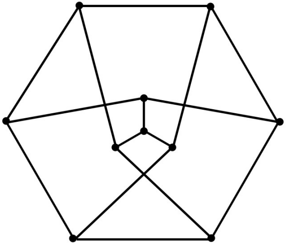

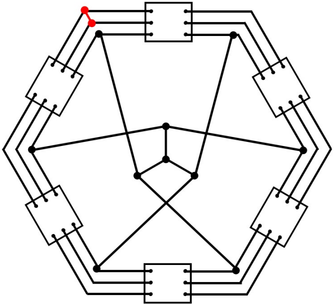

We use supervertices A or A'.

We use superedges $H_{x,y}$, where H is any snark and $d(x, y) \geq 3$.

Superposition and normal colorings

Our aim is to **extend** a normal 5-edge-coloring of G to $G(\mathcal{V}, \mathcal{E})$.

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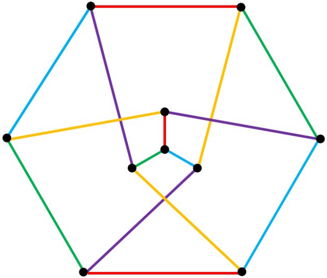

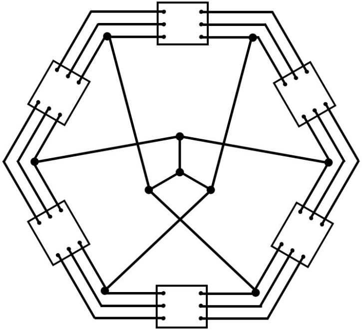

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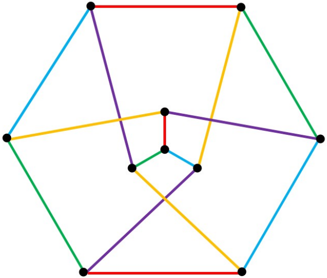

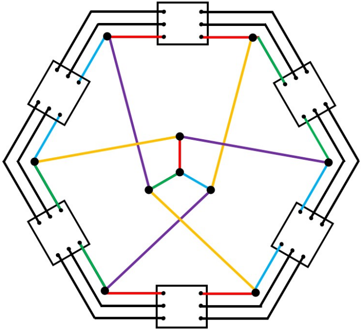

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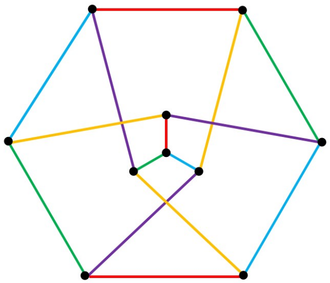

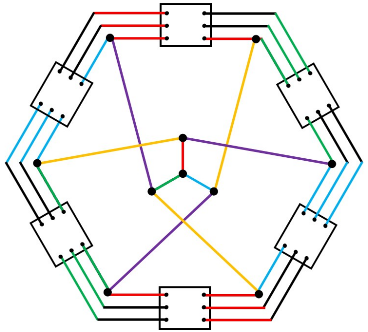

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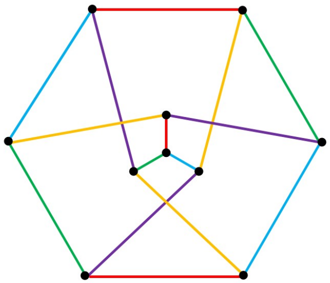

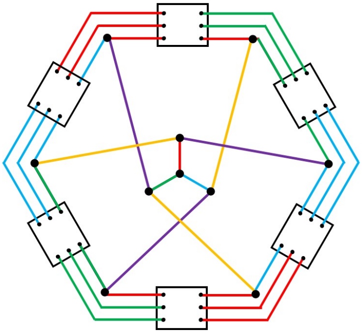

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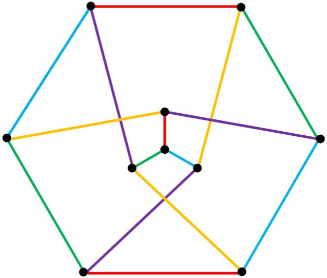

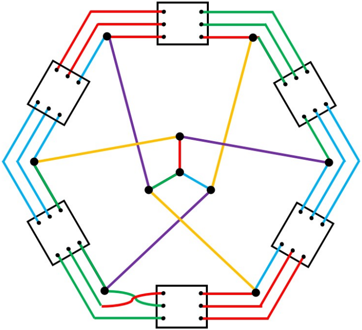

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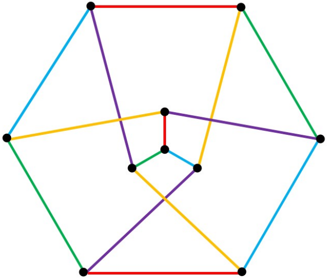

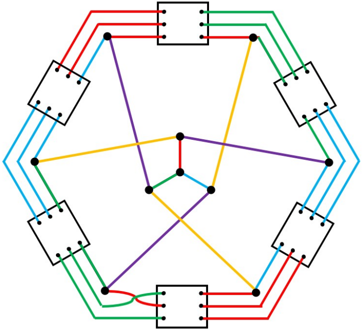

Superposition and normal colorings

Theorem. A superposition $G(\mathcal{V}, \mathcal{E})$ has a normal 5-edge-coloring if $H_{x,y}$ is **fully right**, i.e. it has a normal 5-edge-coloring compatible with all three color schemes below.

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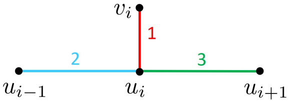

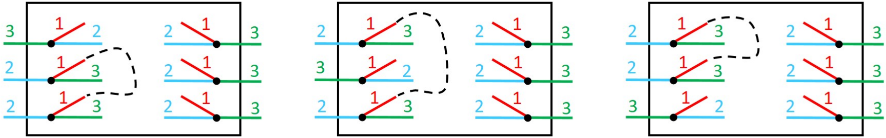

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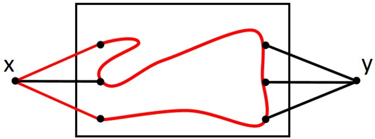

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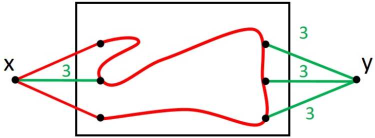

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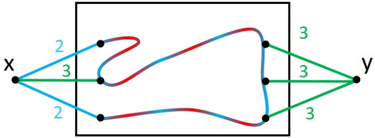

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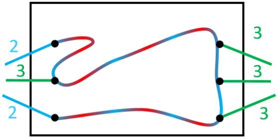

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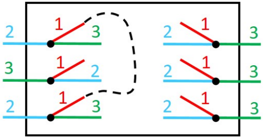

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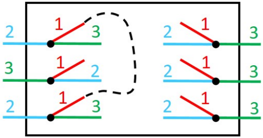

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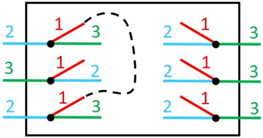

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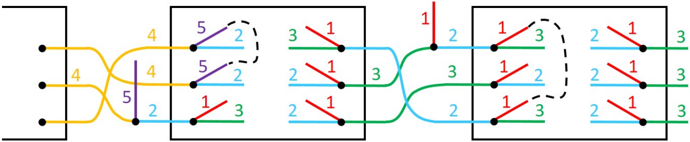
Observation. There exists a superedge $H_{x,y}$, where H is a Flower snark, which is **<u>not</u>** fully right.

Superposition and normal colorings

We approach superedges which are **not** fully right by considering **a pair** of consecutive superedges as a whole.



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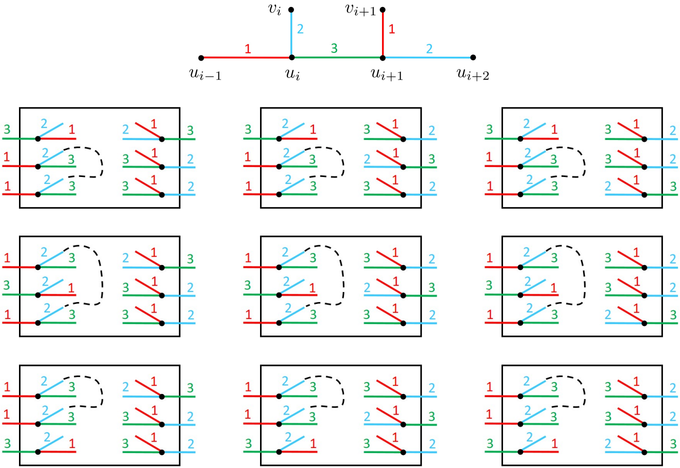

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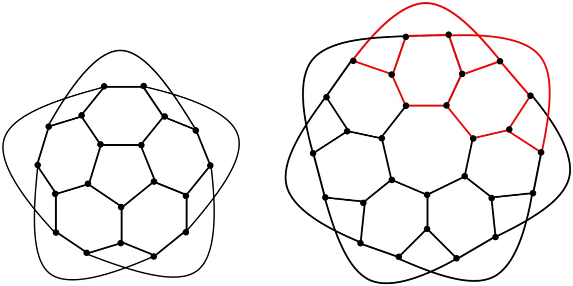

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Thank you for the attention.