Perfect matching in graphs obtained from simplicial complex of tilings

5th Croatian Combinatorial Days
Zagreb, September 19-20, 2024

Edin Liđan
International Burch University
Sarajevo, Bosnia and Herzegovina

This is joint work with Đorđe Baralić

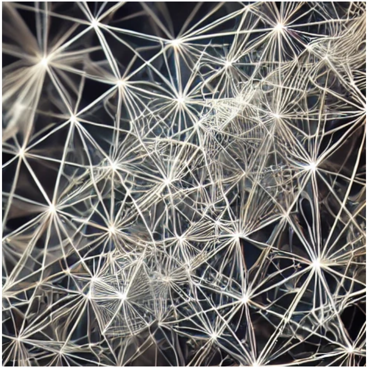
Email: `edin.lidan@ibu.edu.ba`

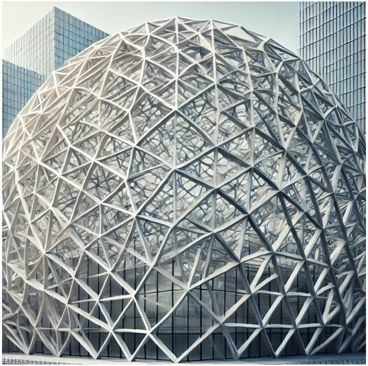
CroCoDays 2024

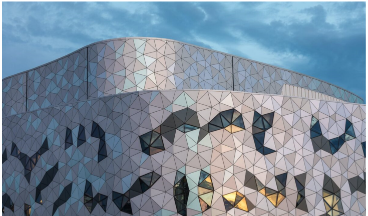
Content

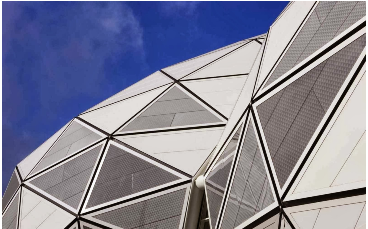
Introduction. Simplicial complex

Around us



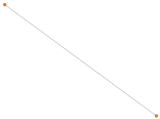




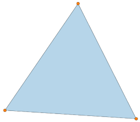


Simplicial complex
Examples

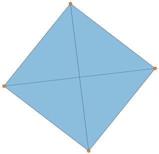
(a) 0-simplex (a point)



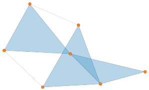
(b) 1-simplex (a segment line)



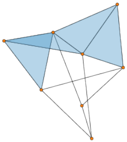
(c) 2-simplex (a triangle)



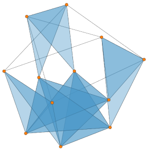
(d) 3-simplex (a tetrahedron)



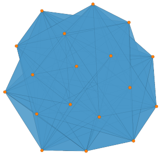
(e) Example



(f) Example



(g) Example



(h) k-simplex ($\mathbb{R}^k$)

- k-simplex is defined as a subset of $\mathbb{R}^n$ spanned by $k+1$ vertices $v_0, \ldots, v_k$,
- that is the set of linear combinations of the vertices with real nonnegative coefficients whose sum is 1.

Simplicial complex
Definition

Definition:

An abstract simplicial complex K on a vertex set $[m] = \{1, 2, ..., m\}$ is a collection of subsets of $[m]$ such that,

i) for each $i \in [m], \{i\} \in [m]$,

ii) for every $\sigma \in K$, if $\tau \subset \sigma$ then $\tau \in K$.

We assume that $\emptyset \in K$.

- The elements of K are called faces.
- A face of K is maximal if it is not contained as a subset in any other face of K.
- The maximal faces are also called facets.
- The dimension of a face σ of a simplicial complex K is defined as $\dim \sigma = |\sigma| - 1$, where $|\sigma|$ denotes the cardinality of σ.

Simplicial complex

Abstract simplicial complex vs. geometric simplicial complex

Simplicial complex

Abstract simplicial complex vs. geometric simplicial complex

- Abstract simplicial complex

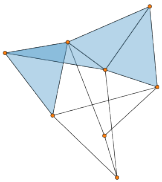
$$K = \{\emptyset, \{1\}, \{2\}, \{3\}, \{4\}, \{5\}, \{6\}, \{7\}, \{8\}, \{1,4\}, \{1,5\}, \{1,6\}, \{1,7\}, \{1,8\}, \{2,5\}, \{2,6\}, \{2,7\}, \{2,8\}, \{3,6\}, \{3,7\}, \{3,8\}, \{4,7\}, \{4,8\}, \{5,8\}, \{1,4,7\}, \{1,4,8\}, \{1,5,8\}, \{2,5,8\}\}$$

Simplicial complex
Abstract simplicial complex vs. geometric simplicial complex

- **Abstract simplicial complex**

$$K = \{\emptyset, \{1\}, \{2\}, \{3\}, \{4\}, \{5\}, \{6\}, \{7\}, \{8\}, \{1,4\}, \{1,5\}, \{1,6\}, \{1,7\}, \{1,8\}, \{2,5\}, \{2,6\}, \{2,7\}, \{2,8\}, \{3,6\}, \{3,7\}, \{3,8\}, \{4,7\}, \{4,8\}, \{5,8\}, \{1,4,7\}, \{1,4,8\}, \{1,5,8\}, \{2,5,8\}\}$$

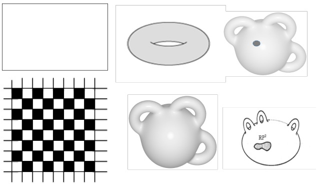
- **Geometrical simplicial complex**



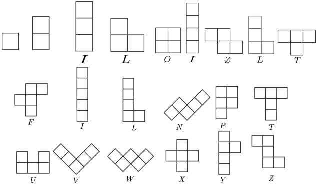
Slika: Simplicial complex $K(D_{I_3})$

Polyomino tilings problem

- Tiling problem
  - A region *M* and finite set Σ of title
  - Does Σ tiles *M*?
- Polyomino type tilings
  - Polyomino tiling problem asks it is possible to properly cover a finite region *M* consisting of cells with polyomino shapes from a given set Σ
- *M* - table in plane, surface, surface with boundary, . . .
- Σ - finite set of polyomino shapes



(a) *M*



(b) Σ

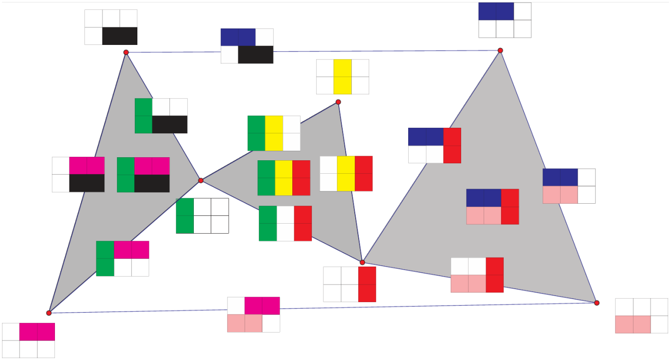
Problem of tilings vs simplicial complex

Problem of tilings vs simplicial complex

- We consider polyomino tiling problem of a finite subset M of square grids by given set of $\mathcal{T}$ of polyomino shapes.

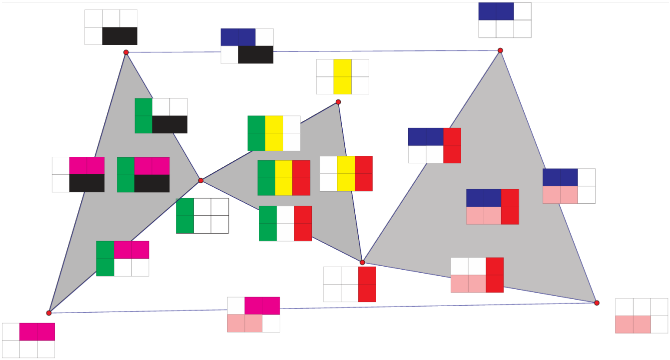
Problem of tilings vs simplicial complex

- We consider polyomino tiling problem of a finite subset M of square grids by given set of $\mathcal{T}$ of polyomino shapes.



Problem of tilings vs simplicial complex

- We consider polyomino tiling problem of a finite subset M of square grids by given set of $\mathcal{T}$ of polyomino shapes.



- $K(M;\mathcal{T})$ is a simplicial complex whose i-faces correspond to a placement of $i+1$ polyomino shapes from $\mathcal{T}$ onto M without overlapping.

Simplicial complex of polyomino type tilings

Proposition

$K(M;\mathcal{T})$ is a flag simplicial complex.

Proposition

Maximal number of polyomino shapes from $\mathcal{T}$ that may be placed on M without overlapping is $\dim(K(M;\mathcal{T}))+1$.

Graph of simplicial complex of polyomino type tilings

Graph of simplicial complex of polyomino type tilings

- Graph of simplicial complex of polyomino tilings

Graph of simplicial complex of polyomino type tilings

- Graph of simplicial complex of polyomino tilings
- Graph which contains all 0-simplex and 1-simplex

Graph of simplicial complex of polyomino type tilings

- Graph of simplicial complex of polyomino tilings
- Graph which contains all 0-simplex and 1-simplex
- The 1-skeleton of this simplicial complex, as a graph.

Graph of simplicial complex of polyomino type tilings

- Graph of simplicial complex of polyomino tilings
- Graph which contains all 0-simplex and 1-simplex
- The 1-skeleton of this simplicial complex, as a graph.

Definition

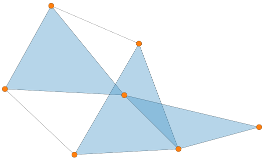
The p–skeleton of a simplicial complex K is denoted by $K(p)$ and is the set of all of the simplices in K of dimension p or less.

Graph of simplicial complex of polyomino type tilings

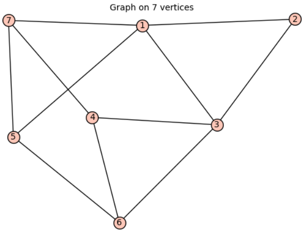
- Graph of simplicial complex of polyomino tilings
- Graph which contains all 0-simplex and 1-simplex
- The 1-skeleton of this simplicial complex, as a graph.

Definition

The p–skeleton of a simplicial complex K is denoted by $K(p)$ and is the set of all of the simplices in K of dimension p or less.



(m) $K_{l_2}(D_{2\times 3})$



(n) $G(K_{l_2}(D_{2\times 3}))$

Perfect matching

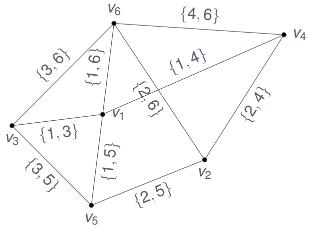
- A perfect matching in graph $G = (V, E)$ is a matching $M(G)$ that covers all the vertices, i.e. for all $v \in V$ there is $e \in M(G)$ such that $v \in e$.

Perfect matching

- A perfect matching in graph $G = (V, E)$ is a matching $M(G)$ that covers all the vertices, i.e. for all $v \in V$ there is $e \in M(G)$ such that $v \in e$.
- A perfect matching in a graph G is a matching in which every vertex of G appears exactly once;

Perfect matching

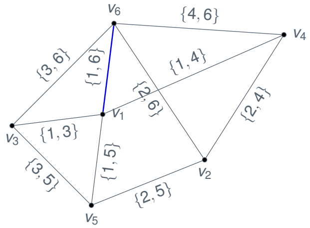
- A perfect matching in graph $G = (V, E)$ is a matching $M(G)$ that covers all the vertices, i.e. for all $v \in V$ there is $e \in M(G)$ such that $v \in e$.
- A perfect matching in a graph G is a matching in which every vertex of G appears exactly once;



- Perfect matching of $G(K_{l_2}(D_{1\times 7}))$
- A matching M is called maximal if $M \cup \{e\}$ is not a matching for any $e \in E(G)$.
- A matching is called maximum if no other matching in G has a larger size.

Perfect matching

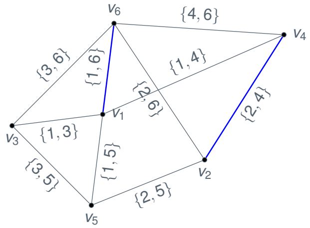
- A perfect matching in graph $G = (V, E)$ is a matching $M(G)$ that covers all the vertices, i.e. for all $v \in V$ there is $e \in M(G)$ such that $v \in e$.
- A perfect matching in a graph G is a matching in which every vertex of G appears exactly once;



- Perfect matching of $G(K_{l_2}(D_{1\times 7}))$
- A matching M is called maximal if $M \cup \{e\}$ is not a matching for any $e \in E(G)$.
- A matching is called maximum if no other matching in G has a larger size.

Perfect matching

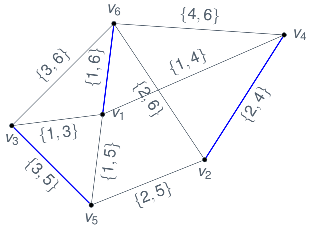
- A perfect matching in graph $G = (V, E)$ is a matching $M(G)$ that covers all the vertices, i.e. for all $v \in V$ there is $e \in M(G)$ such that $v \in e$.
- A perfect matching in a graph G is a matching in which every vertex of G appears exactly once;



- Perfect matching of $G(K_{l_2}(D_{1\times 7}))$
- A matching M is called maximal if $M \cup \{e\}$ is not a matching for any $e \in E(G)$.
- A matching is called maximum if no other matching in G has a larger size.

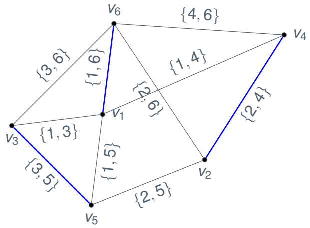
Perfect matching

- A perfect matching in graph $G = (V, E)$ is a matching $M(G)$ that covers all the vertices, i.e. for all $v \in V$ there is $e \in M(G)$ such that $v \in e$.
- A perfect matching in a graph G is a matching in which every vertex of G appears exactly once;



Perfect matching

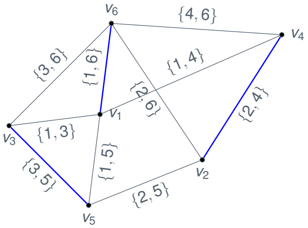
- A perfect matching in graph $G = (V, E)$ is a matching $M(G)$ that covers all the vertices, i.e. for all $v \in V$ there is $e \in M(G)$ such that $v \in e$.
- A perfect matching in a graph G is a matching in which every vertex of G appears exactly once;



- Perfect matching of $G(K_{l_2}(D_{1\times 7}))$
- A matching M is called maximal if $M \cup \{e\}$ is not a matching for any $e \in E(G)$.

Perfect matching

- A perfect matching in graph $G = (V, E)$ is a matching $M(G)$ that covers all the vertices, i.e. for all $v \in V$ there is $e \in M(G)$ such that $v \in e$.
- A perfect matching in a graph G is a matching in which every vertex of G appears exactly once;



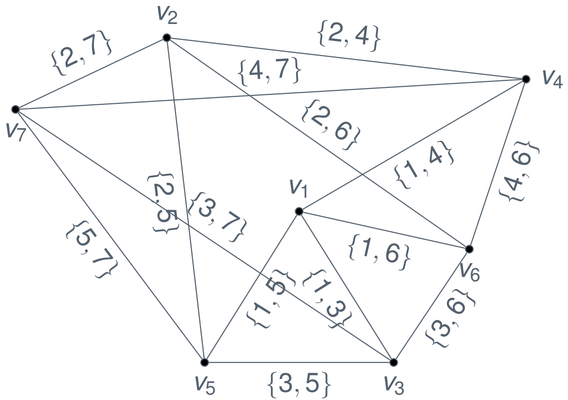
- Perfect matching of $G(K_{l_2}(D_{1\times 7}))$
- A matching M is called maximal if $M \cup \{e\}$ is not a matching for any $e \in E(G)$.
- A matching is called maximum if no other matching in G has a larger size.

A near-perfect matching

- A near-perfect matching in a graph G is a matching in which exactly one vertex is unmatched;

A near-perfect matching

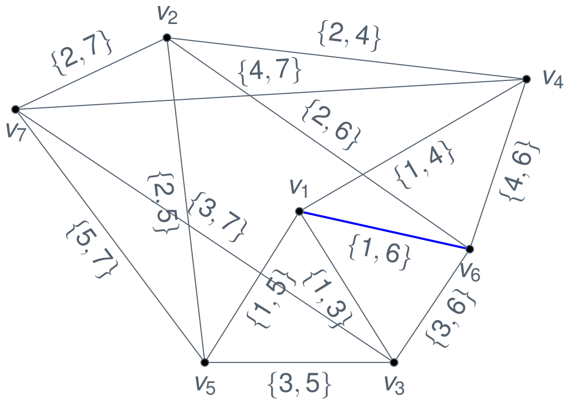
- A near-perfect matching in a graph G is a matching in which exactly one vertex is unmatched;



- A near-perfect matching of $G(K_{I_2}(\mathbb{T}_{1\times 7}))$

A near-perfect matching

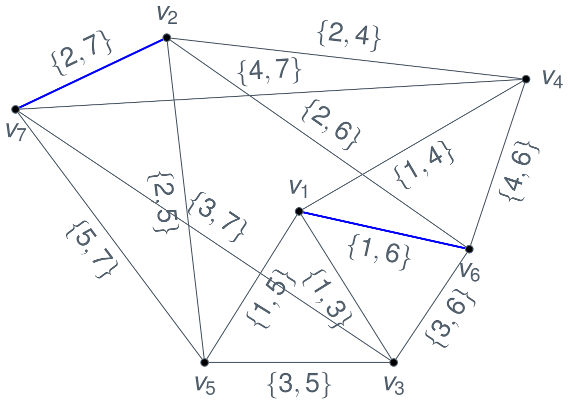
- A near-perfect matching in a graph G is a matching in which exactly one vertex is unmatched;



- A near-perfect matching of $G(K_{I_2}(\mathbb{T}_{1\times 7}))$

A near-perfect matching

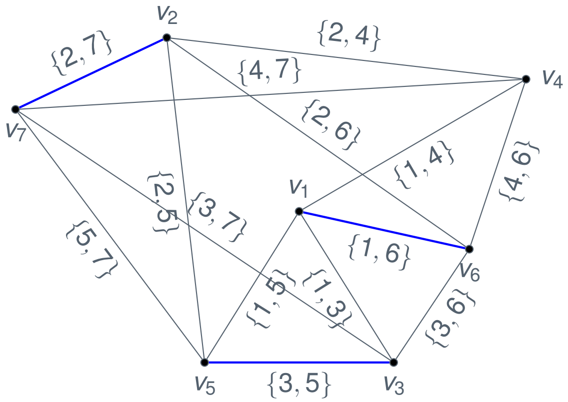
- A near-perfect matching in a graph G is a matching in which exactly one vertex is unmatched;



- A near-perfect matching of $G(K_{I_2}(\mathbb{T}_{1\times 7}))$

A near-perfect matching

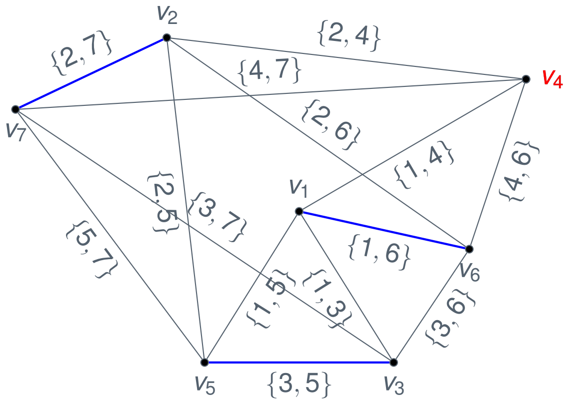
- A near-perfect matching in a graph G is a matching in which exactly one vertex is unmatched;



- A near-perfect matching of $G(K_{I_2}(\mathbb{T}_{1\times 7}))$

A near-perfect matching

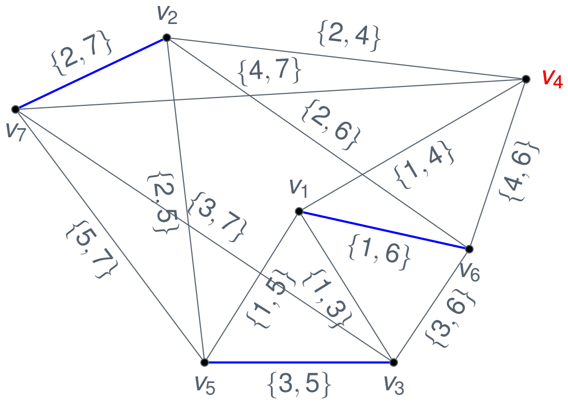
- A near-perfect matching in a graph G is a matching in which exactly one vertex is unmatched;



- A near-perfect matching of $G(K_{I_2}(\mathbb{T}_{1\times 7}))$

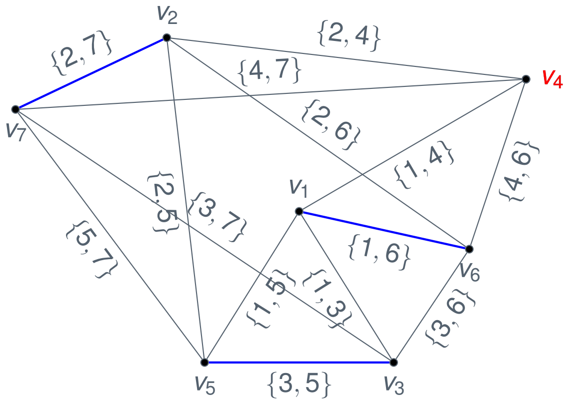
A near-perfect matching

- A near-perfect matching in a graph *G* is a matching in which exactly one vertex is unmatched;



A near-perfect matching

- A near-perfect matching in a graph G is a matching in which exactly one vertex is unmatched;

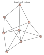


- A near-perfect matching of $G(K_{I_2}(\mathbb{T}_{1\times 7}))$

Some results

n	$G(K_{l_2}(D_{1\times n}))$	Maximal matching	(A-near) perfect matching
5		(1, 3), (2, 4)	perfect matching
6		(1, 5), (2, 4)	a near-perfect matching
7		(1, 6), (2, 4), (3, 5)	perfect matching
8		(2, 4), (3, 6), (1, 7)	a near-perfect matching
9		(5, 6), (3, 7), (2, 4), (8, 1)	perfect matching
10		(8, 3), (5, 7), (2, 4), (9, 1)	a near-perfect matching
11		(2, 7), (8, 10), (1, 4), (3, 6), (9, 5)	perfect matching
12		(4, 6), (9, 2), (8, 10), (3, 7), (1, 11)	a near-perfect matching

Some results

n	$G(K_{l_3}(D_{1\times n}))$	Maximal matching	(A-near) perfect matching
7		(2, 5), (1, 4)	a near-perfect matching
8		(1, 4), (2, 5), (3, 6)	perfect matching
9		(2, 5), (3, 6), (1, 7)	a near-perfect matching
10		(8, 1), (2, 5), (3, 6), (4, 7)	perfect matching
11		(2, 5), (9, 1), (3, 6), (8, 4)	a near-perfect matching
12		(8, 2), (10, 7), (1, 4), (3, 6), (9, 5)	perfect matching
13		(8, 3), (10, 7), (9, 2), (1, 4), (11, 6)	a near-perfect matching
14		(10, 7), (9, 3), (2, 11), (12, 6), (1, 5), (8, 4)	perfect matching

Conclusion

- $G(K_{I_k}(D_{1\times n}))$
  - k even
    - for $n > 2k$ and n even $G(K_{I_k}(D_{1\times n}))$ has a near-perfect matching;
    - for $n > 2k$ and n odd $G(K_{I_k}(D_{1\times n}))$ has a perfect matching;
  - k odd
    - for $n > 2k$ and n even $G(K_{I_k}(D_{1\times n}))$ has a perfect matching;
    - for $n > 2k$ and n odd $G(K_{I_k}(D_{1\times n}))$ has a near-perfect matching;
- $G(K_{I_k}(\mathbb{T}_{1\times n}))$
  - for $n \geq 2k$ and n even $G(K_{I_k}(\mathbb{T}_{1\times n}))$ has a perfect matching;
  - for $n \geq 2k$ and n odd $G(K_{I_k}(\mathbb{T}_{1\times n}))$ has a near-perfect matching;
- $G(K_{I_k}(D_{n\times m}))$
- $G(K_{I_k}(\mathbb{T}_{n\times m}))$

Interesting results

Interesting results

- D. Kozlov (1999)

Interesting results

- D. Kozlov (1999)
  - paths and cycles

Interesting results

- D. Kozlov (1999)
  - paths and cycles
  - homotopy type, contractible, wedge spheres

$n \geq 1$

$$K_{I_2}(D_{1\times n}) = \begin{cases} \mathbb{S}^{k-1} & \text{if } n = 3k, \\ pt & \text{if } n = 3k+1, \\ \mathbb{S}^{k} & \text{if } n = 3k+2. \end{cases}$$

$n \geq 3$

$$\mathbb{T}_{I_2}(D_{1\times n}) = \begin{cases} \mathbb{S}^{k-1} \vee \mathbb{S}^{k-1} & \text{if } r = 3k, \\ \mathbb{S}^{k-1} & \text{if } r = 3k \pm 1. \end{cases}$$

Interesting results

Interesting results

- Matsushita

Interesting results

- Matsushita
  - $K_{I_2}(D_{2\times n})$
  - homotopy type of bouquet of spheres

Interesting results

- Matsushita
  - $K_{I_2}(D_{2\times n})$
  - homotopy type of bouquet of spheres
- S. Goyal, S. Shukla, A. Singh

Interesting results

- Matsushita
  - $K_{I_2}(D_{2\times n})$
  - homotopy type of bouquet of spheres
- S. Goyal, S. Shukla, A. Singh
  - $K_{I_2}(D_{3\times n})$

Interesting results

- Matsushita
  - $K_{I_2}(D_{2\times n})$
  - homotopy type of bouquet of spheres
- S. Goyal, S. Shukla, A. Singh
  - $K_{I_2}(D_{3\times n})$
- Marija Jelić Milutinović

Interesting results

- Matsushita
  - $K_{I_2}(D_{2\times n})$
  - homotopy type of bouquet of spheres
- S. Goyal, S. Shukla, A. Singh
  - $K_{I_2}(D_{3\times n})$
- Marija Jelić Milutinović
  - specific trees (caterpillar, binary tress)

Interesting results

- Matsushita
  - $K_{I_2}(D_{2\times n})$
  - homotopy type of bouquet of spheres
- S. Goyal, S. Shukla, A. Singh
  - $K_{I_2}(D_{3\times n})$
- Marija Jelić Milutinović
  - specific trees (caterpillar, binary tress)
- generalization of Kozlov results

- Liđan, Baralić (2022)

Theorem
Simplicial complex of tilings $K_{l_1,\ldots,l_k}(D_{1\times n})$ has a homotopy type of bouquet of spheres.

Theorem
Simplicial complex of tilings $K_{l_1,\ldots,l_k}(\mathbb{T}_{1\times n})$ has a homotopy type of bouquet of spheres.

- For natural numbers m and n and finite set of polyomino shapes $\mathcal{T}$ simplicial complex $K_{\mathcal{T}}(D_{m\times n})$ and $K_{\mathcal{T}}(\mathbb{T}_{m\times n})$ has homotopy type of bouquet of spheres.
- matching complex

References

[1] D. Kozlov *Complexes of Directed Trees*, J. Comb. Theory, Ser. A88(1): 112–122 (1999)

[2] E. Liđan (2022) *Topological characteristics of generalized polyomino tilings*, PhD thesis, Faculty of Mathematics and Natural Science, Universty of Montenegro, Podgorica.

[3] T. Matsushita, *Matching Complexes of Small Grids*, The Electronic Journal of Combinatorics (2019), Volume **26**, Issue 3

[4] M. Jelić Milutinović, *Kombinatorna topologija i grafovski kompleksi*, Doktorska disertacija, MF, Beograd, 2021.

[5] M. Jelić Milutinović, , H. Jenne, A. McDonough, and J.Vega, *Matching complexes of trees and applications of the matching tree algorithm*, preprint

[6] S. Goyal, S. Shukla, A. Singh, *Matching complexes of* $3 \times n$ *grid graphs*, In: ArXiv e–prints 2106.09915 (June 2021). `https://arxiv.org/pdf/2106.09915.pdf`

[7] Sage 9.0 documentation

Edin Liđan
International Burch University
Sarajevo, Bosnia and Herzegovina

This is joint work with Đorđe Baralić

edin.lidan@ibu.edu.b

Thank you for your attention.
Questions?