Projective curves and generalized rascal triangles

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Rascal triangle

A rascal triangle (A077028 in the Online Encyclopedia of Integer Sequences) is a number triangle generated by formula

$$R(n,k) = k(n-k) + 1, \quad n \geq 0, 0 \leq k \leq n. \tag{1}$$

$n=0$:								1								
$n=1$:							1		1							
$n=2$:						1		2		1						
$n=3$:					1		3		3		1					
$n=4$:				1		4		**5**		4		1				
$n=5$:			1		5		**7**		**7**		5		1			
$n=6$:		1		6		9		**10**		9		6		1		
$n=7$:	1		7		11		13		13		11		7		1	

Figure: The rascal triangle $R(n,k)$

- Product rule, *the diamond formula*

$$\text{South}=(\text{East}\cdot\text{West}+1)/\text{North}, \qquad 10 = \frac{7\cdot 7+1}{5}$$

- Addition rule

$$\text{South}=\text{East}+\text{West}-\text{North}+1, \qquad 10 = 7+7+1-5$$

Hyperplanes over projective curves

$\mathcal{C} \subseteq \mathbb{P}^{t-1}$, $\dim \mathcal{C} = 1$
$\mathcal{P} = \mathbb{Q}[X_0, \ldots, X_{t-1}]$ the ring of polynomials in t variables
$\mathcal{P}_d = \mathbb{Q}[X_0, \ldots, X_{t-1}]_d$ subring of homogeneous polynomials of degree d
graded ring structure

$$\mathcal{P} = \bigoplus_{d \geq 0} \mathcal{P}_d$$

$I(\mathcal{C}) \subseteq \mathcal{P}$ homogeneous ideal of the curve $\mathcal{C}$ consisting of all homogenous polynomials that vanish on $\mathcal{C}$.
graded structure

$$I(\mathcal{C}) = \bigoplus_{d \geq 0} I(\mathcal{C})_d$$

with finitely generated vector spaces $I(\mathcal{C})_d = \mathcal{P}_d \cap I(\mathcal{C})$

Dimensions

Indexing set of the set of monomials of degree d

$$I = \{(i_0, \dots, i_{t-1}) : 0 \leq i_0, \dots, i_{t-1} \leq d, i_0 + \cdots + i_{t-1} = d\}.$$

$$\dim \mathcal{P}_d = |I| = \binom{d+t-1}{d} \tag{2}$$

Hilbert function

$$HF_{\mathcal{C}_{N,m}}(d) = HF_{\mathcal{P}/I(\mathcal{C}_{N,m})}(d) = \dim \mathcal{P}_d - \dim I_d \tag{3}$$

Hilbert polynomial of projective curve is linear polynomial

$$HP_{\mathcal{C}_{N,m}}(d) = \deg(\mathcal{C}_{N,m})d + 1 - g \tag{4}$$

regularity index ...

$$\min\left\{d : HF_{\mathcal{C}_{N,m}}(d) = HP_{\mathcal{C}_{N,m}}(d)\right\} \leq 2$$

Pascal triangle as numbers of homogeneous polynomials

$$P(n,k) = \binom{n+k}{k}, \quad n \geq 0, k \geq 0. \tag{5}$$

$n=0$:								1								
$n=1$:							1		1							
$n=2$:						1		2		1						
$n=3$:					1		3		3		1					
$n=4$:				1		4		6		4		1				
$n=5$:			1		5		10		10		5		1			
$n=6$:		1		6		15		20		15		6		1		
$n=7$:	1		7		21		35		35		21		7		1	

Figure: The Pascal triangle $P(n-k,k), \quad n \geq 0, 0 \leq k \leq n$

Now we can interpret the diagonals of the Pascal's triangle as numbers of linearly independent homogeneous polynomials, so $P(n,k)$ is the number of homogeneous polynomials with $n+1$ variables of degree k.

Smooth nondegenerate projective curves

Let $\mathcal{C} \subset \mathbb{P}^n$ be a nondegenerate smooth projective curve of degree $\mathbf{deg}\,\mathcal{C}$ and genus g. Genus is an intrinsic invariant of the curve, but degree depends on the embedding of the curve in projective space.
First, we must have

$$\mathbf{deg}\,\mathcal{C} \geq n,$$

and curves of minimal degree $\mathbf{deg}\,\mathcal{C} = n$ are rational normal curves of genus 0.
Second, as a consequence of the Riemann-Roch Theorem we have the Castelnouovo's bound for genus:

$$\mathbf{deg}\,\mathcal{C} - 1 = m(n-1) + \varepsilon, \quad m \geq 1,\ 0 \leq \varepsilon \leq n-1 \quad \Longrightarrow \qquad (6)$$
$$g \leq (n-1)\frac{m(m-1)}{2} + m\varepsilon$$

Hilbert polynomial as generalized rascal triangle

We define generalized rascal triangles $\mathcal{T}_{m,\varepsilon,g}$ for smooth nondegenerate projective curves with genus g as

$$\mathcal{T}_{m,\varepsilon,g} = \left\{ T_{m,\varepsilon,g}(1-g, 1-m+\varepsilon, 0, m) : 0 \leq g \leq (n-1)\frac{m(m-1)}{2} + m\varepsilon \right\} \tag{7}$$

where

$$T_{m,\varepsilon,g}(n,k) = (m(n-1)+1+\varepsilon)k + 1 - g, \quad n,k \geq 0. \tag{8}$$

The rascal triangle is $T_{1,0,0}(1,0,0,1)$ and is the only member of $\mathcal{T}_n$ related to rational normal curves of degree n.

Theorem

All triangles $T_{m,\varepsilon,g}$ can be reproduced from the rascal triangle $T_{1,0,0}$.

i) *A triangle $T_{m,\varepsilon,g}$ can be obtained from $T_{m,\varepsilon,0}$ by subtracting g from every entry.*

ii) *A triangle $T_{m,\varepsilon,g}$ can be obtained from $T_{m,\varepsilon-1,g}$ if $\varepsilon \leq n-1$ or from $T_{m-1,0,g}$ if $m \geq 2$ and $\varepsilon = n-1$ by adding k to each entry.*

An example

$$
\begin{array}{ccccccccccccccccccc}
 & & & & & & & & & 2 & & & & & & & & & \\
 & & & & & & & & 1 & & 1 & & & & & & & & \\
 & & & & & & & 0 & & 2 & & 0 & & & & & & & \\
 & & & & & & -1 & & 3 & & 3 & & -1 & & & & & & \\
 & & & & & -2 & & 4 & & 6 & & 4 & & -2 & & & & & \\
 & & & & -3 & & 5 & & \mathbf{9} & & 9 & & 5 & & -3 & & & & \\
 & & & -4 & & 6 & & \mathbf{12} & & \mathbf{14} & & 12 & & 6 & & -4 & & & \\
 & & -5 & & 7 & & 15 & & \mathbf{19} & & 19 & & 15 & & 7 & & -5 & & \\
 & -6 & & 8 & & 18 & & 24 & & 26 & & 24 & & 18 & & 8 & & -6 & \\
-7 & & 9 & & 21 & & 29 & & 33 & & 33 & & 29 & & 21 & & 9 & & -7
\end{array}
$$

Figure: $T_{2,0,n-1}(n, k)$ for curves of degree $2n - 1$ and $g = n - 1$

- rascal-like multiplication rule

$$T(n, k) = \frac{T(n-1, k)T(n, k-1) + D}{T(n-1, k-1)}, \quad \text{with } D = ad - bc \quad (9)$$

- rascal-like addition rule

$$T(n, k) = T(n, k-1) + T(n-1, k) - T(n-1, k-1) + d. \quad (10)$$

The addition rule

$$\Delta HF_{\mathcal{C}}(k) = HF_{\mathcal{C}}(k) - HF_{\mathcal{C}}(k-1)$$

This is the rascal-like addition formula, with
$T_{a,b,c,d} = T_{m,\varepsilon,g}(n,k) = HF_{\mathcal{C}}(k)$

$$T(n,k) - T(n,k-1) = T(n-1,k) - T(n-1,k-1) + d \qquad (11)$$
$$\Delta HF_{\mathcal{C}_n}(k) = \Delta HF_{\mathcal{C}_{n-1}}(k) + d$$

where $d = m$, (6). This is also the difference in arithmetic series that are diagonals of our generalized rascal triangles, and the increase in the degree of curves that the antidiagonals represent.

Numbers of polynomials

Definition

Let $\mathcal{C} \subset \mathbb{P}^n$ be a nondegenerate smooth projective curve of degree $\deg \mathcal{C}$ *and let $m \geq 1$, $0 \leq \varepsilon \leq n-1$. We define triangles $\mathcal{N}_{m,\varepsilon,g}$ for smooth nondegenerate projective curves with fixed genus g as*

$$\mathcal{N}_{m,\varepsilon,g}(n,k) = \binom{n+k}{k} - T_{m,\varepsilon,g}(n,k), \quad n,k \geq 0. \tag{12}$$

Corollary 1

Antidiagonals of the triangle $\mathcal{N}_{m,\varepsilon,g}$ are numbers of linearly independent homogeneous polynomials of degree $k \geq ri(HF_{\mathcal{C}})$ that vanish on a smooth nondegenerate curve $\mathcal{C} \subset \mathbb{P}^n$ of genus g and degree $\deg \mathcal{C} = m(n-1)+1+\varepsilon$ *i.e., the number of hypersufaces of degree k that contain the curve $\mathcal{C}$.*

Canonical curves

We define triangles $\mathcal{N}_{m,\varepsilon,g}$ for smooth nondegenerate projective curves with fixed genus g as

$$\mathcal{N}_{m,\varepsilon,g}(n,k) = \binom{n+k}{k} - T_{m,\varepsilon,g}(n,k), \quad n,k \geq 0. \tag{13}$$

1									
2	1								
3	1	1							
4	1	0	1						
5	1	**0**	−1	1					
6	1	**1**	**0**	−2	1				
7	1	**3**	**5**	**1**	−3	1			
8	1	**6**	**15**	**14**	**3**	−4	1		
9	1	**10**	**31**	**42**	**29**	**6**	−5	1	
10	1	**15**	**54**	**91**	**90**	**51**	**10**	−6	1

Figure: Canonical curves $\mathcal{N}_{2,1,n+1}(n,k)$

Thank you for your attention!