Generalized Continuant Polynomials

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$$
\begin{aligned}
\frac{a}{b} &= z_1 + \frac{r_1}{b} \\
&= z_1 + \frac{1}{z_2 + \frac{r_2}{r_1}} \\
&= z_1 + \frac{1}{z_2 + \frac{1}{z_3 + \frac{1}{z_4}}}
\end{aligned}
$$

...going backwords

$$
\frac{a}{b} = z_1 + \frac{1}{z_2 + \frac{z_4}{z_3 q_4 + 1}} = z_1 + \frac{z_3 z_4 + 1}{z_2 z_3 z_4 + z_2 + z_4}
$$

$$\frac{z_1z_2z_3z_4 + z_1z_2 + z_1z_4 + z_3z_4 + 1}{z_2z_3z_4 + z_2 + z_4},$$

$$z_1z_2z_3z_4z_5 + z_1z_2z_3 + z_1z_4z_5 + z_3z_4z_5 + z_1 + z_3 + z_5.$$

Definition

$$K_0() = 1, \quad K_1(x_1) = x_1$$

$$K_n(x_1, \ldots, x_n) = K_{n-1}(x_1, \ldots, x_{n-1})x_n + K_{n-2}(x_1, \ldots, x_{n-2}) \quad (1)$$

$$\begin{aligned} K_2(x_1, x_2) &= x_1 x_2 + 1 \\ K_3(x_1, x_2, x_3) &= x_1 x_2 x_3 + x_1 + x_3 \\ K_4(x_1, x_2 . x_3, x_4) &= x_1 x_2 x_3 x_4 + x_1 x_2 + x_1 x_4 + x_3 x_4 + 1 \end{aligned}$$

Definition

$$K_0() = 1, \;\; K_1(x_1) = x_1$$

$$K_n(x_1, \ldots, x_n) = K_{n-1}(x_1, \ldots, x_{n-1})x_n + K_{n-2}(x_1, \ldots, x_{n-2}) \quad (1)$$

$$\begin{aligned} K_2(x_1, x_2) &= x_1 x_2 + 1 \\ K_3(x_1, x_2, x_3) &= x_1 x_2 x_3 + x_1 + x_3 \\ K_4(x_1, x_2.x_3, x_4) &= x_1 x_2 x_3 x_4 + x_1 x_2 + x_1 x_4 + x_3 x_4 + 1 \end{aligned}$$

Identities...

$$K_n(x_1, \ldots, x_{k-1}, 0, x_{k+1}, \ldots, x_n)$$
$$= K_{n-2}(x_1, \ldots, x_{k-2}, x_{k-1} + x_{k+1}, x_{k+2}, \ldots, x_n)$$

$$K_n(x_1, \ldots, x_{n-1}, x_n + y)$$
$$= K_n(x_1, \ldots, x_{n-1}, x_n) + K_{n-1}(x_1, \ldots, x_{n-1})y$$

Euler:

$$K_{m+n}(x_1, \ldots, x_{m+n})K_k(x_{m+1}, \ldots, x_{m+k})$$
$$= K_{m+k}(x_1, \ldots, x_{m+k}) + K_n(x_{m+1}, \ldots, x_{m+n})y$$
$$+(-1)^k K_{m-1}(x_1, \ldots, x_{m-1})K_{n-k-1}(x_{m+k+2}, \ldots, x_{m+n})$$

Euler's interpretation

$$K_n(z, z, \ldots, z) = \sum_{k=0}^{n} \binom{n-k}{k} z^{n-2k}$$

$$K_n(1, 1, \ldots, 1) = F_{n+1}$$

$x_1 x_2 x_3$ $\quad$ x_1 $\quad$ x_3

$$K_3(x_1.x_2.x_3) = x_1 x_2 x_3 + x_1 + x_3$$

Proposition

For the integer $n \geq 0$ we have

$$x_1 \cdots x_{n+2} + \sum_{0 \leq j \leq n} K_j(x_1, \ldots, x_j) x_{j+3} \cdots x_{n+2} = K_{n+2}(x_1, \ldots, x_{n+2})$$

Proof. By induction. By a combinatorial argument.

$$\begin{aligned} K_5(x_1, x_2, x_3, x_4, x_5) &= x_1 x_2 x_3 x_4 x_5 + K_0() x_3 x_4 x_5 + K_1(x_1) x_3 x_4 \\ &\quad + K_2(x_1 x_2) x_5 + K_3(x_1, x_2, x_3) \end{aligned}$$

Consequences of Proposition 1

Corollary

For the Fibonacci sequences we have

$$1 + \sum_{k=0}^{n} F_k = F_{n+2}$$

Corollary

For the binomial coefficients we have ("upper summation")

$$\sum_{k \le j \le n} \binom{j}{k} = \binom{n+1}{k+1}.$$

Proofs: evaluation of Proposition 1.

Proposition

For the integer $n \geq 0$ we have

$$[n\ \mathit{even}] + \sum_{1 \leq j,\ j\ \mathit{odd}} K_{n-j}(x_1, \ldots, x_{n-j}) x_{n-j+1} = K_n(x_1, \ldots, x_n).$$

Proof:

$$\begin{aligned} K_n(x_1, \ldots, x_n) &= K_{n-1}(x_1, \ldots, x_{n-1}) x_n + K_{n-2}(x_1, \ldots, x_{n-2}) \\ &= K_{n-1}(x_1, \ldots, x_{n-1} x_n + [(n-2)\ \text{even}] \\ &\quad + \sum_{1 \leq j,\ j\ \text{odd}} K_{n-2-j}(x_1, \ldots, x_{n-2}) x_{n-j-1} \\ &= [n\ \text{even}] + \sum_{1 \leq j,\ j\ \text{odd}} K_{n-j}(x_1, \ldots, x_{n-j}) x_{n-j+1} \end{aligned}$$

Consequences of Proposition 2

Corollary

For the Fibonacci sequences we have

$$\sum_{k=0}^{n-1} F_{2k+1} = F_{2n}$$

$$1 + \sum_{k=1}^{n} F_{2k} = F_{2n+1}$$

Corollary

For the binomial coefficients we have ("parallel summation")

$$\sum_{0\le k\le n} \binom{m+k}{k} = \binom{m+n+1}{n}.$$

Definition

$$
\begin{aligned}
& K_n(x_1,\ldots,x_n,y_1,\ldots,y_{n-1},z_1,\ldots,z_{n-1}) \\
= \quad & x_nK_{n-1}(x_1,\ldots,x_{n-1},y_1,\ldots,y_{n-2},z_1,\ldots,z_{n-2}) \quad (2) \\
& -y_{n-1}z_{n-1}K_{n-2}(x_1,\ldots,x_{n-2},y_1,\ldots,y_{n-3},z_1,\ldots,z_{n-3})
\end{aligned}
$$

$$
\begin{aligned}
K_0() &= 1 \\
K_1(x_1) &= x_1 \\
K_2(x_1,x_2,y_1,z_1) &= x_1x_2 - y_1z_1 \\
K_3(x_1,x_2,x_3,y_1,y_2,z_1,z_2) &= x_1x_2x_3 - x_1y_2z_2 - x_3y_1z_1 \\
K_4(\cdots) &= x_1x_2x_3x_4 - x_1x_2y_3z_3 - x_1x_4y_2z_2 \\
&\quad -x_3x_4y_1z_1 + y_1y_3z_1z_3
\end{aligned}
$$

Definition

$$\begin{aligned} & K_n(x_1,\ldots,x_n,y_1,\ldots,y_{n-1},z_1,\ldots,z_{n-1}) \\ = \quad & x_nK_{n-1}(x_1,\ldots,x_{n-1},y_1,\ldots,y_{n-2},z_1,\ldots,z_{n-2}) \\ & -y_{n-1}z_{n-1}K_{n-2}(x_1,\ldots,x_{n-2},y_1,\ldots,y_{n-3},z_1,\ldots,z_{n-3}) \end{aligned} \quad (2)$$

$$\begin{aligned} K_0() &= 1 \\ K_1(x_1) &= x_1 \\ K_2(x_1,x_2,y_1,z_1) &= x_1x_2 - y_1z_1 \\ K_3(x_1,x_2,x_3,y_1,y_2,z_1,z_2) &= x_1x_2x_3 - x_1y_2z_2 - x_3y_1z_1 \\ K_4(\cdots) &= x_1x_2x_3x_4 - x_1x_2y_3z_3 - x_1x_4y_2z_2 \\ & \quad -x_3x_4y_1z_1 + y_1y_3z_1z_3 \end{aligned}$$

Theorem 1

Theorem

For the integer $n \geq 0$ we have

$$K_{n+2}(x_1, \dots, x_{n+2}, y_1, \dots, y_{n+1}, z_1, \dots, z_{n+1}) = x_1 x_2 \cdots x_{n+2}$$
$$- \sum_{0 \leq j \leq n} y_{j+1} z_{j+1} K_j(x_1, \dots, x_j, y_1, \dots, y_{j-1}, z_1, \dots, z_{j-1}) x_{j+3} \cdots x_{n+2}$$

The close form of this summation we shall prove by induction. Clearly, the equality holds true when n is equal to zero. In that case the left hand side of the equality gives $x_1 x_2 - y_1 z_1$, and this is exactly $K_2(x_1, x_2; y_1; z_1)$. The induction step is provided through the recurrence relation.

Proof of the Theorem 1

$$\begin{aligned}
K_{n+2}(\cdots) &= x_{n+2}K_{n+1}(\cdots) - y_{n-1}z_{n-1}K_n(\cdots) \\
&= \left(x_1x_2\cdots x_{n+1} + \sum_{0\le j\le n-1} y_{j+1}z_{j+1}K_j(\cdots)x_{j+3}\cdots x_{n+1} \right) x \\
&\quad - y_{n-1}z_{n-1}K_n(\cdots) \\
&= x_1x_2\cdots x_{n+2} - \sum_{0\le j\le n-1} y_{j+1}z_{j+1}K_j(\cdots)x_{j+3}\cdots x_{n+2} \\
&\quad - y_{n-1}z_{n-1}K_n(\cdots) \\
&= x_1x_2\cdots x_{n+2} - \sum_{0\le j\le n} y_{j+1}z_{j+1}K_j(\cdots)x_{j+3}\cdots x_{n+2}.
\end{aligned}$$

Consequences of Theorem 1

Corollary

For the integer $n \geq 0$ we have

$$\sum_{k=0}^{n} J_k = \frac{1}{2}(J_{n+2} - 1).$$

Corollary

For nonnegative integers n and k we have

$$\sum_{k \leq j \leq n} \binom{j}{k} = \binom{n+1}{k+1}.$$

Consequences of Theorem 1 ...

Corollary

For the integer $n \geq 0$ we have

$$\det\begin{pmatrix} 1 & 1 & & & & \\ 1 & 2 & 1 & & & \\ & 1 & 3 & 1 & & \\ & & \ddots & \ddots & \ddots & \\ & & & & & 1 \\ & & & & 1 & n \end{pmatrix} = n! - \sum_{j=0}^{n-2} n^{\underline{n-j-2}} \cdot \det\begin{pmatrix} 1 & 1 & & & \\ 1 & 2 & 1 & & \\ & \ddots & \ddots & \ddots & \\ & & & & 1 \\ & & & 1 & j \end{pmatrix}$$

$$\begin{aligned} \det\begin{pmatrix} 1 & 1 & & & \\ 1 & 2 & 1 & & \\ & 1 & 3 & 1 & \\ & & 1 & 4 & 1 \\ & & & 1 & 5 \end{pmatrix} &= K(1,2,3,4,5,1,1,1,1,1,1,1,1) \\ &= 5! - 5^{\underline{3}} - 5^{\underline{2}} \cdot K(1) - 5^{\underline{1}} \cdot K(1,2,1,1) - K(1,2,3,1,1,1,1) \\ &= 40 - 5 \cdot \det\begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} + 5 \cdot \det\begin{pmatrix} 1 & 1 & \\ 1 & 2 & 1 \\ & 1 & 3 \end{pmatrix} \\ &= 33 \end{aligned}$$

Consequences of Theorem 1 ...

$$\det\begin{pmatrix} 1 & F_1 & & & \\ - & 1 & F_2 & & \\ & - & 1 & F_3 & \\ & & - & 1 & F_4 \\ & & & - & 1 \end{pmatrix} = K(1,1,1,1,1,F_1,F_2,F_3,F_4,-,-,-,-)$$

$$= 1 + F_1K() + F_2K(1) + F_3K(1,1,F_1,-) + F_4K(1,1,1,F_1,F_2,-,-)$$

$$= 1 + F_1 + F_2|1| + F_3 \cdot \det\begin{pmatrix} 1 & F_1 \\ - & 1 \end{pmatrix} + F_4 \cdot \det\begin{pmatrix} 1 & F_1 & \\ - & 1 & F_2 \\ & - & 1 \end{pmatrix}$$

$$= 1 + 1 + 1 + 4 + 9$$

$$= 16$$

Theorem 2

Theorem

For the integer $n \geq 0$ we have

$$K_n(x_1, \dots, x_n, y_1, \dots, y_{n-1}, z_1 \dots, z_{n-1}) =$$
$$\sum_{j \ \text{even}, \ j=0}^{n-1} (-1)^{(n-j-1)/2} K_j(x_1, \dots, x_j, y_1, \dots, y_{j-1}, z_1 \dots, z_{j-1}) \cdot x_{j+1} \cdot \prod_{k \ \text{even}, \ k=j+2}^{n-1} y_k z_k$$

when n is odd, and

$$K_n(x_1, \dots, x_n, y_1, \dots, y_{n-1}, z_1 \dots, z_{n-1}) =$$
$$\sum_{j \ \text{odd}, \ j=-1}^{n-1} (-1)^{(n-j-1)/2} K_j(x_1, \dots, x_j, y_1, \dots, y_{j-1}, z_1 \dots, z_{j-1}) \cdot x_{j+1} \cdot \prod_{k \ \text{odd}, \ k=j+2}^{n-1} y_k z_k$$

when n is even.

$$K_0()x_1 y_2 z_2 y_4 z_4 - K_2(x_1, x_2, y_1, z_1) x_3 y_4 z_4 + K_4(x_1, x_2, x_3, x_4, y_1, y_2, y_3, z_1, z_2, z_3) x_5 =$$
$$K_5(x_1, x_2, x_3, x_4, x_5, y_1, y_2, y_3, y_4, z_1, z_2, z_3, z_4)$$

Consequences of Theorem 2 ...

Corollary

For the integer $n \geq 0$ we have

$$\sum_{k=0}^{n-1} F_{2k+1} = F_{2n}$$

$$1 + \sum_{k=1}^{n} F_{2k} = F_{2n+1}$$

$$\begin{aligned} F_6 &= K(1,1,1,1,1,-,-,-,-,1,1,1,1) \\ &= 1 + K(1,1,-,1) + K(1,1,1,1,-,-,-,1,1,1) \\ &= 1 + F_3 + F_5 \end{aligned}$$

Jacobsthal numbers

Corollary

For the integer $n \geq 0$ we have

$$\sum_{k=0}^{n-1} 2^{n-k-1} \cdot J_{2k+1} = J_{2n}$$

$$2^n + \sum_{k=1}^{n} 2^{n-k} \cdot J_{2k} = J_{2n+1}$$

Proof: Evaluation of Theorem 2 for $K_n(1, 1, \ldots, 1, -2, -2, \ldots, -2, 1, 1, \ldots, 1)$.

Parallel summation

Corollary

For nonnegative integers m and n we have

$$\sum_{0 \le k \le n} \binom{m+k}{k} = \binom{m+n+1}{n}.$$

Proof: By evaluation of Theorem 2...

Corollary

For the integer $n \geq 0$ we have

$$\det\begin{pmatrix}1 & 1 & & & & \\ 1 & 2 & 1 & & & \\ & 1 & 3 & 1 & & \\ & & \ddots & \ddots & \ddots & \\ & & & & & 1 \\ & & & & 1 & n\end{pmatrix} = \sum_{j=-1}^{n-1} P(j)(-1)^{\frac{n-j-1}{2}} \cdot j \cdot \det\begin{pmatrix}1 & 1 & & & \\ 1 & 2 & 1 & & \\ & \ddots & \ddots & \ddots & \\ & & & & 1 \\ & & & 1 & j\end{pmatrix}$$

where $P(j) = [j \text{ even}]$ if n is odd, and $P(j) = [j \text{ odd}]$ if n is even.

$$\begin{aligned}\det\begin{pmatrix}1 & 1 & & & \\ 1 & 2 & 1 & & \\ & 1 & 3 & 1 & \\ & & 1 & 4 & 1 \\ & & & 1 & 5\end{pmatrix} &= K(1,2,3,4,5,1,1,1,1,1,1,1,1) \\ &= 1 - 3\cdot K(1,2,1,1) + 5 \cdot K(1,2,3,4,1,1,1,1,1,1) \\ &= 1 - 3 \cdot \det\begin{pmatrix}1 & 1 \\ 1 & 2\end{pmatrix} + 5 \cdot \det\begin{pmatrix}1 & 1 & & \\ 1 & 2 & 1 & \\ & 1 & 3 & 1 \\ & & 1 & 4\end{pmatrix} \\ &= 33\end{aligned}$$

$$
\begin{aligned}
\det\begin{pmatrix} 1 & F_1 & & & \\ - & 1 & F_2 & & \\ & - & 1 & F_3 & \\ & & - & 1 & F_4 \\ & & & - & 1 \end{pmatrix} &= K(1,1,1,1,1,F_1,F_2,F_3,F_4,-,-,-,-) \\
&= F_2F_4 - F_4 \cdot K(1,1,F_1,-) \\
&\quad + \cdot K(1,1,1,1,F_1,F_2,F_3,-,-,-) \\
&= F_2F_4 - F_4 \cdot \det\begin{pmatrix} 1 & F_1 \\ - & 1 \end{pmatrix} \\
&\quad + \det\begin{pmatrix} 1 & F_1 & & \\ - & 1 & F_2 & \\ & - & 1 & F_3 \\ & & - & 1 \end{pmatrix} \\
&= 16
\end{aligned}
$$

$K_n(1,\dots,1;0,\dots,0;0,\dots,0)$	$1,1,1,\dots,1,\dots$	$A000012$
$K_n(-1,\dots,-1;0,\dots,0;0,\dots,0)$	$1,-1,1,\dots,(-1)^n,\dots$	$A033999$
$K_n(2,\dots,2;1,\dots,1;1,\dots,1)$	$1,2,3,\dots,n+1,\dots$	$A000027$
$K_n(-2,\dots,-2;1,\dots,1;1,\dots,1)$	$1,-2,3,-4,\dots,(-1)^n(n+1),\dots$	$A181983$
$K_n(a,\dots,a;0,\dots,0;0,\dots,0)$	$1,a,a^2,a^3,\dots,a^n,\dots$	
$K_n(-a,\dots,-a;0,\dots,0;0,\dots,0)$	$1,-a,a^2,-a^3,\dots,(-1)^n a^n,\dots$	
$K_n(3,\dots,3;2,\dots,2;1,\dots,1)$	$1,3,7,15,\dots,2^{n+1}-1,\dots$	$A000225$
$K_n(-3,\dots,-3;2,\dots,2;1,\dots,1)$	$1,-3,7,-15,\dots,(-1)^n(2^{n+1}-1),\dots$	$A225883$
$K_n(1,\dots,1;-1,\dots,-1;1,\dots,1)$	$1,1,2,3,5\dots,F_{n+1},\dots$	$A000045$
$K_n(-1,\dots,-1;-1,\dots,-1;1,\dots,1)$	$1,-1,2,-3,5\dots,(-1)^n F_{n+1},\dots$	$A039834$
$K_n(1,2,3,\dots,n;0,\dots,0;0,\dots,0)$	$1,2,6,24,\dots,n!,\dots$	$A000142$
$K_n(1,2,3,\dots,n;1\dots,1;1,\dots,1)$	$1,1,1,2,7,33,191,\dots$	$A058797$
$K_n(1,1,1,\dots,1;1,1,2,\dots,F_n;-,\dots,-)$	$1,1,2,3,7,16,51,179,\dots,$	$A089125$
$K_n(3,\dots,3;1,\dots,1;3,\dots,3)$	$1,3,6,9,9,0,-27,-81,-162,\dots$	$A057083$
$K_n(5,\dots,5;1,\dots,1;5,\dots,5)$	$1,5,20,75,275,1000,3625,\dots$	$A030191$
$K_n(3,\dots,3;2,\dots,2;3,\dots,3)$	$1,3,3,-9,-45,-81,27,567,\dots$	$A190960$

Thank you for your attention!

$K_n(1,\ldots,1;0,\ldots,0;0,\ldots,0)$	$1,1,1,\ldots,1,\ldots$	$A000012$
$K_n(-1,\ldots,-1;0,\ldots,0;0,\ldots,0)$	$1,-1,1,\ldots,(-1)^n,\ldots$	$A033999$
$K_n(2,\ldots,2;1,\ldots,1;1,\ldots,1)$	$1,2,3,\ldots,n+1,\ldots$	$A000027$
$K_n(-2,\ldots,-2;1,\ldots,1;1,\ldots,1)$	$1,-2,3,-4,\ldots,(-1)^n(n+1),\ldots$	$A181983$
$K_n(a,\ldots,a;0,\ldots,0;0,\ldots,0)$	$1,a,a^2,a^3,\ldots,a^n,\ldots$	
$K_n(-a,\ldots,-a;0,\ldots,0;0,\ldots,0)$	$1,-a,a^2,-a^3,\ldots,(-1)^n a^n,\ldots$	
$K_n(3,\ldots,3;2,\ldots,2;1,\ldots,1)$	$1,3,7,15,\ldots,2^{n+1}-1,\ldots$	$A000225$
$K_n(-3,\ldots,-3;2,\ldots,2;1,\ldots,1)$	$1,-3,7,-15,\ldots,(-1)^n(2^{n+1}-1),\ldots$	$A225883$
$K_n(1,\ldots,1;-1,\ldots,-1;1,\ldots,1)$	$1,1,2,3,5\ldots,F_{n+1},\ldots$	$A000045$
$K_n(-1,\ldots,-1;-1,\ldots,-1;1,\ldots,1)$	$1,-1,2,-3,5\ldots,(-1)^n F_{n+1},\ldots$	$A039834$
$K_n(1,2,3,\ldots,n;0,\ldots,0;0,\ldots,0)$	$1,2,6,24,,\ldots,n!,\ldots$	$A000142$
$K_n(1,2,3,\ldots,n;1\ldots,1;1,\ldots,1)$	$1,1,1,2,7,33,191,\ldots$	$A058797$
$K_n(1,1,1,\ldots,1;1,1,2,\ldots,F_n;-,\ldots,-)$	$1,1,2,3,7,16,51,179,\ldots,$	$A089125$
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$K_n(3,\ldots,3;2,\ldots,2;3,\ldots,3)$	$1,3,3,-9,-45,-81,27,567,\ldots$	$A190960$

Thank you for your attention!