On hyperfibonacci and hyperlucas numbers and their weighted sums

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1. Hypersequences $(a_n^{(r)})_{n\in\mathbb{N}_0}$, $r\in\mathbb{N}_0$, of an arbitrary sequence $(a_n)_{n\in\mathbb{N}_0}$, $a_n^{(0)} := a_n$

1.1. Applications to the Fibonacci and Lucas sequences

2. Weighted sums of the type $\sum\limits_{k=0}^{n} k^l a_k^{(r)}$, $l,n,r\in\mathbb{N}_0$

3. Main results

3.1 Applications to the Fibonacci and Lucas sequences ("Ledin and Brousseau's summation problems")

4. Conclusions

Reference:

D. PARISSE, On Hypersequences of an Arbitrary Sequence and Their Weighted Sums, Integers 24 (2024), #A70

1. Hypersequences

Def.: The hypersequence of the rth generation, $r \in \mathbb{N}_0$, of an arbitrary sequence $(a_n)_{n\in\mathbb{N}_0}$ is defined recursively as

$$a_n^{(0)} := a_n \,, \quad a_n^{(r)} := \sum_{k=0}^{n} a_k^{(r-1)} \,, \quad r \geq 1$$

Examples For $r=1$:
$$a_n^{(1)} = \sum_{k=0}^{n} a_k^{(0)} = \sum_{k=0}^{n} a_k$$

is the sequence of partial sums of $(a_n)_{n\in\mathbb{N}_0}$

For $r=2$:
$$a_n^{(2)} = \sum_{k=0}^{n} a_k^{(1)} = \sum_{k=0}^{n}\left(\sum_{j=0}^{k} a_j\right)$$

is the sequence of partial sums of $(a_n^{(1)})_{n\in\mathbb{N}_0}$ and so on.

$r \backslash n$	0	1	2	3	4	...
0	a_0	a_1	a_2	a_3	a_4	...
1	a_0	a_0+a_1	$a_0+a_1+a_2$	$a_0+a_1+a_2+a_3$	$a_0+a_1+a_2+a_3+a_4$	...
2	a_0	$2a_0+a_1$	$3a_0+2a_1+a_2$	$4a_0+3a_1+2a_2+a_3$	$5a_0+4a_1+3a_2+2a_3+a_4$	
3	a_0	$3a_0+a_1$	$6a_0+3a_1+a_2$	$10a_0+6a_1+3a_2+a_3$	$15a_0+10a_1+6a_2+3a_1+a_4$	
⋮	⋮	⋮	⋮	⋮	⋮	

Theorem 1 (Dil, Mező, 2008)

For all $r \in \mathbb{N}$ and $n \in \mathbb{N}_0$:

$$a_n^{(r)} = \sum_{k=0}^{n} \binom{n+r-1-k}{r-1} a_k = \sum_{k=0}^{n} \binom{r+k-1}{k} a_{n-k}$$

Examples 0) Constant sequence $a_n = 1$, $n \in \mathbb{N}_0$

$r \backslash n$	0	1	2	3	4	5	6	7	8	9	10
0	1	1	1	1	1	1	1	1	1	1	1
1	1	2	3	4	5	6	7	8	9	10	11
2	1	3	6	10	15	21	28	36	45	55	66
3	1	4	10	20	35	56	84	120	165	220	286
4	1	5	15	35	70	126	210	330	495	715	1001
5	1	6	21	56	126	252	462	792	1287	2002	3003
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮

É. Lucas called this array "Le carré arithmétique de Fermat"

By Theorem 1 we have the general term

$$a_n^{(r)} = \sum_{k=0}^{n} \binom{n+r-1-k}{r-1} \cdot 1 = \binom{n+r}{r}, \quad n, r \in \mathbb{N}_0$$

1.1. Applications to the Fibonacci and Lucas sequences

1) Fibonacci sequence F_n defined by

$F_0 = 0,\ F_1 = 1\ ,\ F_{n+2} = F_{n+1} + F_n\ ,\ n \geq 0$

By Theorem 1 we obtain

$$F_n^{(r)} = \sum_{k=0}^{n} \binom{n+r-1-k}{r-1} F_k = \sum_{k=0}^{n} \binom{r+k-1}{k} F_{n-k}$$

r\n	0	1	2	3	4	5	6	7	8	9 …
0	0	1	1	2	3	5	8	13	21	34 …
1	0	1	2	4	7	12	20	33	54	88 …
2	0	1	3	7	14	26	46	79	133	221 …
3	0	1	4	11	25	51	97	176	309	530 …
4	0	1	5	16	41	92	189	365	674	1204 …
5	0	1	6	22	63	155	344	709	1383	2587 …
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮

Hyperfibonacci numbers $F_n^{(r)}$

Theorem 2 Let $r, n \in \mathbb{N}_0$. Then

$$F_n^{(r)} = F_{n+2r} - \sum_{k=0}^{r-1} \binom{n+k-1}{k} F_{2(r-k)} \qquad (1)$$

Equating the formulas in Theorem 1 and Theorem 2 we obtain

Corollary 1 For all $r, n \in \mathbb{N}_0$, we have

$$F_{n+2r} = \sum_{k=0}^{n} \binom{n+r-1-k}{r-1} F_k + \sum_{k=0}^{r-1} \binom{n+k-1}{k} F_{2(r-k)}$$

$$= \sum_{k=0}^{n} \binom{r+k-1}{k} F_{n-k} + \sum_{k=0}^{r-1} \binom{n+k-1}{k} F_{2(r-k)}$$

The hyperfibonacci numbers of the rth generation satisfy a second-order linear inhomogeneous recurrence relation

Theorem 3 (Cristea, Martinjak, Urbiha, 2016)

For all $r, n \in \mathbb{N}_0$, we have

$$F_0^{(r)} = 0, \quad F_1^{(r)} = 1, \quad F_{n+2}^{(r)} = F_{n+1}^{(r)} + F_n^{(r)} + \binom{n+r}{r-1}, \quad n \geq 0$$

2) <u>Lucas sequence L_n</u> defined by

$$L_0 = 2,\ L_1 = 1\ ,\ L_{n+2} = L_{n+1} + L_n\ ,\ n \geq 0$$

By Theorem 1 we obtain

$$L_n^{(r)} = \sum_{k=0}^{n} \binom{n+r-1-k}{r-1} L_k = \sum_{k=0}^{n} \binom{r+k-1}{k} L_{n-k}$$

r\n	0	1	2	3	4	5	6	7	8	9 ...
0	2	1	3	4	7	11	18	29	47	76 ...
1	2	3	6	10	17	28	46	75	122	198 ...
2	2	5	11	21	38	66	112	187	309	507 ...
3	2	7	18	39	77	143	255	442	751	1258 ...
4	2	9	27	66	143	286	541	983	1734	2992 ...
5	2	11	38	104	247	553	1074	2057	3791	6783 ...
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮

Hyperlucas numbers $L_n^{(r)}$

<u>Theorem 4</u> Let $r, n \in \mathbb{N}_0$. Then

$$L_n^{(r)} = L_{n+2r} - \sum_{k=0}^{r-1} \binom{n+k}{k} L_{2(r-k)-1} \qquad (2)$$

Equating the formulas in Theorem 1 and Theorem 4 we obtain

<u>Corollary 2</u> For all $r, n \in \mathbb{N}_0$, we have

$$L_{n+2r} = \sum_{k=0}^{n} \binom{n+r-1-k}{r-1} L_k + \sum_{k=0}^{r-1} \binom{n+k}{k} L_{2(r-k)-1}$$

$$= \sum_{k=0}^{n} \binom{r+k-1}{k} L_{n-k} + \sum_{k=0}^{r-1} \binom{n+k}{k} L_{2(r-k)-1}$$

The hyperlucas numbers of the rth generation satisfy also a second-order linear inhomogeneous recurrence relation

<u>Theorem 5</u> For all $r, n \in \mathbb{N}_0$, we have

$$L_0^{(r)} = 2, \quad L_1^{(r)} = 2r+1, \quad L_{n+2}^{(r)} = L_{n+1}^{(r)} + L_n^{(r)} + \frac{n+2r}{n+2}\binom{n+r}{r-1}, \quad n \geq 0$$

<u>Proof</u>: $L_n^{(r)} = \sum_{k=0}^{n} \binom{r+k-1}{k} L_{n-k}$ satisfies the above recurrence relation

2. Weighted sums of the type $\sum_{k=0}^{n} k^l \cdot a_k^{(r)}$, $l, n, r \in \mathbb{N}_0$

Def.: $t_l^{(r)}(n) := \sum_{k=0}^{n} k^l \cdot a_k^{(r)}$, $l, n, r \in \mathbb{N}_0$

Obviously, for $r=0$: $t_l^{(0)}(n) = t_l(n) = \sum_{k=0}^{n} k^l \cdot a_k$

and for $l=0$: $\boxed{t_0(n) = \sum_{k=0}^{n} a_k = a_n^{(1)}}$ (*)

By Theorem 1 for $r=2$:

$$a_n^{(2)} = \sum_{k=0}^{n} \binom{n+1-k}{1} a_k = (n+1) \sum_{k=0}^{n} a_k - \sum_{k=0}^{n} k a_k$$

or

$$\boxed{t_1(n) = \sum_{k=0}^{n} k a_k = (n+1) a_n^{(1)} - a_n^{(2)}} \quad (**)$$

Similarly, for $r=3$:

$$a_n^{(3)} = \sum_{k=0}^{n} \binom{n+2-k}{2} a_k = \frac{(n+2)(n+1)}{2} \sum_{k=0}^{n} a_k - \frac{2n+3}{2} \sum_{k=0}^{n} k a_k + \frac{1}{2} \sum_{k=0}^{n} k^2 a_k$$

or

$$\boxed{t_2(n) = \sum_{k=0}^{n} k^2 a_k = (n+1)^2 a_n^{(1)} - (2n+3) a_n^{(2)} + 2 a_n^{(3)}} \quad (***)$$

Similarly, for $r=4$ we obtain

$$\boxed{t_3(n) = \sum_{k=0}^{n} k^3 a_k = (n+1)^3 a_n^{(1)} - (3n^2+9n+7)\,a_n^{(2)} + 6(n+2)\,a_n^{(3)} - 6\,a_n^{(4)},} \tag{****}$$

and so on.

In matrix form:

$$\begin{pmatrix} t_0(n) \\ t_1(n) \\ t_2(n) \\ t_3(n) \\ \vdots \end{pmatrix} = C(n) \cdot \begin{pmatrix} a_n^{(1)} \\ a_n^{(2)} \\ a_n^{(3)} \\ a_n^{(4)} \\ \vdots \end{pmatrix},$$

where $C(n)$ is the infinite lower triangular matrix

$$C(n) := \begin{pmatrix} 1 & 0 & 0 & 0 & \cdots \\ n+1 & -1 & 0 & 0 & \cdots \\ (n+1)^2 & -(2n+3) & 2 & 0 & \cdots \\ (n+1)^3 & -(3n^2+9n+7) & 6(n+2) & -6 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

Equations (*), (**), (***), and (****) are special cases of <u>Abel's summation by parts</u>

<u>Theorem 5</u> If $(a_n)_{n\in\mathbb{N}_0}$ and $(b_n)_{n\in\mathbb{N}_0}$ are two arbitrary sequences (of real or complex numbers), then

$$\sum_{k=0}^{n} a_k b_k = a_n^{(1)} b_{n+1} + \sum_{k=0}^{n} a_n^{(1)} \left(b_k - b_{k+1}\right)$$

3. Main results

For $b_k := k^\ell$, $\ell \in \mathbb{N}_0$, $k \in \mathbb{N}_0$, we obtain the recurrence relation for $t_\ell^{(r)}(n)$ with respect to ℓ:

<u>Theorem 6</u> For all $n, r \in \mathbb{N}_0$, we have

$$t_\ell^{(r)}(n) = (n+1)^\ell a_n^{(r+1)} - \sum_{j=0}^{\ell-1} \binom{\ell}{j} t_j^{(r+1)}(n)$$

<u>Proof</u> Binomial theorem and Abel's summation by parts.

<u>Def</u>: $$c_{\ell,m}(n) := \sum_{k=0}^{m} \binom{m}{k} (-1)^k \cdot (k+n+1)^{\ell}$$

or, more generally,

$$p_{\ell,m}(x,y) := \sum_{k=0}^{m} \binom{m}{k} (-1)^k (kx+y)^{\ell}, \quad x,y \in \mathbb{C}$$

Note that for $x=1$ and $y=n+1$ we have

$$p_{\ell,m}(1,n+1) = c_{\ell,m}(n).$$

The next theorem shows that $c_{\ell,m}(n)$ are the entries of the matrix $C(n)$.

<u>Theorem 7</u> Let $\ell, n, r \in \mathbb{N}_0$. Then the solution of the recurrence relation for $t_\ell^{(r)}(n)$ is given by

$$\begin{aligned} t_\ell^{(r)}(n) &= \sum_{k=0}^{n} k^\ell a_k^{(r)} \\ &= \sum_{m=0}^{\ell} c_{\ell,m}(n) \cdot a_n^{(r+m+1)} \\ &= \sum_{j=0}^{\ell} \binom{\ell}{j} \left(\sum_{m=0}^{j} (-1)^m \cdot m! \left\{ {j \atop m} \right\} a_n^{(r+m+1)} \right) (n+1)^{\ell-j}, \end{aligned}$$

where $\left\{ {j \atop m} \right\}$ are the Stirling numbers of the second

For $r=0$ we have

Corollary 3

$$t_\ell(n)=\sum_{k=0}^{n} k^\ell a_k=\sum_{m=0}^{\ell} c_{\ell,m}(n)\,a_n^{(m+1)}$$

$$=\sum_{j=0}^{\ell}\binom{\ell}{j}\left(\sum_{m=0}^{j}(-1)^m\cdot m!\left\{ {j \atop m} \right\} a_n^{(m+1)}\right)(n+1)^{\ell-j}$$

Some properties of $P_{\ell,m}(x,y)$ and $c_{\ell,m}(n)$, respectively

Proposition 1 Let $x,y\in\mathbb{C}$, $\ell,m\in\mathbb{N}_0$. Then

1) (Boyadzhiev, 2012)

$$P_{\ell,m}(x,y)=(-1)^m\cdot m!\sum_{j=0}^{\ell}\binom{\ell}{j}\left\{ {j \atop m} \right\} x^j y^{\ell-j}$$

2) (Katsuura, 2009)

$$P_{\ell,m}(x,y)=\begin{cases} 0, & \text{for } 0\le \ell\le m-1\\ (-1)^m\cdot m!\,x^m, & \text{for } \ell=m\end{cases}$$

3) (Carlitz, 1980)

$$c_{\ell,m}(n) = \sum_{k=0}^{m} (-1)^k \cdot \binom{m}{k} (k+n+1)^{\ell} = (-1)^m \cdot m! \left\{ {\ell+n+1 \atop m+n+1} \right\}_{n+1},$$

where $\left\{ {m \atop n} \right\}_r$ are the "<u>r-Stirling numbers of the second kind</u>" first introduced in 1984 by A. Z. Broder as the number of set partitions of $\{1, 2, \dots, m\}$ into n nonempty, unordered parts ~~such~~ that $1, 2, \dots, r$ are in different parts.

4) (Some entries)

$$P_{\ell,0}(x,y) = y^{\ell}$$

$$P_{\ell,1}(x,y) = y^{\ell} - (x+y)^{\ell}$$

$$\dots\dots\dots\dots$$

$$P_{\ell,\ell-1}(x,y) = (-1)^{\ell-1} \cdot \ell!\, x^{\ell-1} \left(y + \frac{\ell-1}{2} x \right)$$

$$P_{\ell,\ell}(x,y) = (-1)^{\ell} \cdot \ell!\, x^{\ell}$$

$$P_{0,m}(x,y) = \sum_{k=0}^{m} (-1)^k \binom{m}{k} = 0^m$$

<u>Row sums</u>

$$\sum_{m=0}^{\ell} P_{\ell,m}(x,y) = (y-x)^{\ell}$$

(For $x=1$, $y=n+1$:

$$\sum^{\ell} c_{\ell,m}(n) = n^{\ell}$$

The first few entries of $P(x,y) := \left(p_{\ell,m}(x,y)\right)_{\ell,m=0,\dots,n}$ are

$$P(x,y) = \begin{pmatrix} 1 & 0 & 0 & 0 & \cdots \\ y & -x & 0 & 0 & \cdots \\ y^2 & -x(x+2y) & 2x^2 & 0 & \cdots \\ y^3 & -x(x^2+3xy+3y^2) & 2x^2(3x+3y) & -6x^3 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

5) • $p_{\ell-1,m}(x,y) = \frac{1}{\ell} \cdot \frac{\partial}{\partial y} p_{\ell,m}(x,y)$

(in particular, for $x=1$ and $y=n+1$:

$$\boxed{c_{\ell-1,m}(n) = \frac{1}{\ell} \cdot \frac{d}{dn} c_{\ell,m}(n)} \qquad \text{(A)}$$

• $p_{\ell,m}(x,y) = (mx+y)\, p_{\ell-1,m}(x,y) - mx\, p_{\ell-1,m-1}(x,y)$

(in particular, for $x=1$ and $y=n+1$:

$$\text{(B)} \quad \boxed{c_{\ell,m}(n) = (m+n+1)\, c_{\ell-1,m}(n) - m \cdot c_{\ell-1,m-1}(n)}$$

Together with $c_{0,0}(n)=1$ this is a recurrence relation for $c_{\ell,m}(n)$. $(c_{k,0}(n)=0$ for $k \neq 0)$

3.1 Applications to the Fibonacci and Lucas sequences ("Ledin and Brousseau's summation problems")

a) Let $a_n := F_n$ (Fibonacci). Then by Theorem 2

$$F_n^{(r)} = F_{n+2r} - \sum_{k=0}^{r-1} \binom{n+k-1}{k} F_{2(r-k)} \qquad (+)$$

Therefore, by Theorem 7

$$\sum_{k=0}^{n} k^\ell F_k^{(r)} = \sum_{m=0}^{\ell} c_{\ell,m}(n)\, F_n^{(r+m+1)}$$

For $r=0$:

$$\sum_{k=0}^{n} k^\ell F_k = \sum_{m=0}^{\ell} c_{\ell,m}(n)\, F_n^{(m+1)}$$

$$\overset{(+)}{=} \sum_{m=0}^{\ell} c_{\ell,m}(n)\, F_{n+2(m+1)} - \sum_{m=0}^{\ell} c_{\ell,m}(n) \sum_{k=0}^{m} \binom{n+k-1}{k} F_{2(m+1-k)}$$

Proposition 2

$$\sum_{m=0}^{\ell} c_{\ell,m}(n) \sum_{k=0}^{m} \binom{n+k-1}{k} F_{2(m+1-k)} = \sum_{m=0}^{\ell} c_{\ell,m}(0)\, F_{2(m+1)}$$

and $c_{\ell,m}(0) = (-1)^m \cdot m! \left\{ {\ell+1 \atop m+1} \right\}$

<u>Theorem 8</u> For all $l, n \in \mathbb{N}_0$, we have

$$\sum_{k=0}^{n} k^l \cdot F_k = \sum_{m=0}^{l} c_{l,m}(n)\, F_{n+2(m+1)} - \sum_{m=0}^{l} \underbrace{c_{l,m}(0)}_{=(-1)^m \cdot m! \cdot \left\{ {l+1 \atop m+1} \right\}} F_{2(m+1)}$$

The constant term on the right-hand side above can be expressed as follows:

<u>Conjecture 1</u> For all $l \in \mathbb{N}_0$, we have

$$\sum_{m=0}^{l} (-1)^m \cdot m! \left\{ {l+1 \atop m+1} \right\} F_{2(m+1)} = (-1)^l \sum_{m=0}^{l} m! \left\{ {l \atop m} \right\} F_{m+2}$$

b) Let $a_n := L_n$ (Lucas). Then by Theorem 4

$$L_n^{(r)} = L_{n+2r} - \sum_{k=0}^{r-1} \binom{n+k}{k} L_{2(r-k)-1} \qquad (++)$$

Therefore, by Theorem 7

$$\sum_{k=0}^{n} k^l L_k^{(r)} = \sum_{m=0}^{l} c_{l,m}(n) \cdot L_n^{(r+m+1)}$$

For $r=0$:

$$\sum_{k=0}^{n} k^{\ell} L_k = \sum_{m=0}^{\ell} c_{\ell,m}(n)\, L_n^{(m+1)}$$

$$\underset{(++)}{=} \sum_{m=0}^{\ell} c_{\ell,m}(n)\, L_{n+2(m+1)} - \underbrace{\sum_{m=0}^{\ell} c_{\ell,m}(n) \sum_{k=0}^{m} \binom{n+k}{k} L_{2(m+1-k)-}}_{=: T}$$

<u>Proposition 3</u>

$$T = \sum_{m=0}^{\ell} c_{\ell,m}(0) \underbrace{\sum_{k=0}^{m} L_{2(m+1-k)-1}}_{= \sum_{k=0}^{m} L_{2k+1} = L_{2(m+1)} - L_0}$$

$$= \sum_{m=0}^{\ell} c_{\ell,m}(0)\, L_{2(m+1)} - L_0 \underbrace{\sum_{m=0}^{\ell} c_{\ell,m}(0)}_{=0^{\ell}}$$

<u>Theorem 9</u> For all $l, n \in \mathbb{N}_0$, we have

$$\sum_{k=0}^{n} k^l L_k = \sum_{m=0}^{l} c_{l,m}(n) L_{n+2(m+1)} - \sum_{m=0}^{l} \underbrace{c_{l,m}(0)}_{=(-1)^m \cdot m! \left\{ {l+1 \atop m+1} \right\}} L_{2(m+1)} + L_0 \cdot 0^l$$

The constant term on the right-hand side above can be expressed as follows:

<u>Conjecture 2</u> For all $l, n \in \mathbb{N}_0$, we have

$$\sum_{m=0}^{l} (-1)^m \cdot m! \left\{ {l+1 \atop m+1} \right\} L_{2(m+1)} = (-1)^l \sum_{m=0}^{l} m! \left\{ {l \atop m} \right\} L_{m+2}$$

4. Conclusions

Equation (A) at p. 13 shows how to calculate the coefficients of $t_{l-1}^{(r)}(n)$ knowing those of $t_l^{(r)}(n)$, whereas Equation (B) at p. 13 solves the inverse problem, that is, it shows how to calculate the coefficients of $t_l^{(r)}(n)$ knowing those of $t_{l-1}^{(r)}(n)$. This was still an open ~~a~~ problem (see T. Koshy, Fibonacci and Lucas Numbers with Applications, 2nd ed., 2018, p. 519)

Example: Knowing $t_3(n)$ calculate $t_4(n)$.

$$t_3(n)=\sum_{k=0}^{n} k^3 F_k = (n+1)^3 F_{n+2} - (3n^2+9n+7)F_{n+4} + (6n+12)F_{n+6} - 6F_{n+8} + 50$$

By Equation (B) we have

$$c_{4,0}(n) = (n+1)\, c_{3,0}(n) = (n+1)^4$$
$$c_{4,1}(n) = (n+2)\, c_{3,1}(n) - c_{3,0}(n) = -(4n^3+18n^2+28n+15)$$
$$c_{4,2}(n) = (n+3)\, c_{3,2}(n) - 2c_{3,1}(n) = 12n^2+48n+50$$
$$c_{4,3}(n) = (n+4)\, c_{3,3}(n) - 3c_{3,2}(n) = -(24n+60)$$
$$c_{4,4}(n) = (n+5)\, c_{3,4}(n) - 4c_{3,3}(n) = 24$$

$$\Rightarrow t_4(n)=\sum_{k=0}^{n} k^4 F_k = (n+1)^4 F_{n+2} - (4n^3+18n^2+28n+15)F_{n+4} + (12n^2+48n+50)F_{n+6} - (24n+60)F_{n+8} + 24F_{n+10} - \underbrace{(F_2 - 15F_4 + 50F_6 - 60F_8 + 24F_{10})}_{=416}$$

Conversely, by Equation (A) we obtain from $t_4(n)$

$$t_3(n) = (n+1)^3 F_{n+2} - (3n^2+9n+7)F_{n+4} + (6n+12)F_{n+6} - 6F_{n+8} - (F_2 - 7F_4 + 12F_6 - 6F_8).$$