A NEW THEOREM FROM THE NUMBER THEORY AND ITS APPLICATION FOR A 3-ADIC EVALUATION FOR LARGE SCHRÖDER NUMBERS

A FORMULA FOR A BINOMIAL THEOREM IS WELL-KNOWN:

$$\sum_{k=0}^{n} \binom{n}{k} a^{n-k} \cdot b^{k} = (a+b)^{n}, \qquad (1)$$

WHERE a, b ARE INTEGERS

THEREFORE,

$$\omega_{a+b}\left(\sum_{k=0}^{n} \binom{n}{k} a^{n-k} \cdot b^{k}\right) = n, \text{ WHERE}$$

$\omega_{x}(y) = l$ IF x^{l} DIVIDES y, AND x^{l+1} DO NOT DIVIDE y, $x \neq \pm 1$

LET US CONSIDER

THE SUM:

$$B(n, a, b) = \sum_{k=0}^{n} \binom{n}{k}^2 a^{n-k} \cdot b^k$$

, WHERE a AND b REL. PRIME ARE INTEGERS

A QUESTION ARISES:

„HOW MANY TIMES IS a+b CONTAINED IN THE SUM $B(n, a, b)$?

A NEW THEOREM FROM THE NUMBER THEORY:

$$\omega_{a+b}\left(\sum_{k=0}^{2n}\binom{2n}{k}^2 a^{2n-k} b^{k}\right) = \omega_{a+b}\left(\binom{2n}{n}\right) \quad (EQ:2)$$

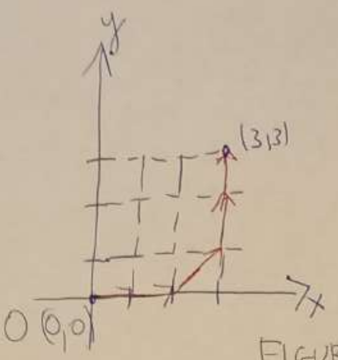
AND

$$\omega_{a+b}\left(\sum_{k=0}^{2n+1}\binom{2n+1}{k}^2 a^{2n+1-k} b^{k}\right) = 1 + \omega_{a+b}\left((2n+1)\binom{2n}{n}\right) \quad (EQ:3)$$

THE CENTRAL DELANNOY NUMBERS

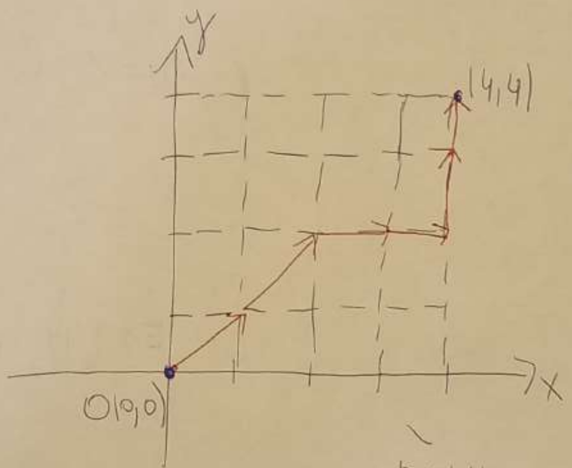
THE CENTRAL DELANNOY NUMBERS $D(n)$ REPRESENT THE NUMBER OF LATTICE PATHS (WAYS) STARTING FROM THE POINT $(0,0)$ TO (n,n) BY USING ONLY THREE TYPES OF STEPS: HORIZONTAL $(1,0)$, VERTICAL $(0,1)$ AND DIAGONAL $(1,1)$.

THE CENTRAL DELANNOY NUMBERS COUNT ALL POSSIBLE WAYS OF ROYAL WALKS



— ROYAL WALK #1

FIGURE 1



ROYAL WALK #2

THE FORMULA FOR CENTRAL DELANNOY NUMBERS:

$$D(n) = \sum_{k=0}^{n} \binom{n}{k}^2 2^k$$

BY NEW THEOREM (EQNS (2) AND (3)) IT FOLLOWS THAT: ($a := 1$, $b := 2$)

$$\omega_3 (D(2n)) = \omega_3 \left(\binom{2n}{n} \right) \text{ OR}$$

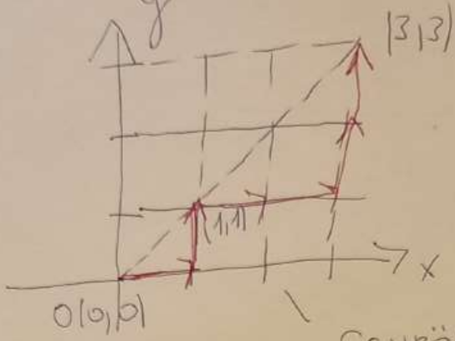
$$\boxed{V_3 (D(2n)) = V_3 \left(\binom{2n}{n} \right)} \quad (EQ:4) \text{ AND}$$

$$\boxed{V_3 (D(2n+1)) = 1 + V_3 (2n+1) + V_3 \left(\binom{2n}{n} \right) \quad (EQ:5)}$$

LARGE SCHRÖDER NUMBERS

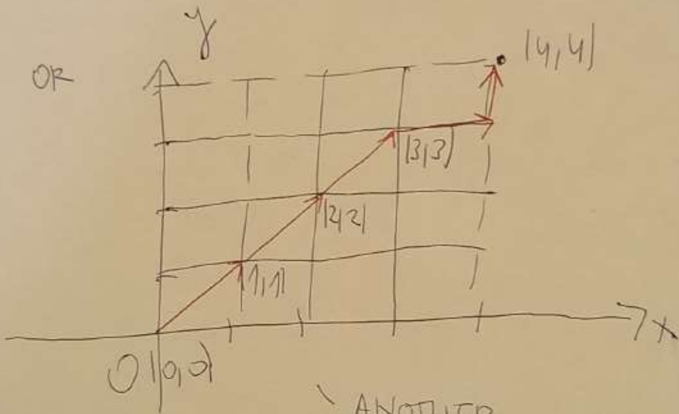
LARGE SCHRÖDER NUMBER S_n COUNTS ALL LATTICE WALKS STARTING FROM THE POINT $(0,0)$ TO THE POINT (n,n) BY USING HORIZONTAL $(1,0)$, VERTICAL $(0,1)$ AND DIAGONAL STEPS SUCH THAT NEVER RISE ABOVE THE MAIN DIAGONAL

$$y = x$$



SCHRÖDER PATH

OR



ANOTHER

SCHRÖDER PATH

A FORMULA FOR THE LARGE SCHRÖDER NUMBERS

IT IS ALSO KNOWN:

$$S_n = \sum_{k=0}^{n} \frac{1}{k+1} \binom{n+k}{k} \binom{n}{k} \quad \text{AND}$$

$$D_n = \sum_{k=0}^{n} \binom{n+k}{k} \binom{n}{k}$$

A 3-ADIC EVALUATION OF LARGE SCHRÖDER NUMBERS

$$\boxed{V_3(S_{2n+1}) = V_3(C_n)} \quad \text{(EQ: 6)} \quad \text{AND}$$

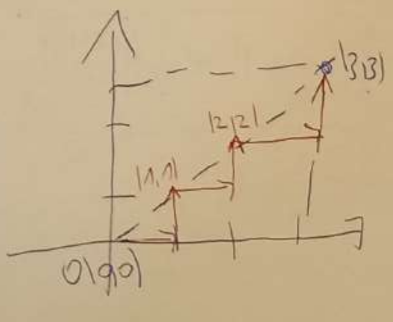
$$\boxed{V_3(S_{2n+2}) = 1 + V_3(2n+1) + V_3(C_n)} \quad \text{(EQ: 7)}$$

WHERE $n \in \mathbb{N}_0$ AND C_n IS THE n-TH CATALAN NUMBER:

$$C_n = \frac{1}{n+1}\binom{2n}{n}$$

THE CATALAN NUMBERS:

C_n IS THE NUMBER OF ALL LATTICE WALKS IN THE PLANE STARTING FROM THE POINT (0,0) TO THE POINT (n,n) THAT NEVER RISE ABOVE THE MAIN DIAGONAL



A PROOF OF
A FORMULA FOR
A 3-ADIC EVALUATION
FOR S_n, A SKETCH

$$\boxed{S_n = \frac{1}{2}\left(-D_{n-1} + 6\cdot D_n - D_{n+1}\right)} \quad \text{(EQ.8)}$$

$$\boxed{V_3\left(\left(D_{2u}\right)\right) = V_3\left(\binom{2u}{u}\right)} \quad \text{(EQ.4)} \quad \text{AND}$$

$$\boxed{V_3\left(D_{2n+1}\right) = 1 + V_3\left(2n+1\right) + V_3\left(\binom{2u}{u}\right)} \quad \text{(EQ.5)}$$

ÁN EXAMPLE #1

CENTRAL DELANNOY NUMBERS

D(n) [SEQUENCE A001850]

$D(0)=1$

$D(1)=3$

$D(2)=13$

$D(3)=63$

$D(4)=321$

$D(5)=1683$

$D(6)=8989$

$D(7)=48639$

$D(8)=265729$

AN EXAMPLE #1.

LET US CALCULATE $V_3(D_8)$!

BY THE (EQ: 4)

$$V_3(D_8) = V_3\left(\binom{8}{4}\right)$$

BY USING <u>KUMMER'S</u> THEOREM FOR A P-ADIC EVALUATION FOR BC, IT FOLLOWS: $V_3\left(\binom{8}{4}\right) = 0$

SINCE $4 = (11)_3$ AND $(11)_3 + (11)_3 = (22)_3$.
THERE ARE NO CARRY OVERS.

IT FOLLOWS THAT:

$V_3(D_8) = V_3(265729) = 0$, OR

$\boxed{V_3(D_8) = 0}$ WHICH MEANS 3^0 DIVIDES D_8 AND 3^1 DOES NOT DIVIDE D_8

AN EXAMPLE #2.

LET US CALCULATE $V_3(D_7)$.

BY THE (EQ:5), IT FOLLOWS:

$$\boxed{V_3(D_{2n+1}) = 1 + V_3(2n+1) + V_3\left(\binom{2n}{n}\right)} \quad (EQ:5)$$

$n := 3$

$$V_3(D_7) = 1 + V_3(7) + V_3\left(\binom{6}{3}\right)$$
$$= 1 + 0 + 0 = 1, \text{ SINCE}$$

$3 = (10)_3$, AND $(10)_3 + (10)_3 = (20)_3$
(THERE ARE NO CARRY OVERS)

IT FOLLOWS:

$\boxed{V_3(D_7) = 1}$, OR

$\boxed{V_3(48639) = 1}$ $\Leftarrow$ 3^1 DIVIDES 48639 AND 3^2 DOES NOT DIVIDE 48639

AN EXAMPLE #2+1

THE LARGE SCHRÖDER NUMBERS
S(n) [SEQUENCE A006318]

S(0) = 1
S(1) = 2
S(2) = 6
S(3) = 22
S(4) = 90
S(5) = 394
S(6) = 1806
S(7) = 8558

AN EXAMPLE # 3

LET US FIND THE $V_3(S(7))$.

BY THE EQ. (6), $V_3(S_{2n+1}) = V_3(C_n)$.

THEREFORE,

$$V_3(S_7) = V_3(C_3);$$

$$V_3(C_3) = V_3\left(\frac{1}{4} \cdot \binom{6}{3}\right) = V_3\left(\binom{6}{3}\right) - V_3(4) = 0 - 0 = 0$$

$\Rightarrow V_3(C_3) = 0$

IT FOLLOWS THAT:

$V_3(S_7) = 0$, OR

$V_3(8558) = 0$, SO 3^0 DIVIDE 8558

AND 3^1 DOES NOT DIVIDE 8558

AN EXAMPLE #4

LET US FIND $V_3(S_6)$.

BY THE (EQ: 7), IT FOLLOWS THAT:

$$\boxed{V_3(S_{2n+2}) = 1 + V_3(2n+1) + V_3(C_n)}$$

BY SETTING $n := \cancel{4}\ 2 \Rightarrow$

$$\begin{aligned} V_3(S_6) &= 1 + V_3(5) + V_3(C_2) \\ &= 1 + 0 + V_3\left(\tfrac{1}{3}\binom{4}{2}\right) \\ &= 1 + 0 + V_3\left(\binom{4}{2}\right) - V_3(3) \\ &= 1 + \cancel{1-1} = 1 \end{aligned}$$

IT FOLLOWS THAT: $V_3(S_6) = 1$, OR

$V_3(1806) = 1$.

SO 3^1 DIVIDES 1806, AND

3^2 DOES NOT DIVIDE 1806.

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