Characterizing bipartite distance-regularized graphs with vertices of eccentricity at most 4

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Terminology and notations

In this talk, $\Gamma = (X, \mathcal{R})$ will denote a finite, undirected, connected graph, without loops and multiple edges.

Definition

Let $\Gamma = (X, \mathcal{R})$ and $x, y \in X$.

- The **distance** between x and y, denoted by $\partial(x, y)$, is the length of a shortest walk from x to y.
- **Eccentricity of** x is the greatest distance between x and any other vertex. That is $\varepsilon(x) = \max_{z \in X} \partial(x, z)$.
- **Diameter of** Γ: $D = \max\{\varepsilon(x) \mid x \in X\}$.

Definition

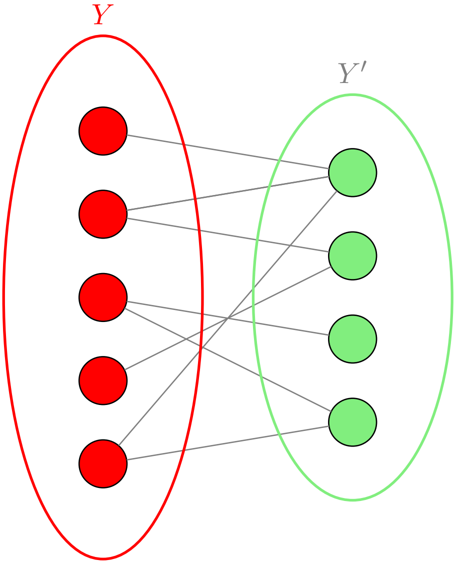
Let $\Gamma = (X, \mathcal{R})$.

- For an integer i we represent with $\Gamma_i(x)$ the collection of all vertices that are at distance i from vertex x. That is

$$\Gamma_i(x) = \{y \in X \mid \partial(x, y) = i\}.$$

- $\Gamma(x) = \Gamma_1(x)$.
- Γ is k-**regular** iff $|\Gamma(x)| = k$ for every vertex $x \in X$.
- The collection of all the subsets $\Gamma_i(x)$, for $0 \leq i \leq \varepsilon(x)$, makes up a partition of the vertex set X that is called the **distance partition of** Γ **relative to** x.

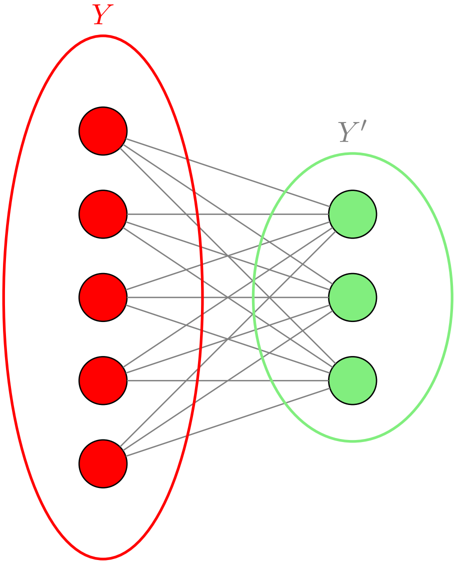
(Y, Y')-bipartite graph



Definition

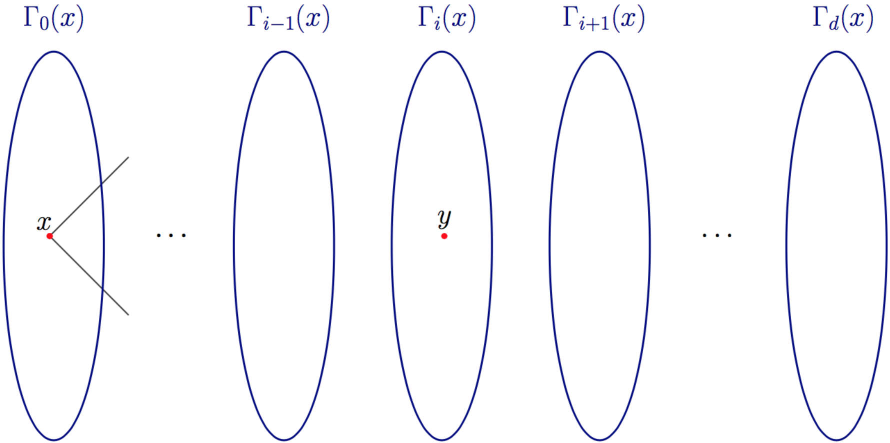
A **bipartite** (or (Y, Y')-**bipartite**) graph is a graph whose vertex set can be partitioned into two subsets Y and Y' such that each edge has one end in Y and one end in Y'. The vertex sets Y and Y' in such a partition are called **color partitions** of the graph.

Biregular graph



Definition

A bipartite graph Γ with color partitions Y and Y' is said to be **biregular** if the valency of a vertex only depends on the color partition where it belongs to.


$\Gamma_0(x)$
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x
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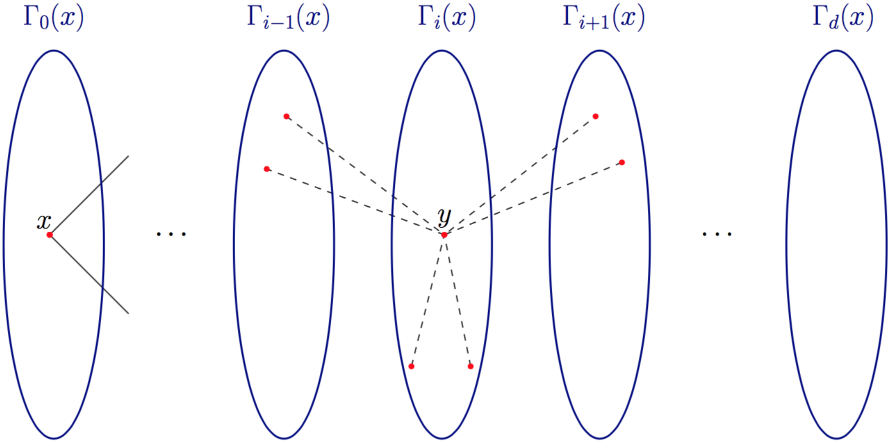
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y
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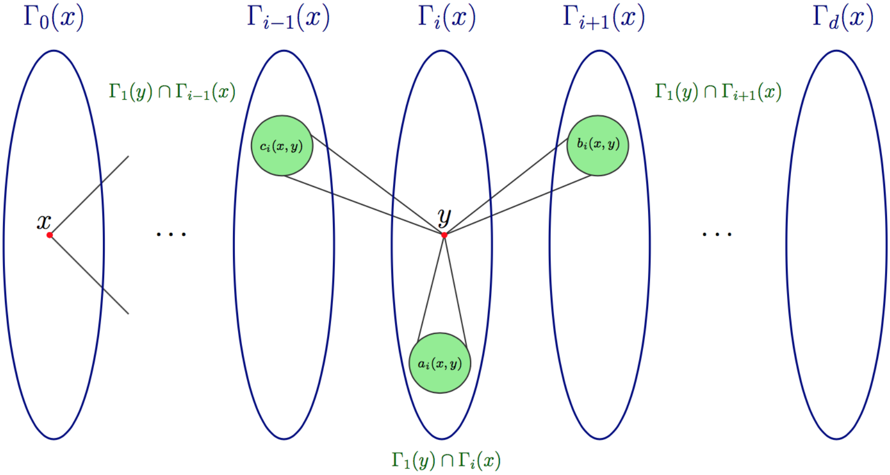
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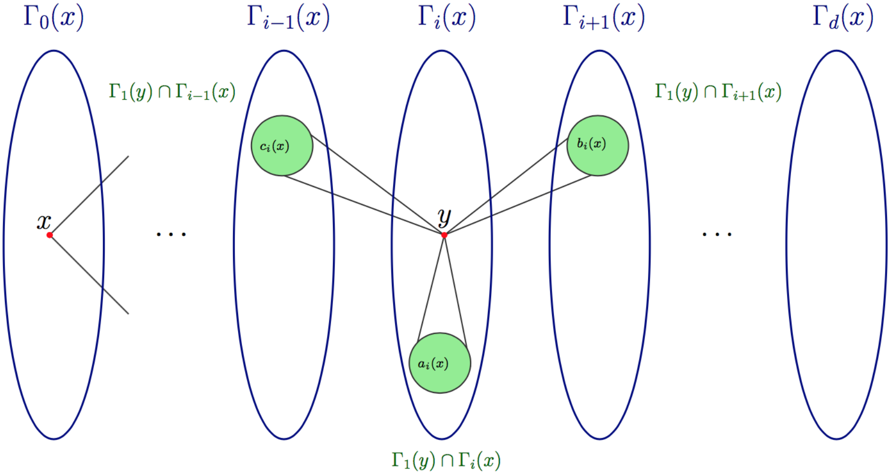
$c_i(x,y)$
$b_i(x,y)$
x
y
$a_i(x,y)$
$\Gamma_1(y) \cap \Gamma_i(x)$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$\dots$
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$\Gamma_1(y) \cap \Gamma_i(x)$

Local regularity property

Let $x \in X$. Assume that $y \in \Gamma_i(x)$ for some $0 \leq i \leq \varepsilon(x)$ and let z be a neighbour of y. Then

$$\partial(x, z) \in \{i-1, i, i+1\}$$

and so $z \in \Gamma_{i-1}(x) \cup \Gamma_i(x) \cup \Gamma_{i+1}(x)$. For $y \in \Gamma_i(x)$ we therefore define the following numbers:

$$a_i(x, y) = |\Gamma_i(x) \cap \Gamma(y)|, \qquad b_i(x, y) = |\Gamma_{i+1}(x) \cap \Gamma(y)|,$$

$$c_i(x, y) = |\Gamma_{i-1}(x) \cap \Gamma(y)|.$$

Local regularity property

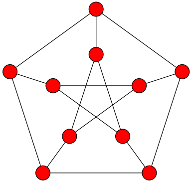
Definition (GODSIL AND SHAWE-TAYLOR, 1987.)

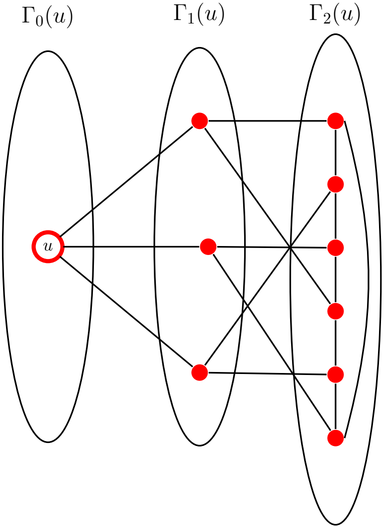
We say that $x \in X$ is **distance-regularized** if the numbers $a_i(x,y), b_i(x,y)$ and $c_i(x,y)$ do not depend on the choice of $y \in \Gamma_i(x), (0 \leq i \leq \varepsilon(x))$.

In this case, the numbers $a_i(x,y), b_i(x,y)$ and $c_i(x,y)$ are simply denoted by $a_i(x), b_i(x)$ and $c_i(x)$ respectively, and are called the **intersection numbers of** x.

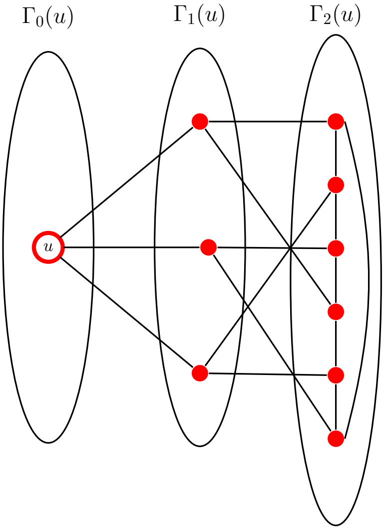
Example

Let u be the vertex of Petersen graph.





Example

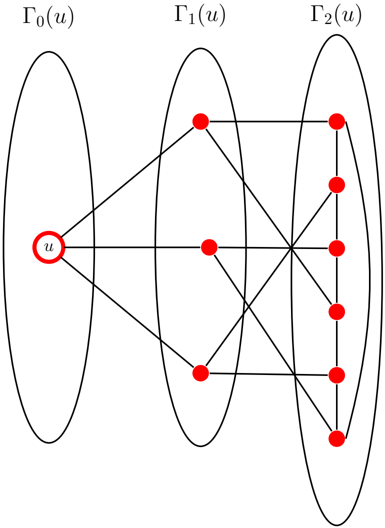


Let u be the vertex of Petersen graph.

$$\begin{array}{lllll} a_0 & = & 0, & a_1 = 0, & a_2 = 2 \\ b_0 & = & 3, & b_1 = 2, & b_2 = 0 \\ c_0 & = & 0, & c_1 = 1, & c_2 = 1 \end{array}$$

u is distance regularized.

Example



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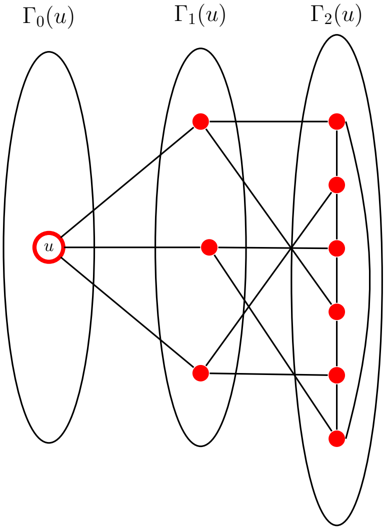

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u is distance regularized.

Distance-regularized graphs

Definition

- A connected graph in which every vertex is distance-regularized is called a **distance-regularized** graph.
- A **distance-regular graph** is distance-regularized graph where all its vertices have the same intersection array
- A distance-regularized graph is said to be **distance-biregular** if
  - is bipartite
  - vertices in the same color partition have the same intersection numbers
  - vertices in the different color partition have different intersection numbers.

Theorem (GODSIL AND SHAWE-TAYLOR, 1987.)

Every distance-regularized graph is either distance-regular or distance-biregular.

Distance-semiregular graphs with respect to Y

Definition

A connected bipartite graph Γ with color partitions Y and Y' is called **distance-semiregular with respect to** Y if it is distance-regular around all vertices in Y, with the same parameters.

Design

Definition

An incidence structure $\mathcal{D} = (\mathcal{P}, \mathcal{B}, I)$ such. that

- $|\mathcal{P}| = v$, $|\mathcal{B}| = b$,
- each block $B \in \mathcal{B}$ is incident with exactly k points,
- every t-tuple of distinct points from $\mathcal{P}$ is incident with exactly λ blocks
- each point is incident with exactly r blocks

is called a t-(v, b, r, k, λ) **design** or a t-(v, k, λ) **design**.

Definition

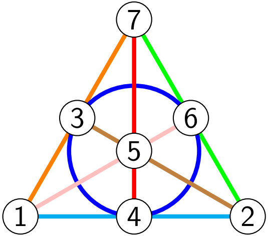
Let x and y be non-negative integers with $x < y$. A design $\mathcal{D}$ is called a (proper) **quasi-symmetric design** with intersection numbers x and y if any two distinct blocks of $\mathcal{D}$ intersect in either x or y points, and both intersection numbers are realized.

Incidence graph of a design

Definition

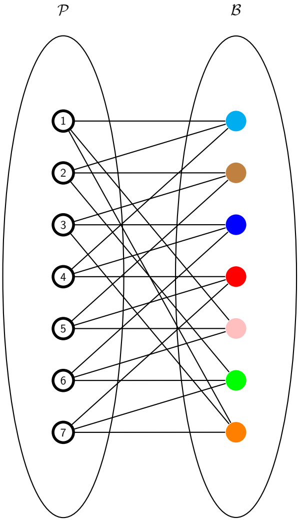
The incidence graph of a design $\mathcal{D} = (\mathcal{P}, \mathcal{B}, \mathcal{I})$ is a $(\mathcal{P}, \mathcal{B})$-bipartite graph where the point $x \in \mathcal{P}$ is adjacent to the block $B \in \mathcal{B}$ if and only if x is incident with B.

Example



$\mathcal{P} = \{1, 2, 3, 4, 5, 6, 7\}$

$\mathcal{B} = \{\{1, 2, 4\}, \{2, 3, 5\}, \{3, 4, 6\}, \{4, 5, 7\},$
$\{1, 5, 6\}, \{2, 6, 7\}, \{1, 3, 7\}\}$



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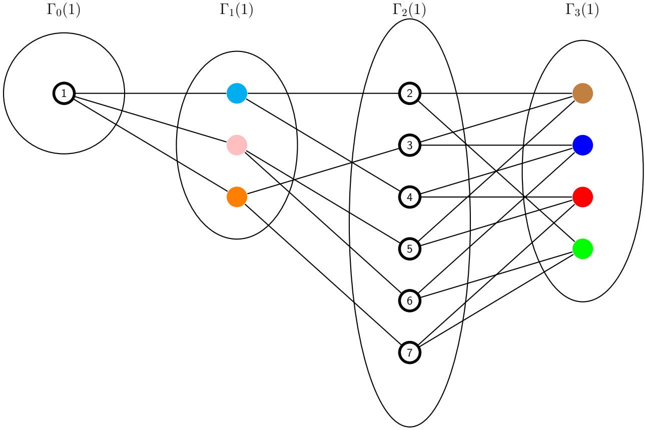

$$
\begin{aligned}
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b_0 &= 3, \quad b_1 = 2, \quad b_2 = 2, \quad b_3 = 0 \\
c_0 &= 0, \quad c_1 = 1, \quad c_2 = 1, \quad c_3 = 3
\end{aligned}
$$

Bipartite distance-regularized graphs (BDRG)

If $\Gamma = (X, \mathcal{R})$ is (Y, Y')-bipartite distance-regularized graph then Γ is either a bipartite distance-regular graph or Γ is a distance-biregular graph.

Let $\Gamma = (X, \mathcal{R})$ be bipartite distance-regularized graph. Then

- $a_i(x) = 0$ for $0 \leq i \leq \varepsilon(x)$
- All vertices from Y (Y', respectively) have the same eccentricity D (D', respectively)
- All vertices from Y (Y', respectively) have the same the same valency k (k', respectively)
- For $x \in Y$, $y \in Y'$ and an integer i we abbreviate $c_i := c_i(x)$, $b_i := b_i(x)$, $c_i' := c_i(y)$ and $b_i' := b_i(y)$.

Also it holds:

Theorem (DELORME, 1994.)

The difference $D - D'$ is at most 1. If $D < D'$ then D is odd.

BDRG with $D \leq 3$

Let Γ be (Y, Y')-bipartite distance-regularized graph with:

- $D = 1$ then there is one-to-one correspondence between the incidence graph of 1-$(1, 1, b)$ designs and Γ.
- $D = 2$ then there exists a one-to-one correspondence between the incidence graphs of 2-(v, v, b) designs and Γ.
- $D = 3$ then there is one-to-one correspondence between the incidence graphs of 2-designs and distance-semiregular graphs with distance-regularized vertices of eccentricity 3. Moreover:
  - incidence graphs of symmetric 2-designs are equivalent to bipartite distance-regular graphs with vertices of eccentricity 3 (Brouwer Cohen, Neumaier, 1989.)
  - incidence graphs of quasi-symmetric 2-designs with one intersection number zero are equivalent to distance-biregular graphs with $D = 3$ and $D' = 4$.
- What about $D = 4$?
  - B. Fernández, M.Maksimović, S. Rukavina, *Characterizing bipartite distance-regularized graphs with vertices of eccentricity 4*, Bull. Malays. Math. Sci. Soc. (2), 47, 2024.

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SPBIBD

Definition

Let $\mathcal{D}$ be a 1-(v, b, r, k, λ) design and let (s, t) be a pair of non-negative integers. A flag (a non-flag) of $\mathcal{D}$ is a point-block pair (p, B) such that $p \in B$ ($p \notin B$). We say that $\mathcal{D}$ is a **special partially balanced incomplete block design** (SPBIBD for short) of type (s, t) if there are constants λ_1 and λ_2 with the following properties:

(1) Any two points are contained in either λ_1 or λ_2 blocks.

(2) If a point-block pair (p, B) is a flag, then the number of points in B which occur with p in λ_1 blocks is s.

(3) If a point-block pair (p, B) is a non-flag, then the number of points in B which occur with p in λ_1 blocks is t.

In this case, we say that $\mathcal{D}$ is a $(v, b, r, k, \lambda_1, \lambda_2)$ SPBIBD of type (s, t).

The incidence graph of a SPBIBD

Lemma

Let $\mathcal{D} = (\mathcal{P}, \mathcal{B}, \mathcal{I})$ *be a* $(v, b, r, k, \lambda_1, 0)$ *SPBIBD of type* $(k-1, t)$*. Let* Γ *denote the incidence graph of* $\mathcal{D}$*. Then, every vertex* $p \in \mathcal{P}$ *has eccentricity* 4 *in* Γ*.*

Lemma

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$$\begin{array}{lllll} c_0 = 0, & c_1 = 1, & c_2 = \lambda_1, & c_3 = t, & c_4 = r. \\ b_0 = r, & b_1 = k-1, & b_2 = r - \lambda_1, & b_3 = k - t & b_4 = 0. \end{array}$$

The incidence graph of a SPBIBD

Lemma

Let $\mathcal{D} = (\mathcal{P}, \mathcal{B}, \mathcal{I})$ be a quasi-symmetric $(v, b, r, k, \lambda_1, 0)$ SPBIBD of type $(k-1, t)$ with intersection numbers $x = 0$ and $y > 0$. Let Γ denote the incidence graph of $\mathcal{D}$. Then, every vertex $B \in \mathcal{B}$ has eccentricity 4 in Γ.

The incidence graph of a SPBIBD

Theorem

Let $\mathcal{D} = (\mathcal{P}, \mathcal{B}, \mathcal{I})$ be a quasi-symmetric $(v, b, r, k, \lambda_1, 0)$ SPBIBD of type $(k-1, t)$ with intersection numbers $x = 0$ and $y > 0$. Let Γ denote the incidence graph of $\mathcal{D}$. Then, Γ is a $(\mathcal{P}, \mathcal{B})$-bipartite distance-regularized graph. Moreover, every vertex $p \in \mathcal{P}$ has eccentricity equals 4 and the following intersection numbers:

$$\begin{aligned} c_0 &= 0, \quad c_1 = 1, \quad c_2 = \lambda_1, \quad c_3 = t, \quad c_4 = r. \\ b_0 &= r, \quad b_1 = k-1, \quad b_2 = r - \lambda_1, \quad b_3 = k - t \quad b_4 = 0. \end{aligned}$$

In addition, every vertex $B \in \mathcal{B}$ has eccentricity equals 4 and the following intersection numbers:

$$\begin{aligned} c'_0 &= 0, \quad c'_1 = 1, \quad c'_2 = y, \quad c'_3 = \frac{t\lambda_1}{y}, \quad c'_4 = k. \\ b'_0 &= k, \quad b'_1 = r-1, \quad b'_2 = k - y, \quad b'_3 = r - \frac{t\lambda_1}{y}, \quad b'_4 = 0. \end{aligned}$$

Distance-semiregular graphs and SPBIBDs

Lemma

Let Γ be a (Y, Y')-distance semiregular graph with respect to Y. Assume every vertex in Y has eccentricity $D = 4$. Let b_i, c_i $(0 \leq i \leq 4)$ denote the intersection numbers of every vertex in Y. Then, Γ is the incidence graph of a $(1 + \frac{b_0 b_1}{c_2} + \frac{b_0 b_1 b_2 b_3}{c_2 c_3 c_4}, b_0 + \frac{b_0 b_1 b_2}{c_2 c_3}, b'_0, b_0, c_2, 0)$ SPBIBD of type (b_1, c_3).

Theorem

There is a one-to-one correspondence between the incidence graph of SPBIBDs with parameters $(v, b, r, k, \lambda_1, 0)$ of type $(k-1, t)$ and distance-semiregular graphs with distance-regularized vertices of eccentricity 4.

Lemma

Let Γ be a (Y, Y')-bipartite distance-regularized graph with vertices of eccentricity 4. Let $b_i, c_i; b_i', c_i'$ $(0 \leq i \leq 4)$ denote the intersection numbers of every vertex in Y and in Y' respectively. Then, Γ is the incidence graph of a $(1 + \frac{b_0 b_1}{c_2} + \frac{b_0 b_1 b_2 b_3}{c_2 c_3 c_4}, b_0 + \frac{b_0 b_1 b_2}{c_2 c_3}, b_0', b_0, c_2, 0)$ SPBIBD of type (b_1, c_3) which is quasi-symmetric with intersection numbers $x = 0$ and $y = c_2'$.

Theorem

There is a one-to-one correspondence between the incidence graph of quasi-symmetric SPBIBDs with parameters $(v, b, r, k, \lambda_1, 0)$ of type $(k - 1, t)$ with intersection numbers $x = 0$ and $y > 0$, and bipartite distance-regularized graphs with vertices of eccentricity 4.

THANK YOU!