Triangle of numbers arisen from modular forms

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Previous work

- I. Kodrnja, H. Koncul, *Polynomials vanishing on a basis of* $S_m(\Gamma_0(N))$, Glasnik matematički, to appear
- compute the homogeneous polynomials that vanish on cuspidal modular forms
- how many are there

Basics

- $N > 1$ and $m \geq 2$ even number
- $f_0, \dots, f_{t-1} \in S_m(\Gamma_0(N))$ basis of the space of cuspidal modular forms for the congruence subgroup $\Gamma_0(N)$ of weight m, $\dim S_m(\Gamma_0(N)) = t$
- $X_0(N)$ modular curve for $\Gamma_0(N)$
- holomorphic map $X_0(N) \to \mathbb{P}^{t-1}$

$$\mathfrak{a}_z \mapsto (f_0(z) : \cdots : f_{t-1}(z))$$

- image curve $\mathcal{C}(N, m) \subseteq \mathbb{P}^{t-1}$
- $g = f_{t-1}$

$$\mathfrak{a}_z \mapsto (f_0(z)/g(z) : \cdots : f_{t-1}(z)/g(z))$$

Algorithm

- $P \in \mathbb{Q}[x_0, \dots, x_{t-1}]$ homogeneous polynomial of degree d

$$P(x_0, \dots, x_{t-1}) = \sum_{\substack{0 \le i_0, \dots, i_{t-1} \le d \\ i_0 + \cdots + i_{t-1} = d}} a_{i_0, \dots, i_{t-1}} x_0^{i_0} \cdots x_{t-1}^{i_{t-1}}.$$

$$P(f_0(z), \cdots, f_{t-1}(z)) = \sum_{\substack{0 \le i_0, \dots, i_{t-1} \le d \\ i_0 + \cdots + i_{t-1} = d}} a_{i_0, \dots, i_{t-1}} f_0^{i_0} \cdots f_{t-1}^{i_{t-1}} = 0$$

$$I = \{(i_0, \dots, i_{t-1}) : 0 \le i_0, \dots, i_{t-1} \le d, i_0 + \cdots + i_{t-1} = d\}$$

- cardinality of $|I|$ is the weak composition problem

$$d' = \dim \mathcal{P}_d = |I| = \binom{d + t - 1}{d}$$

- for fixed values of d, N, m solve a homogeneous system of equations
- unknowns are coefficients of a polynomial P, $a_0, \ldots, a_{d'-1}$
- coefficients of the system are values of q-expansions of evaluated monomials $f_0^{i_0} \ldots f_t^{i_{t-1}}$ over the indexing set I,

$$
\begin{aligned}
P(f_0, \ldots, f_{t-1}) &= \sum_{\substack{0 \le i_0, \ldots, i_{t-1} \le d \\ i_0 + \cdots + i_{t-1} = d}} a_{i_0, \ldots, i_{t-1}} f_0^{i_0} \cdots f_{t-1}^{i_{t-1}} \\
&= \sum_{\substack{0 \le i_0, \ldots, i_{t-1} \le d \\ i_0 + \cdots + i_{t-1} = d}} a_{i_0, \ldots, i_{t-1}} \left(a_0^{(i_0, \ldots, i_{t-1})} + a_1^{(i_0, \ldots, i_{t-1})} q + \ldots \right) \\
&= p_0 + p_1 q + p_2 q^2 + \cdots
\end{aligned}
$$

- homogeneous system $p_0 = p_1 = \cdots = p_{B_{md}} = 0$
- solutions are obtained as the basis of the right kernel of the transpose of $d' \times B_{md}$ matrix whose rows are made of coefficients of $f_0^{i_0} \ldots f_{t-1}^{i_{t-1}}$, after ordering the index set I

For a given N and weight m, with the use of lexicographic ordering on the set of monomials of degree d:

Input: q-expansions of $f_0, \ldots, f_{t-1}$ basis of $S_m(\Gamma_0(N))$

- for a degree $d \geq 0$:
  - for each monomial index $(i_0, \ldots, i_{t-1}) \in I$ in the ordered set of monomials of degree d:

    compute $f_0^{i_0} \ldots f_{t-1}^{i_{t-1}}$
  - create a $d' \times B_{md}$ matrix A, whose rows are first B_{md} coefficients of q-expansion of $f_0^{i_0} \ldots f_{t-1}^{i_{t-1}}$
  - return the dimension (or the elements) of the right kernel of A

Output: beginning part of an array containing the number of linearly independent homogeneous polynomials of degree $d \geq 0$ vanishing on all forms, i.e. such that $P(f_0, \ldots, f_{t-1}) = 0$.

Table: (N, m) for $2 \le \dim S_m(\Gamma_0(N)) \le 8$

t	g	(N,m)
2	0	(2,12), (2,14), (3,10), (4,8)
	1	(11,4)
	2	(22,2), (23,2), (26,2), (28,2), (29,2), (31,2), (37,2), (50,2)
3	0	(2,16), (2,18), (3,12), (3,14), (4,10), (5,8), (5,10), (6,6), (7,6), (7,8), (8,6), (9,6), (10,4), (12,4), (13,4), (16,4)
	3	(30,2), (33,2), (34,2), (35,2), (39,2), (40,2), (41,2), (43,2), (45,2), (48,2), (64,2)
4	0	(2,20), (2,22), (3,16), (4,12)
	1	(14,4), (15,4), (17,4), (19,4), (11,6)
	4	(38,2), (44,2), (47,2), (53,2), (54,2), (61,2), (81,2)
5	0	(2,24), (2,26), (3,18), (3,20), (4,14), (5,12), (5,14), (6,8), (7,10), (8,8), (9,8), (10,6), (13,6), (18,4), (25,4)
	2	(23,4)
	5	(42,2), (46,2), (51,2), (52,2), (55,2), (56,2), (57,2) (59,2), (63,2), (65,2), (67,2), (72,2), (73,2), (75,2)
6	0	(2,28),(2,30), (3,22), (4,16)
	1	(11,8), (17,6), (20,4), (21,4), (27,4)
	6	(58,2), (71,2), (79,2)
7	0	(2,32), (2,34), (3,24), (3,26), (4,18), (5,16), (5,18) (6,10), (7,12), (7,14), (8,10), (9,10), (12,6), (13,8), (16,6)
	2	(22,4), (29,4), (31,4)
	7	(60,2), (62,2), (68,2), (69,2), (77,2), (80,2), (83,2), (85,2), (89,2), (91,2), (97,2), (98,2)
8	0	(2,36), (2,38), (3,28), (4,20)
	1	(11,10), (14,6), (15,6), (19,6), (24,4), (32,4)
	8	(74,2), (76,2)

Table: Number of polynomials for $2 \leq t \leq 8$ and $2 \leq d \leq 10$

t	g	degree d of P: 2	3	4	5	6	7	8	9	10
2	0	0	0	0	0	0	0	0	0	0
	1	0	0	0	0	0	0	0	0	0
	2	0	0	0	0	0	0	0	0	0
3	0	1	3	6	10	15	21	28	36	45
	3	0	0	1	3	6	10	15	21	28
4	0	3	10	22	40	65	98	140	192	255
	1	2	8	19	36	60	92	133	184	246
	4	1	5	14	29	51	81	120	169	220
5	0	6	22	53	105	185	301	462	678	960
	2	4	18	47	97	175	289	448	662	942
	5	3	15	42	90	166	278	435	646	925
6	0	10	40	105	226	431	756	1246	1956	2952
	1	9	38	102	222	426	750	1239	1948	2943
	6	6	31	91	207	407	727	1212	1917	2908
7	0	15	65	185	431	887	1673	2954	4950	7947
	2	13	61	179	423	877	1661	2940	4934	7929
	7	10	54	168	408	858	1638	2913	4903	7894
8	0	21	98	301	756	1673	3382	6378	11376	19377
	1	20	96	298	752	1668	3376	6371	11368	19368
	8	15	85	281	729	1639	3341	6330	11321	19315

Table: Number of polynomials for $g = 0$

t	d								
	2	3	4	5	6	7	8	9	10
3	1	3	6	10	15	21	28	36	45
4	3	10	22	40	65	98	140	192	255
5	6	22	53	105	185	301	462	678	960
6	10	40	105	226	431	756	1246	1956	2952
7	15	65	185	431	887	1673	2954	4950	7947
8	21	98	301	756	1673	3382	6378	11376	19377

- notation $T(n, m)$ or $T_{n,m}$
- OEIS database - Number sequence A124326
- Pascal's triangle − rascal triangle
- $T(n, m) = A007318(n, m) - A077028(n, m)$
- $T(n, m) = \binom{n}{m} - m(n - m) - 1, n, m \geq 0$
- resulting from adding the row number (starting with 0) of Pascal's triangle to each entry in that row, subtracting the corresponding entries in the triangle formed by taking the finite diagonals in the multiplication table in order of increasing length (A003991), and removing the outer two layers

							0							
						0	**1**	0						
					0	**2**	1	**2**	0					
				0	**3**	3	**4**	3	**3**	0				
			0	**4**	6	**6**	10	**6**	6	**4**	0			
		0	**5**	10	**8**	22	**9**	22	**8**	10	**5**	0		
	0	**6**	15	**10**	40	**12**	53	**12**	40	**10**	15	**6**	0	
0	**7**	21	**12**	65	**15**	105	**16**	105	**15**	65	**15**	21	**7**	0

Figure: Extended triangle

$$N(t,d) = N(t-1,d) + N(t,d-1) + (t-2)(d-1), \qquad t \geq 3, d \geq 2$$

$$N(t,d) = N(t-1,d) + N(t,d-1) + td, \qquad t, d \geq 0$$

Pascal triangle

$$N_h(t,d) = N_h(t-1,d) + N_h(t,d-1)$$

$$N_t(x) = \sum_{k\geq 0} N_h(t,d)x^k = N_{t-1}(x) + xN_t(x) \Rightarrow$$

$$N_t(x) = \frac{1}{1-x}N_{t-1}(x) = \frac{1}{(1-x)^t}$$

where $N_h(t,d)$ is the coefficient of x^t in $N_t(x)$ is

$$N_h(t,d) = \binom{t+d-1}{d},$$

particular solution:

$$\begin{aligned} N_p(t,d) &= (C_1 t + C_2)(D_1 d + D_2) + E \\ &= C_1 D_1 td + C_1 D_2 t + D_1 C_2 d + C_2 D_2 + E \end{aligned}$$

$$C_1 = -1,\ C_2 = 1,\ D_1 = 1,\ D_2 = 0,\ E = -1.$$

substitution $n = t + d - 1,\ m = d \rightarrow$ OEIS

Theorem

Numbers in Table for genus 0 satisfy the formula:

$$N(t,d) = \binom{t+d-1}{d} - (t-1)d - 1$$

Theorem

Let $t = \dim S_m(\Gamma_0(N)) \geq 3$*,* g *be genus of* $\Gamma_0(N)$*. Let* $N \geq 2$ *be such that* $g = 0$ *or* $g = t$ *and* $X_0(N)$ *is hyperelliptic. Then the number* $N(t,d)$ *of homogeneous polynomials of degree* $d \geq 0$ *that vanish on cuspidal modular forms is equal to*

$$N(t,d) = \binom{t+d-1}{d} - (t-1)d - 1.$$

Theorem

Let $t = \dim S_m(\Gamma_0(N)) \geq 3$*,* g *be genus of* $\Gamma_0(N)$*. Then the number* $N(t,d)$ *of homogeneous polynomials of degree* $d \geq 0$ *that vanish on cuspidal modular forms is equal to*

- $g = \{1, 2\}$

$$N(t,d) = \binom{t+d-1}{d} - (g+t-1)d - 1 + g \quad (1)$$

- $g > 2$

$$N(t,d) = \binom{t+d-1}{d} - (g+t-2)d - 1 + g \quad (2)$$

Rascal triangle

1
1 1
1 2 1
1 3 3 1
1 4 5 4 1
1 5 7 7 5 1
1 6 9 10 9 6 1
1 7 11 13 13 11 7 1

- diamond rule
- $S \times N = E \times W - 1$
- $S + N = E + W - 1$
- $T(n, m) = m(n - m) + 1,\ n \geq 0,\ 0 \leq m \leq n$

Generalized rascal triangle

Definition

Let $\overline{d}, D \in \mathbb{Z}$ and let T be a number triangle. If the interior numbers satisfy

$$T_{t,d} \times T_{t-1,d-1} = T_{t-1,d} \times T_{t,d-1} + D$$

we call this a **rascal-like multiplication rule** *with multiplicative constant D; and if the interior numbers satisfy*

$$T_{t,d} + T_{t-1,d-1} = T_{t-1,d} + T_{t,d-1} + \overline{d}$$

that will be called **a rascal-like addition rule** *with additive constant $\overline{d}$.*

Definition

Let $c, \overline{d}, d_1, d_2 \in \mathbb{Z}$. A number triangle $T(c, \overline{d}, d_1, d_2)$ is called generalized rascal triangle if

$$T_{t,d} = c + dd_1 + td_2 + td\overline{d}$$

for all $t, d \geq 0$.

- $c = T_{0,0}$
- $d_1 = T_{0,d+1} - T_{0,d}$
- $d_2 = T_{t+1,0} - T_{t,0}$
- $\overline{d} = (T_{t+1,d+1} - T_{t+1,d}) - (T_{t,d+1} - T_{t,d}) = (T_{t+1,d+1} - T_{t,d+1}) - (T_{t+1,d} - T_{t,d})$
- rascal triangle $T(1, 1, 0, 0)$

Teorem (Hotchkiss 2020)

Let $c, \overline{d}, d_1, d_2 \in \mathbb{Z}$ *and* T *be a number triangle with arithmetic sequences* $T_{0,d} = c + dd_1$ *and* $T_{t,0} = c + td_2$ *on the exterior diagonals. Th* T *is the generalized rascal triangle* $T(c, \overline{d}, d_1, d_2)$ *and if and only if*

$$T_{t,d} = c + dd_1 + td_2 + td\overline{d} \quad \forall t, d \geq 0,$$

$$T_{t,d} + T_{t-1,d-1} = T_{t,d-1} + T_{t-1,d} + \overline{d},$$

and whenever $T_{t-1,d-1} \neq 0$

$$T_{t,d} \times T_{t-1,d-1} = T_{t,d-1} \times T_{t-1,d} + D$$

where $D = c\overline{d} - d_1 d_2$.

Theorem

Let $t = \dim S_m(\Gamma_0(N)) \geq 3$*,* g *be genus of* $\Gamma_0(N)$*. Then the number* $N(t,d)$ *of homogeneous polynomials of degree* $d \geq 0$ *is equal to*

- $g = \{1, 2\}$

$$N(t,d) = \binom{t+d-1}{d} - (g+t-1)d - 1 + g \quad (1)$$

- $g > 2$

$$N(t,d) = \binom{t+d-1}{d} - (g+t-2)d - 1 + g \quad (2)$$

$(1) \to c = 1-g,\ \overline{d} = 1,\ d_1 = g-1,\ d_2 = 0$

- $g = 0,\ D = 1,\ \ T(1,1,0,0)$
- $g = 1,\ D = 0,\ T_{td} = T_{td-1} + T_{t-1d} + td - 1$
- $g = 2,\ D = -1\ T_{td} = T_{td-1} + T_{t-1d} + (t+1)d - 2$
- $g = 3,\ D = -2\ T_{td} = T_{td-1} + T_{t-1d} + (t+2)d - 3$
- $\cdots$
- $g = A,\ D = 1-A,\ T_{td} = T_{td-1} + T_{t-1d} + (t+A-1)d - A$
- $T_{t,0} = g,\ T_{0,d} = g\{1, 0, -1, -2, -3\cdots\}$ arithmetic sequence starting with g and decreasing by g

					1					
				1	**-1**	0				
			1	**-1**	0	**0**	-1			
		1	**-1**	0	**1**	-1	**1**	2		
	1	**-1**	0	**2**	0	**3**	-2	**2**	-3	
1	**-1**	0	**3**	2	**5**	1	**5**	-3	**3**	-4

Figure: (1) for $g = 1$

					2					
				2	**-2**	0				
			2	**-2**	0	**0**	-2			
		2	**-2**	0	**1**	-2	**2**	-4		
	2	**-2**	0	**2**	-1	**4**	-4	**4**	-6	
2	**-2**	0	**3**	1	**6**	-1	**7**	-6	**6**	-8

Figure: (1) for $g = 2$

$(2) \to c = 1 - g,\ \overline{d} = 1,\ d_1 = g - 2,\ d_2 = 0$

- $g = 3,\ D = -2\ T_{td} = T_{td-1} + T_{t-1d} + (t+1)d - 3$
- $g = 4,\ D = -3\ T_{td} = T_{td-1} + T_{t-1d} + (t+2)d - 2$
- $\cdots$
- $g = B,\ D = 1 - B,\ T_{td} = T_{td-1} + T_{t-1d} + (t+B-2)d - A$
- $T_{t,0} = g,\ T_{0,d} =$ arithmetic sequence starting with g and decreasing by g-1

3

3 **-3** 1

3 **-3** 1 **-1** -1

3 **-3** 1 **0** -1 **1** -3

3 **-3** 1 **1** 0 **3** -3 **3** -5

3 **-3** 1 **2** 2 **5** 0 **6** -5 **5** -7

Figure: (2) for $g = 3$

4

4 **-4** 1

4 **-4** 1 **-1** -2

4 **-4** 1 **0** -2 **2** -5

4 **-4** 1 **1** -1 **4** -5 **5** -8

4 **-4** 1 **2** 1 **6** -2 **8** -8 **8** -11

Figure: (2) for $g = 4$

(1) for $g = t \to c = 1,\ \overline{d} = 2,\ d_1 = -1,\ d_2 = -1 \to D = 1$

- $T_{td} = T_{td-1} + T_{t-1d} + d(2t-1) - t - 1$
- $T_{t,0} = \{1, 2, 3, 4, 5 \cdots\},\ T_{0,d} = \{1, 0, -1, -2, -3 \cdots\}$

(2) for $g = t \to c = 1,\ \overline{d} = 2,\ d_1 = -2,\ d_2 = -1 \to D = 0$

- $T_{td} = T_{td-1} + T_{t-1d} + 2d(t+2) - t - 1$
- $T_{t,0} = \{1, 2, 3, 4 \cdots\},\ T_{0,d} = \{1, 1, 1, 1, 1, \cdots\}$

1
2 **-2** 0
3 **-3** 0 **-1** -1
4 **-4** 0 **0** -2 **0** -2
5 **-5** 0 **1** -2 **3** -4 **1** -3
6 **-6** 0 **2** -1 **6** -3 **6** -6 **2** -4

Figure: (1) for $g = t$

1
2 **-2** 1
3 **-3** 1 **-2** -1
4 **-4** 1 **-1** 0 **-2** 1
5 **-5** 1 **0** 0 **1** -1 **-2** 1
6 **-6** 1 **1** 1 **4** 0 **3** -2 **-2** 1

Figure: (2) for $g = t$

						a						
					a	**0**	a					
				a	**-0**	2a	**0**	a				
			a	**0**	3a	**1**	3a	**0**	a			
		a	**0**	4a	**2**	6a+1	**2**	4a	**0**	a		
	a	**0**	5a	**3**	10a+3	**4**	10a+3	**3**	5a	**0**	a	
a	**0**	6a	**4**	15a+ 6	**6**	20a+10	**6**	15a+6	**4**	6a	**0**	a

- $T_{td} = T_{td-1} + T_{t-1d} + td$
- for $a = 1 \rightarrow$ A130671 $T(n, m) = 2\binom{m}{n} - (1 + n(m - n)),\ n, m \geq 0$
- $T_{td} = (a + 1)\binom{t}{d} - (1 + t(d + t)),\ t, d \geq 0$
- what if outer minor and major diagonals are not equal?
  $T_{t0} = b,\ T_{0d} = a$

Thank you for your attention!

- F. Diamond, J. Shurman, A First Course in Modular Forms, Graduate Texts in Mathematics, Springer-Verlag, New York, 2005
- P. K. Hotchkiss, Generalized Rascal Triangles, Journal of Integer Sequences, Vol. 23 (2020), 20.7.4
- I. Kodrnja, G. Muić, On primitive elements of algebraic fields and models of X0(N), The Ramanujan Journal 55(2) (2021), 393-420
- I. Kodrnja, H. Koncul, Polynomials vanishing on a basis of $S_m(\Gamma_0(N))$, to apper in Glasnik Matematički
- G. Muić, On embeddings of modular curves in projective spaces, Monatsh. Math. 173(2) 2014), 239-256
- OEIS (The Online Encyclopedia of Integer Sequences) , url: `https://oeis.org/`, visited on 25.3.2024