On equienergetic regular graphs by means of their spectral distances

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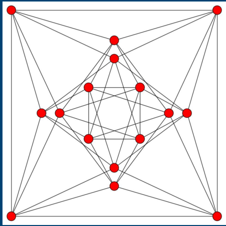
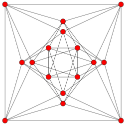


Graph and its spectrum

Let G be a finite simple graph of order n and size m, with the set of vertices $V(G) = \{v_1, v_2, \ldots, v_n\}$, and the set of edges $E(G) = \{e_1, e_2, \ldots, e_m\}$.



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The adjacency matrix of G is the $n \times n$ matrix $A(G) = (a_{ij})$, where $a_{ij} = 1$ if there is an edge between the vertices v_i and v_j, and $a_{ij} = 0$, otherwise.

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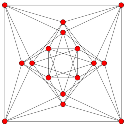
The characteristic polynomial $P_G(x) = \det(xI_n - A(G))$ of G is the characteristic polynomial of its adjacency matrix $A(G)$.



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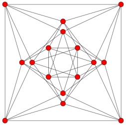
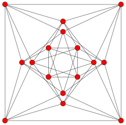
The (adjacency) eigenvalues $\lambda_1(G) \geqslant \cdots \geqslant \lambda_n(G)$ of G are the eigenvalues of $A(G)$, and they form the (adjacency) spectrum of G.





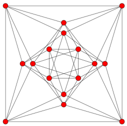
Ivan Gutman, The energy of a graph, Ber. Math. -Statist. Sekt. Forschungsz. Graz, 103 (1978) 1–22.

Definition

The energy $\mathcal{E}(G)$ of a n-vertex graph G is:

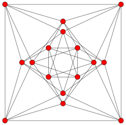
$$\mathcal{E}(G) = \sum_{i=1}^{n} |\lambda_i(G)|,$$

where $\lambda_i(G)$, $i = 1, 2, \ldots, n$, are the adjacency eigenvalues of G.



Definition

Two graphs G_1 and G_2 (with the same number of vertices) are said to be equienergetic if they have the same energy, i.e. $\mathcal{E}(G_1) = \mathcal{E}(G_2)$.



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A research task:

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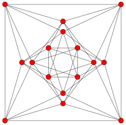

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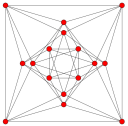
- Two isomorphic or two cospectral graphs are obviously equienergetic.



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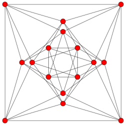
- Two isomorphic or two cospectral graphs are obviously equienergetic.
- Union of a graph G and an arbitrary number of isolated vertices has the same energy as G, and it is not cospectral to G.



There are far less trivial cases of pairs of non-cospectral connected equienergetic graphs, and it is of interest to find examples of such graphs.

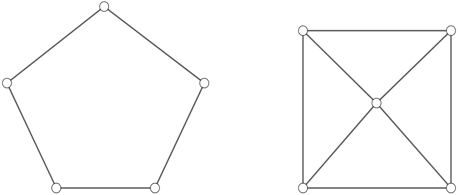
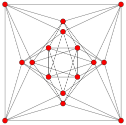


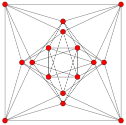
Figure: The smallest pair of non-cospectral connected equienergetic graphs with the same number of vertices.



Definition (D. Stevanović, Energy and NEPS of graphs, Lin. Multilin. Algebra 53 (2005) 67-74.)

Let $\mathcal{B}$ be a set of binary n-tuples, i.e. $\mathcal{B} \subseteq \{0,1\}^n \setminus \{(0,\ldots,0)\}$ such that for every $i = 1,\ldots,n$ there exists $\beta = (\beta_1,\ldots,\beta_n) \in \mathcal{B}$ with $\beta_i = 1$. The non-complete extended p-sum (NEPS) of graphs $G_1,\ldots,G_n$ with basis $\mathcal{B}$, denoted by NEPS$(G_1,\ldots,G_n;\mathcal{B})$, is the graph with the vertex set $V(G_1)\times\cdots\times V(G_n)$, in which two vertices $(u_1,\ldots,u_n)$ and $(v_1,\ldots,v_n)$ are adjacent if and only if there exists $(\beta_1,\ldots,\beta_n) \in \mathcal{B}$ such that u_i is adjacent to v_i in G_i whenever $\beta_i = 1$, and $u_i = v_i$ whenever $\beta_i = 0$.

Graphs $G_1,\ldots,G_n$ are called the factors of NEPS.



The **product** $G_1 \times G_2 \times \cdots \times G_n$ of graphs $G_1, G_2, \ldots, G_n$, also called **direct product**, is NEPS of these graphs with basis $\mathcal{B} = \{(1, 1, \ldots, 1)\}$.

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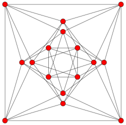
The sum $G_1 + G_2 + \cdots + G_n$ of graphs $G_1, G_2, \ldots, G_n$, also called Cartesian product, is NEPS of these graphs with basis consisting of n-tuples of the form $(0, \ldots, 0, 1, 0, \cdots, 0)$ with 1 on i-th place, for each $i = 1, 2, \ldots, n$.



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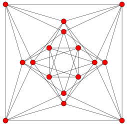
Theorem (D. Cvetković, M. Doob, H. Sachs, Spectra of Graphs - Theory and Application, Johann Ambrosius Barth Verlag, Heidelberg-Leipzig, 1995.)

The spectrum of NEPS$(G_1, \dots, G_n; \mathcal{B})$ consists of all possible values Λ given by $\Lambda = \sum_{\beta \in \mathcal{B}} \lambda_1^{\beta_1} \cdots \lambda_n^{\beta_n}$, where λ_i is an arbitrary eigenvalue of G_i, $i = 1, \dots, n$.



R. Balakrishnan, The energy of a graph, Lin. Algebra Appl. 387 (2004) 287-295.

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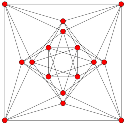
Corollary

For any two graphs G_1 and G_2, it holds

$$\mathcal{E}(G_1 \times G_2) = \mathcal{E}(G_1)\,\mathcal{E}(G_2).$$



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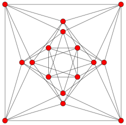

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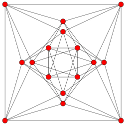
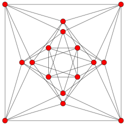
Let G be a pentagon, H a four-sided pyramid, and for each $n \in \mathbb{N}$ and $i = 1, 2, \ldots, n$, let

$$\mathcal{G}_{n,i} = \underbrace{G \times \cdots \times G}_{i \text{ times}} \times \underbrace{H \times \cdots \times H}_{n-i \text{ times}}.$$

Graphs $\mathcal{G}_{n,i}$, $i = 1, 2, \ldots, n$, form a family of n mutually non-cospectral connected equienergetic graphs having the same number of vertices.

H. S. Ramane, H. B. Walikar, S. B. Rao, B. D. Acharya, P. R. Hampiholi, S. R. Jog, I. Gutman, Spectra and energies of iterated line graphs of regular graphs, Applied Mathematics Letters 18 (2005) 679-682.





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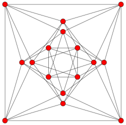
The iterated line graphs $L^k(G)$, $k \geqslant 0$, of a graph G are defined recursively as $L^2(G) = L(L(G))$, $L^3(G) = L(L^2(G))$, $\ldots$, $L^k(G) = L(L^{k-1}(G))$ where we assume that $L^0(G) = G$ and $L^1(G) = L(G)$, and where $L(G)$ is the line graph of G.

Theorem (H. S. Ramane, H. B. Walikar, S. B. Rao, B. D. Acharya, P. R. Hampiholi, S. R. Jog, I. Gutman, Spectra and energies of iterated line graphs of regular graphs, Applied Mathematics Letters 18 (2005) 679-682.)

Let G_1 and G_2 be two non-cospectral regular graphs of the same order and of the same degree $r \geqslant 3$. Then for $k \geqslant 2$ the iterated line graphs $L^k(G_1)$ and $L^k(G_2)$ form a pair of non-cospectral equienergetic graphs of equal order and with the same number of edges. If, in addition, G_1 and G_2 are chosen to be connected, then also $L^k(G_1)$ and $L^k(G_2)$ are connected.

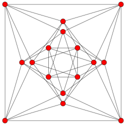


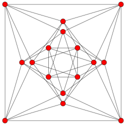
Corollary (H. S. Ramane, I. Gutman, H. B. Walikar, S. B. Halkarni, Equienergetic complement graphs, Kragujevac J. Sci. 27 (2005) 67-74.)

Let G_1 and G_2 be two non-cospectral regular graphs on n vertices, and of degree $r \geqslant 3$. Then, for any $k \geqslant 2$, both graphs $\overline{L^k(G_1)}$ and $\overline{L^k(G_2)}$ are regular, non-cospectral, possessing the same number of vertices and the same number of edges, and they are equienergetic, i.e. $\mathcal{E}(\overline{L^k(G_1)}) = \mathcal{E}(\overline{L^k(G_2)})$.

$\mathcal{E}(G) = \mathcal{E}(\overline{G})$

H. S. Ramane, B. Parvathalu, D. D. Patil, K. Ashoka, Graphs equienergetic with their complements, MATCH Commun. Math. Comput. Chem. 82 (2019) 471-480.





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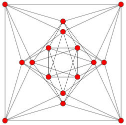
Theorem

The following holds:

- $\mathcal{E}(n\,K_n) = \mathcal{E}(\overline{n\,K_n})$, for $n \geqslant 2$;
- $\mathcal{E}(L(K_{p,q})) = \mathcal{E}(\overline{L(K_{p,q})})$, where $p, q \geqslant 2$;
- $\mathcal{E}(\overline{n\,K_n} + K_m) = \mathcal{E}(\overline{\overline{n\,K_n} + K_m})$, for $n \geqslant 2$ and $m \leqslant n$;
- if G is a strongly regular graph with parameters $(4n^2, 2n^2 - n, n^2 - n, n^2 - n)$, $n > 1$, then $\mathcal{E}(G) = \mathcal{E}(\overline{G})$;
- if G is a strongly regular graph with parameters $(n^2, 3n - 3, n, 6)$, $n > 2$, then $\mathcal{E}(G) = \mathcal{E}(\overline{G})$.

H. S. Ramane, B. Parvathalu, D. D. Patil, K. Ashoka, Graphs equienergetic with their complements, MATCH Commun. Math. Comput. Chem. 82 (2019) 471-480.

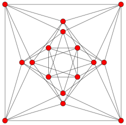
1. Discussion of the non-self-complementary, non-regular graphs satisfying $\mathcal{E}(G) = \mathcal{E}(\overline{G})$.
2. Finding structural and spectral properties of graphs satisfying $\mathcal{E}(G) = \mathcal{E}(\overline{G})$.



Let G_1 and G_2 be two non isomorphic graphs of order n, whose spectra with respect to the adjacency matrix A are:

$$\lambda_1(G_1) \geqslant \lambda_2(G_1) \geqslant \cdots \geqslant \lambda_n(G_1), \text{ and}$$
$$\lambda_1(G_2) \geqslant \lambda_2(G_2) \geqslant \cdots \geqslant \lambda_n(G_2).$$

Spectral distances of graphs



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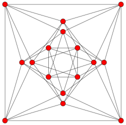
Definition (D. Stevanović, Research problems from the Aveiro workshop on graph spectra, Linear Algebra and its Applications, 423 (2007) 172-181.)

The spectral distance of G_1 and G_2 is the Manhattan distance between their spectra:

$$\sigma(G_1, G_2) = \sum_{i=1}^{n} |\lambda_i(G_1) - \lambda_i(G_2)|.$$

Spectral distance related parameters

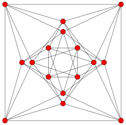


Definition (I. M. Jovanović, Z. Stanić, Spectral distances of graphs, Linear Algebra and its Application, 436 (2012), 1425-1435.)

Let $\mathcal{G}$ be an arbitrary set of graphs of order n.

- The cospectrality of $G \in \mathcal{G}$ is:
  $\mathsf{cs}_{\mathcal{G}}(G) = \min\{\sigma(G, H) : H \in \mathcal{G},\ H \neq G\}$.
- The cospectrality measure of $\mathcal{G}$ is:
  $\mathsf{cs}(\mathcal{G}) = \max\{\mathsf{cs}_{\mathcal{G}}(G) : G \in \mathcal{G}\}$.
- The spectral eccentricity of $G \in \mathcal{G}$ is:
  $\mathsf{secc}_{\mathcal{G}}(G) = \max\{\sigma(G, H) : H \in \mathcal{G},\ H \neq G\}$.
- The spectral diameter of $\mathcal{G}$ is:
  $\mathsf{sdiam}(\mathcal{G}) = \max\{\mathsf{secc}_{\mathcal{G}}(G) : G \in \mathcal{G}\}$.

A spectral distance related conjecture

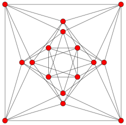


Conjecture No. 2 by Z. Stanić

If R_1 and R_2 are graphs having the maximum spectral distance among the connected regular graphs of order n, then one of them is K_n, i.e.

$$\mathsf{sdiam}(\mathcal{R}_n) = \mathsf{secc}_{\mathcal{R}_n}(K_n),$$

where $\mathcal{R}_n$ is the set of all connected regular graphs with n vertices.

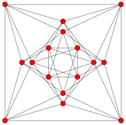


Theorem (I. M. Jovanović, Z. Stanić, Spectral distances of graphs, Linear Algebra and its Application, 436 (2012), 1425-1435.)

Let G be a n-vertex graph with n^* adjacency eigenvalues which are greater than or equal to -1. Then:

$$\sigma(G, K_n) = 2\left(n^* - 1 + \sum_{i=2}^{n^*} \lambda_i(G)\right). \qquad (1)$$

Regular graph and its complement



Let G be a regular graph of degree r with n vertices whose adjacency spectrum is: $r = \lambda_1(G) \geqslant \lambda_2(G) \geqslant \cdots \geqslant \lambda_n(G)$.

Theorem (D. Cvetković, P. Rowlinson, S. Simić, An Introduction to the Theory of Graph Spectra, Cambrige University Press, Cambridge, 2010.)

$$P_{\overline{G}}(x) = (-1)^n \frac{x-n+r+1}{x+r+1} P_G(-x-1)$$

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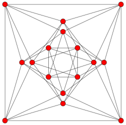
$$P_{\overline{G}}(x) = (-1)^n \frac{x-n+r+1}{x+r+1} P_G(-x-1)$$

relation (2)

$$\sigma(G, K_n) = |n-1-r| + \sum_{i=2}^{n} |-1-\lambda_i(G)| = E(\overline{G})$$



Regular graph and its complement

Let G be a regular graph of degree r with n vertices whose adjacency spectrum is: $r = \lambda_1(G) \geqslant \lambda_2(G) \geqslant \cdots \geqslant \lambda_n(G)$.

Theorem (D. Cvetković, P. Rowlinson, S. Simić, An Introduction to the Theory of Graph Spectra, Cambrige University Press, Cambridge, 2010.)

$$P_{\overline{G}}(x) = (-1)^n \frac{x-n+r+1}{x+r+1} P_G(-x-1)$$

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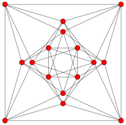
relation (3)

$$\sigma(\overline{G}, K_n) = |n-1-(n-1-r)| + \sum_{i=2}^{n} |-1-(-1-\lambda_i(G))| = E(G)$$

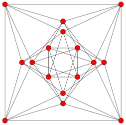


Theorem (I. M. Jovanović, E. Zogić, Some spectral characterizations of equienergetic regular graphs and their complements, MATCH Commun. Math. Comput. Chem., Vol. 86, No. 3, (2021), 559-575.)

Let G be a r-regular graph of order n, with the following adjacency eigenvalues: $\lambda_1(G) \geqslant \lambda_2(G) \geqslant \cdots \geqslant \lambda_n(G)$. Let $n^* = n^*(G)$ be the number of eigenvalues of G which are greater than or equal to -1, among which there are $n_1 = n_1(G)$ non-negative eigenvalues. Then:

$$\mathcal{E}(G) = \mathcal{E}(\overline{G}) \text{ if and only if } n^* - 1 + \sum_{i=n_1+1}^{n^*} \lambda_i(G) = r.$$

Proof: By use of relations (1), (2) and (3).



Definition (D. Cvetković, P. Rowlinson, S. Simić, An Introduction to the Theory of Graph Spectra, Cambrige University Press, Cambridge, 2010.)

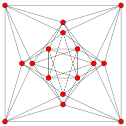
A strongly regular graph G with parameters (n, r, e, f) is a r-regular graph on n vertices in which any two adjacent vertices have exactly e common neighbours, and any two non-adjacent vertices have exactly f common neighbours.

Strongly regular graphs

Definition (D. Cvetković, P. Rowlinson, S. Simić, An Introduction to the Theory of Graph Spectra, Cambrige University Press, Cambridge, 2010.)

A *strongly regular graph* G with parameters (n, r, e, f) is a r-regular graph on n vertices in which any two adjacent vertices have exactly e common neighbours, and any two non-adjacent vertices have exactly f common neighbours.

If $G = SRG(n, r, e, f)$ is a connected strongly regular graph, different from the complete graph K_n, then the adjacency spectrum of G consists of: r, $[s]^k$ and $[t]^l$, where
$s, t = \frac{1}{2}\left((e-f) \pm \sqrt{\Delta}\right)$;
$k, l = \frac{1}{2}\left(n - 1 \mp \frac{2r+(n-1)(e-f)}{\sqrt{\Delta}}\right)$, and
$\Delta = (e-f)^2 + 4(r-f)$.



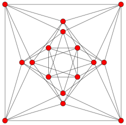
Strongly regular graphs

I. M. Jovanović, E. Zogić, Some spectral characterizations of equienergetic regular graphs and their complements, MATCH Commun. Math. Comput. Chem., Vol. 86, No. 3, (2021), 559-575.

Corollary

Let $G = SRG(n, r, e, f)$ be a connected strongly regular graph different from the complete graph K_n, and with the spectrum: r, $[s]^k$ and $[t]^l$. Then,

$$\mathcal{E}(G) = \mathcal{E}(\overline{G}) \text{ if and only if } k = r.$$



Strongly regular graphs

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Corollary

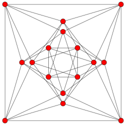
Let $G = SRG(n, r, e, f)$ be a connected strongly regular graph different from the complete graph K_n, and with the spectrum: r, $[s]^k$ and $[t]^l$. Then,

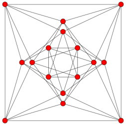
$$\mathcal{E}(G) = \mathcal{E}(\overline{G}) \text{ if and only if } k = r.$$

Corollary

Let G be a connected strongly regular graph different from the complete graph K_n, with parameters (n, r, e, f). Then,

$$\mathcal{E}(G) = \mathcal{E}(\overline{G}) \text{ if and only if } n = 1 + \frac{2r\,(\sqrt{\Delta} + 1)}{\sqrt{\Delta} - e + f}.$$





H. S. Ramane, B. Parvathalu, D. D. Patil, K. Ashoka, Graphs equienergetic with their complements, MATCH Commun. Math. Comput. Chem. 82 (2019) 471-480.

The following holds:

- $\mathcal{E}(C_4 + K_2) = \mathcal{E}(\overline{C_4 + K_2}) = 12$;
- $\mathcal{E}(\overline{G} + K_2) = \mathcal{E}(\overline{\overline{G} + K_2}) = 336$, where $G = SRG(50, 7, 0, 1)$ is Moore graph;
- $\mathcal{E}(\overline{H} + K_2) = \mathcal{E}(\overline{\overline{H} + K_2}) = 80$, where $H = SRG(16, 5, 0, 2)$ is Clebsch graph.

Corollary (I. M. Jovanović, E. Zogić)

Let G be a r-regular graph of order n whose the adjacency eigenvalues are: $\lambda_1(G) \geqslant \cdots \geqslant \lambda_n(G)$. Let us denote $I_{-1} = \{i \in \{1, 2, \ldots, n\} : \lambda_i(G) \in [-2, -1)\}$ and $I_{+1} = \{i \in \{1, 2, \ldots, n\} : \lambda_i(G) \in [0, 1)\}$, and let us suppose that in the spectrum of G there are n_{-2} eigenvalues which are greater than or equal to -2, and that among them there are n_0 non-negative eigenvalues. Then $\mathcal{E}(G + K_2) = \mathcal{E}(\overline{G + K_2})$ if and only if

$$n_{-2} + n_0 + \sum_{i \in I_{-1} \cup I_{+1}} \lambda_i(G) + |I_{-1}| - |I_{+1}| = r + 2,$$

where $|I|$ stands for the cardinality of the set I.

Corollary (I. M. Jovanović, E. Zogić)

If $G = SRG(d^2(d+2), d(d^2+d-1), d(d^2-1), d(d^2-1))$, where $d > 2$, then, $\mathcal{E}(G + K_2) = \mathcal{E}(\overline{G + K_2})$.

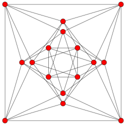
Equienergeticity with respect to some graph operations

Corollary (I. M. Jovanović, E. Zogić)

Let G be a r-regular graph of order n, where $n \geqslant 4$, and with the adjacency eigenvalues: $\lambda_1(G) \geqslant \cdots \geqslant \lambda_n(G)$. Let us suppose that $\lambda_i(G) \geqslant -r+2$, for each $i = 1, 2, \ldots, n$. Then:

$$\mathcal{E}(L(G)) = \mathcal{E}(\overline{L(G)}) \text{ if and only if } r = \frac{n+1}{2}.$$

Equienergetic complements of regular graphs



Corollary (I. M. Jovanović, E. Zogić)

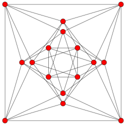
Let G_i, $i = 1, 2$, be two r_i regular graphs of order n whose the adjacency spectra are $\lambda_1(G_i) \geqslant \cdots \geqslant \lambda_n(G_i)$. Let us suppose that:

$$-r_1 + r_2 + n_1^* - n_2^* + \sum_{j=n_1^{**}+1}^{n_1^*} \lambda_j(G_1) - \sum_{j=n_2^{**}+1}^{n_2^*} \lambda_j(G_2) = 0,$$

where n_i^{**} are the numbers of non-negative eigenvalues of G_i, and n_i^* are the numbers of eigenvalues which are greater than or equal to -1. If $\mathcal{E}(G_1) = \mathcal{E}(G_2)$, then $\mathcal{E}(\overline{G_1}) = \mathcal{E}(\overline{G_2})$.

Equienergetic complements of regular graphs

Let $G^{\circ} = 2\left(K_2 + K_{\frac{n}{4}}\right)$, where $n \geqslant 12$ and n is divisible by 4. The spectrum of G° is: $\left[\frac{n}{4}\right]^2$, $\left[\frac{n}{4} - 2\right]^2$, $[0]^{\frac{n}{2}-2}$, $[-2]^{\frac{n}{2}-2}$, and its energy: $\mathcal{E}(G^{\circ}) = 2n - 8$. G° is a regular graph with $r_{G^{\circ}} = \frac{n}{4}$ and $n^{**}_{G^{\circ}} = \frac{n}{2} + 2$, and therefore $-r_{G^{\circ}} + n^{**}_{G^{\circ}} = \frac{n}{4} + 2$.



Let $G^\circ = 2\left(K_2 + K_{\frac{n}{4}}\right)$, where $n \geqslant 12$ and n is divisible by 4. The spectrum of G° is: $\left[\frac{n}{4}\right]^2$, $\left[\frac{n}{4}-2\right]^2$, $[0]^{\frac{n}{2}-2}$, $[-2]^{\frac{n}{2}-2}$, and its energy: $\mathcal{E}(G^\circ) = 2n-8$. G° is a regular graph with $r_{G^\circ} = \frac{n}{4}$ and $n^{**}_{G^\circ} = \frac{n}{2}+2$, and therefore $-r_{G^\circ} + n^{**}_{G^\circ} = \frac{n}{4}+2$.

Let us denote by $V\left(K_{\frac{n}{2},\frac{n}{2}}\right) = V_1 \cup V_2$, where $V_1 = \{x_i \,|\, 1 \leqslant i \leqslant \frac{n}{2}\}$ and $V_2 = \{y_i \,|\, 1 \leqslant i \leqslant \frac{n}{2}\}$, the vertex set of $K_{\frac{n}{2},\frac{n}{2}}$ of order n, such that $n \geqslant 12$ and $n = 4\,q$, for $q \in \mathbb{Z}^+$. Let G be the graph obtained by deleting $\frac{n}{2}$ perfect matching edges $\{x_i y_i \,|\, 1 \leqslant i \leqslant \frac{n}{2}\}$ from $K_{\frac{n}{2},\frac{n}{2}}$, and let H be the graph obtained by deleting $\frac{n}{2}$ perfect matching edges $\{x_i y_{i+1} : i - \text{odd}\} \cup \{x_i y_{i-1} : i - even\}$ from G. The spectrum of H is: $\frac{n}{2}-2$, $[2]^{\frac{n}{4}-1}$, $[0]^{\frac{n}{2}}$, $[-2]^{\frac{n}{4}-1}$, $-\frac{n}{2}+2$, and its energy $\mathcal{E}(H) = 2n-8$. H is a regular graph with $r_H = \frac{n}{2}-2$ and $n^{**}_H = \frac{3n}{4}$, and therefore $-r_H + n^{**}_H = \frac{n}{4}+2$.

Equienergetic complements of regular graphs

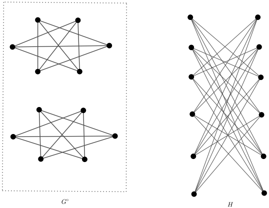


Figure: Equienergetic graphs G° and H of order $n = 12$

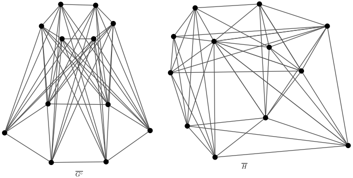
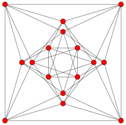


Figure: Equienergetic complements $\overline{G^\circ}$ and $\overline{H}$ for $n = 12$



An upper bound on the graph energy

Theorem (J. Koolen, V. Moulton, Maximal energy graphs, Advances in Applied Mathematics, 26 (2001), 47-52.)

If G is a graph on n vertices, then

$$\mathcal{E}(G) \leqslant \frac{n}{2}(1+\sqrt{n}),$$

where the equality is attained if and only if G is a strongly regular graph with parameters $(n, (n+\sqrt{n})/2, (n+\sqrt{n})/4, (n+\sqrt{n})/4)$.

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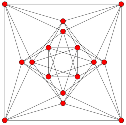
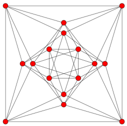
$\Rightarrow$ Conjecture No. 2:

$$\mathsf{sdiam}(\mathcal{R}_n) = \mathsf{secc}_{\mathcal{R}_n}(K_n) = \max_{G\in\mathcal{R}_n, G\neq K_n} \mathcal{E}(\overline{G}) \leqslant \frac{n}{2}(1+\sqrt{n}).$$



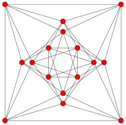


I. M. Jovanović, Spectral distances in some sets of graphs, Revista de la Union Matematica Argentina, Vol. 63, No. 1, (2022), 1-20.

The generalized quadrangle $G = GQ(3,9)$ is the strongly regular graph with parameters (112, 30, 2, 10), whose spectrum is: $30,\ [2]^{90},\ [-10]^{21}$. Since it holds:

$$\sigma(G, \overline{G}) = 690 > 648,648 \approx \frac{112}{2}\left(1+\sqrt{112}\right) \geqslant \mathrm{secc}_{\mathcal{R}_{112}}(K_{112}),$$

and since both graphs, $G = GQ(3,9)$ and its complement $\overline{G}$, are connected, Conjecture No. 2 is disproved.



Thank you for your attention!