UNIVERSITY of SOPRON

FACULTY OF WOOD ENGINEERING
AND CREATIVE INDUSTRIES

INSTITUTE OF BASIC SCIENCES
DEPARTMENT OF MATHEMATICS

Walks on tiled boards

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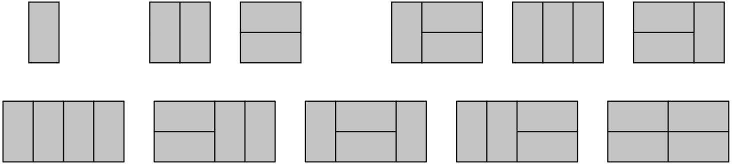
Introduction – tilings, examples

We consider the $(2 \times n)$-board.

- Numbers of tilings with dominoes are the shifted Fibonacci numbers.
  For example in case of $n = 1,\ 2,\ 3,\ 4$:



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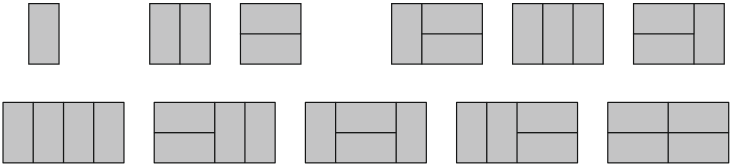
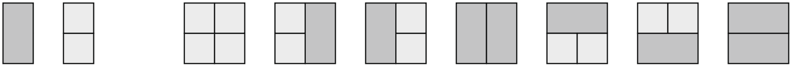
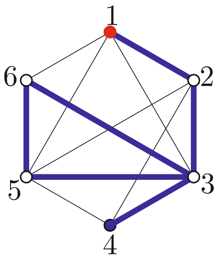


- Numbers of tilings with squares and dominoes are given by $r_n = 3r_{n-1} + r_{n-2} - r_{n-3} \quad (n \geq 3)$, $r_0 = 1$, $r_1 = 2$, and $r_2 = 7$. For example in case of $n = 1,\ 2$:

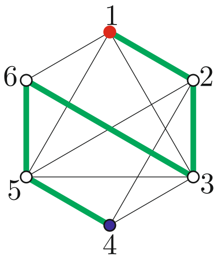


Introduction

A **self-avoiding walk** on a graph is a walk that never visits the same vertex more than once.



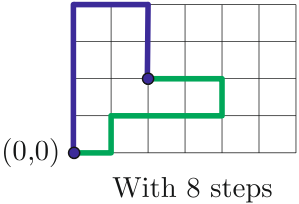
1-2-3-6-5-3-4: walk



1-2-3-6-5-4: self-avoiding walk

We restrict our study of self-avoiding walks to rectangular grid graphs.

Introduction – Manhattan



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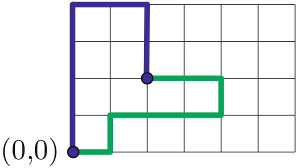
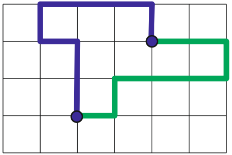


With 8 steps



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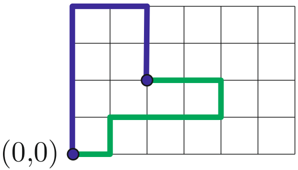

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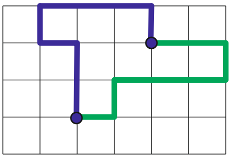

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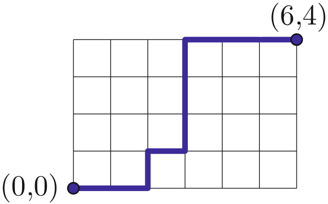
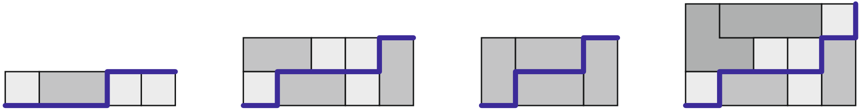


With minimum steps

Examples for tilings and walks

- Examples for walks on tiled boards



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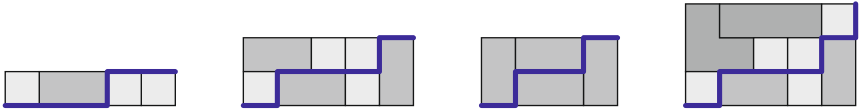
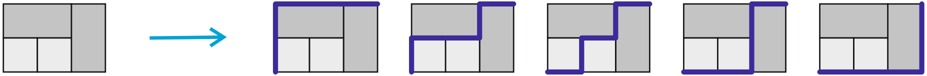


- Walks on a 2×3-board with a given tiling



Tilings and walks on $(1 \times n)$-board

- All the walks on the tiled boards 1×0, 1×1, and 1×2

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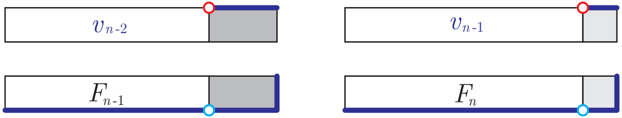
- Tilings and walks on $(1 \times n)$-board with recurrence



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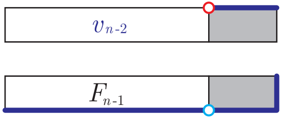
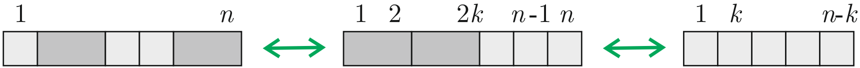


$v_n = v_{n-2} + F_{n-1} + v_{n-1} + F_n = v_{n-1} + v_{n-2} + F_{n+1}$

- Tilings with exactly k dominoes on $(1 \times n)$-board



Theorem

The tiling-walking sequence $(v_n)_{n=0}^{\infty}$ *of the* $(1 \times n)$*-board with squares and dominoes has the recurrence relation*

$$nv_n = (n+1)v_{n-1} + (n+2)v_{n-2}, \quad n \geq 2,$$

where the initial values are $v_0 = 1$, $v_1 = 2$ *(A001629 in OEIS).*

Corollary

The sequence $(v_n)_{n=0}^{\infty}$ *is recursively given by the 4th order linear homogeneous recurrence relation with constant coefficients*

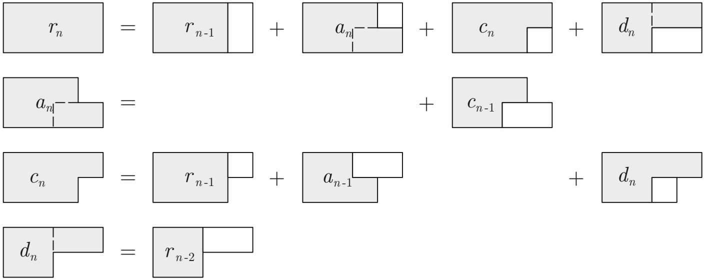
$$v_n = 2v_{n-1} + v_{n-2} - 2v_{n-3} - v_{n-4}, \quad n \geq 4,$$

where the initial values are $v_0 = 1$, $v_1 = 2$, $v_2 = 5$, *and* $v_3 = 10$. *Moreover, for* $n \geq 0$

$$5v_n = 2(n+2)F_{n+1} + (n+1)F_{n+2},$$

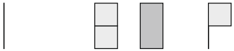
where F_n *is the nth Fibonacci number.*

Tiling with squares and dominoes on $(2 \times n)$-board



$$\begin{aligned} r_n &= r_{n-1} + a_n + c_n + d_n, \\ a_n &= c_{n-1}, \\ c_n &= r_{n-1} + a_{n-1} + d_n, \\ d_n &= r_{n-2}, \end{aligned}$$

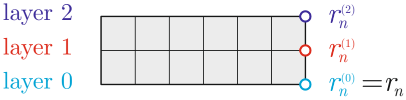
where the initial values are $a_0 = c_0 = d_0 = a_1 = d_1 = 0$, $r_0 = 1$, $r_1 = 2$, and $c_1 = 1$.

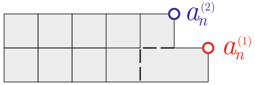


The solution is known: $r_n = 3r_{n-1} + r_{n-2} - r_{n-3} \quad (n \geq 3)$.

An example and initial walks

The walks ending on layer 0, 1, and 2 are denoted by $x^{(0)}$, $x^{(1)}$, and $x^{(2)}$, respectively.





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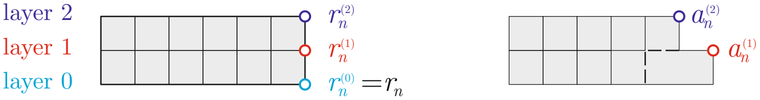
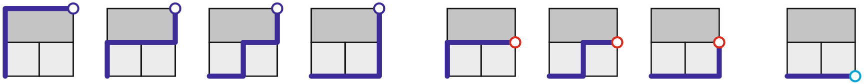


Example: walks on (2×2)-board in case of a given tiling



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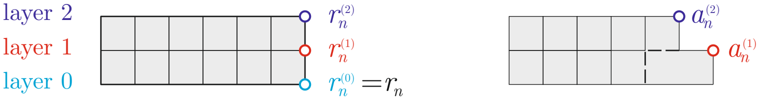

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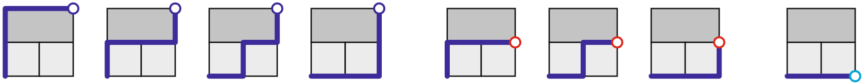
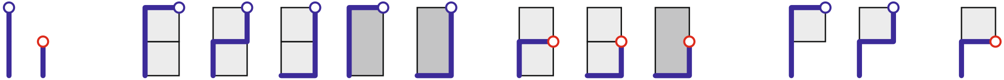


Initial walks (when $n = 0$ and $n = 1$) for the next figure



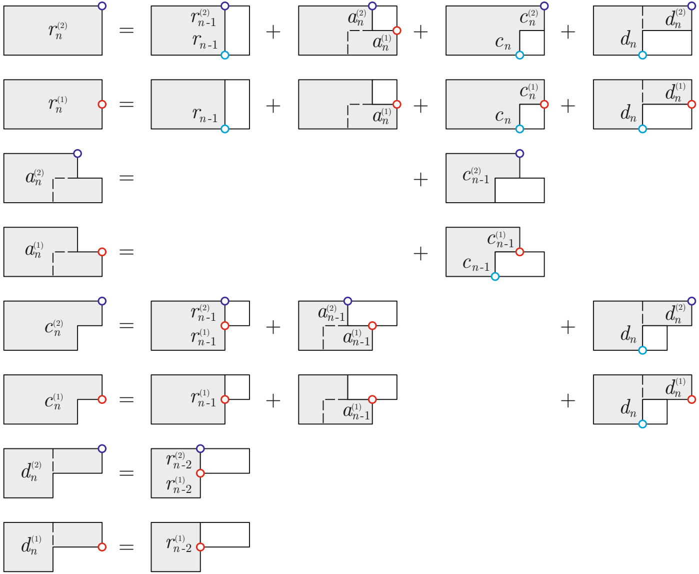


Figure: Recursions of numbers of walks on tiled boards

System of 12 recurrence equations

$$
\begin{aligned}
r_n^{(2)} &= (r_{n-1}^{(2)} + r_{n-1}) + (a_n^{(2)} + a_n^{(1)}) + (c_n^{(2)} + c_n) + (d_n^{(2)} + d_n), \\
r_n^{(1)} &= r_{n-1} + a_n^{(1)} + (c_n^{(1)} + c_n) + (d_n^{(1)} + d_n), \\
a_n^{(2)} &= c_{n-1}^{(2)}, \\
a_n^{(1)} &= c_{n-1}^{(1)} + c_{n-1}, \\
c_n^{(2)} &= (r_{n-1}^{(2)} + r_{n-1}^{(1)}) + (a_{n-1}^{(2)} + a_{n-1}^{(1)}) + (d_n^{(2)} + d_n), \\
c_n^{(1)} &= r_{n-1}^{(1)} + a_{n-1}^{(1)} + (d_n^{(1)} + d_n), \\
d_n^{(2)} &= r_{n-2}^{(2)} + r_{n-2}^{(1)}, \\
d_n^{(1)} &= r_{n-2}^{(1)} \\
r_n &= r_{n-1} + a_n + c_n + d_n, \\
a_n &= c_{n-1}, \\
c_n &= r_{n-1} + a_{n-1} + d_n, \\
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\end{aligned}
$$

Solving the system of recurrence equations

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- $$2r^{(2)}_{n+1} - r^{(2)}_{n} - 6r^{(2)}_{n-1} + r^{(2)}_{n-2} + 6r^{(2)}_{n-3} + 2r^{(2)}_{n-4} = c^{(2)}_{n+2} - c^{(2)}_{n+1} + 5c^{(2)}_{n-1} - 4c^{(2)}_{n-3} - c^{(2)}_{n-4},$$

  $$2r^{(2)}_{n} - 6r^{(2)}_{n-1} - 7r^{(2)}_{n-2} + 14r^{(2)}_{n-3} + 14r^{(2)}_{n-4} - 2r^{(2)}_{n-5} - 3r^{(2)}_{n-6} = c^{(2)}_{n+1} - 3c^{(2)}_{n} - 2c^{(2)}_{n-1} + 6c^{(2)}_{n-2} - 3c^{(2)}_{n-3} - 9c^{(2)}_{n-4} + 2c^{(2)}_{n-6}.$$

Solving the system of recurrence equations

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- $$2r^{(2)}_{n+1} - r^{(2)}_{n} - 6r^{(2)}_{n-1} + r^{(2)}_{n-2} + 6r^{(2)}_{n-3} + 2r^{(2)}_{n-4} = c^{(2)}_{n+2} - c^{(2)}_{n+1} + 5c^{(2)}_{n-1} - 4c^{(2)}_{n-3} - c^{(2)}_{n-4},$$

  $$2r^{(2)}_{n} - 6r^{(2)}_{n-1} - 7r^{(2)}_{n-2} + 14r^{(2)}_{n-3} + 14r^{(2)}_{n-4} - 2r^{(2)}_{n-5} - 3r^{(2)}_{n-6} = c^{(2)}_{n+1} - 3c^{(2)}_{n} - 2c^{(2)}_{n-1} + 6c^{(2)}_{n-2} - 3c^{(2)}_{n-3} - 9c^{(2)}_{n-4} + 2c^{(2)}_{n-6}.$$
- Finally, we managed to solve, and the result is when $w_n = r^{(2)}_n$:

Result of walks on tiled $(2 \times n)$-board

Theorem

The tiling-walking sequence $(w_n)_{n=0}^{\infty}$ *of the* $(2 \times n)$*-board tiled with squares and dominoes is recursively given by the 9th order homogeneous linear recurrence relation* $(n \geq 9)$

$$w_n = 8w_{n-1} - 17w_{n-2} - 7w_{n-3} + 41w_{n-4} + w_{n-5} - 23w_{n-6} + 3w_{n-7} + 4w_{n-8} - w_{n-9}$$

with initial values $1, 5, 28, 130, 569, 2352, 9363, 36183, 136663$ *(*$n = 0, \ldots, 8$*).*
A composed form of it is

$$r_n = 3r_{n-1} + r_{n-2} - r_{n-3},$$

where

$$\begin{aligned} r_n &= x_n - 3x_{n-1} - x_{n-2} + x_{n-3}, \\ x_n &= y_n - 3y_{n-1} + y_{n-2}, \\ y_n &= w_n - w_{n-1}. \end{aligned}$$

Tiling and walking with only dominoes

Theorem

The tiling-walking sequence $(w_n)_{n=0}^{\infty}$ *of the* $(2 \times n)$*-board with only dominoes is recursively given by the 6-th order homogeneous linear recurrence relation* $(n \geq 6)$

$$w_n = 2w_{n-1} + 2w_{n-2} - 4w_{n-3} - 2w_{n-4} + 2w_{n-5} + w_{n-6}$$

with initial values $1, 2, 6, 12, 26, 50$ *for* $n = 0, \ldots, 5$. *(A054454 in OEIS.) Moreover, for* $n \geq 0$, *we obtain the equation*

$$w_n = \frac{1 + (-1)^n}{2} + \frac{3}{5}(1 + n)F_n + \frac{4n}{5}F_{n+1},$$

and for even and odd terms, the equations

$$\begin{aligned} 5\,w_{2n} &= 5 + (3 + 6n)\,F_{2n} + 8n\,F_{2n+1}, \\ 5\,w_{2n+1} &= (6 + 6n)\,F_{2n+1} + (4 + 8n)\,F_{2n+2}, \end{aligned}$$

where F_n *is the nth Fibonacci number.*

Walks on tiled $(2 \times n)$-board – future works

Németh, L.: Walks on tiled boards, *Mathematica Slovaca*, x (2024?), p.15, accepted.

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There are several questions in this topic. They give many works for me, perhaps for you. (I hope.)