TECHNISCHE UNIVERSITÄT BERLIN

The connectivity dimension of a graph

Kurt Klement Gottwald, Tobias Hofmann

Introduction

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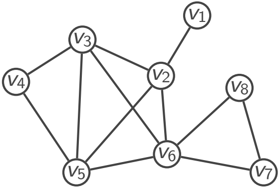
Terminology.

- we consider nonempty, finite, simple, undirected graphs
- $\kappa(x, y)$ denotes the number of independent paths between two vertices x and y
- $\kappa(x, x) := \infty$ for all vertices x.

$W = \{v_1, v_4, v_8\}$



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Given an ordered subset $W = \{w_1, \ldots, w_k\}$ of vertices of a graph G, the

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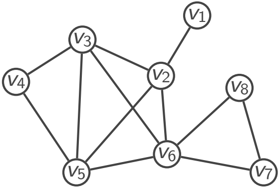

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$$W = \{v_1, v_4, v_8\} \text{ is not resolving}$$
$$r(v_2, W) = [1, 2, 1]$$
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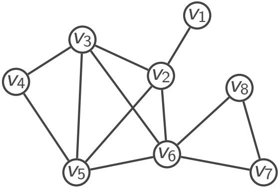
$$X = \{v_2, v_5, v_7\} \text{ is resolving}$$



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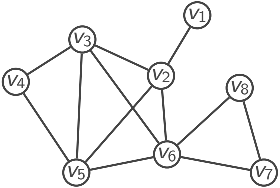

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- A *connectivity basis* is a resolving set of minimum cardinality and the *connectivity dimension* $\mathrm{cdim}(G)$ of G is the cardinality of a basis of G.

$X = \{v_2, v_5, v_7\}$ is not a basis, but $B = \{v_3, v_8\}$ is

$r(v_1, B) = [1, 1]$, $r(v_2, B) = [3, 1]$

$r(v_3, B) = [\infty, 1]$, $r(v_4, B) = [2, 1]$

$r(v_5, B) = [4, 1]$, $r(v_6, B) = [3, 2]$

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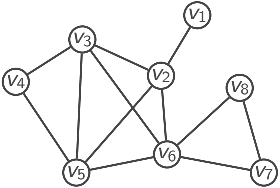

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A Measure of Graph Heterogeneity

Observation.

The set $V(G) \setminus \{w\}$, $w \in V(G)$, is resolving for any connected graph G. Thus

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Proof. Clearly, $\text{cdim}(K_2) = 1$. To show: This is the only such graph.

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$$\begin{aligned} r(v_1, W) &= [\kappa(v_1, w)] = [\infty], \\ r(v_2, W) &= [\kappa(v_2, w)] = [1], \\ &\vdots \\ r(v_n, W) &= [\kappa(v_n, w)] = [n-1], \quad \text{for an appropriate labeling of vertices.} \end{aligned}$$

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- But $\kappa(v_n, w) = n - 1$ says that v_n and w are adjacent to all other vertices.
- We obtain two independent paths $v_2 w$ and $v_2 v_n w$, contradicting $\kappa(v_2, w) = 1$. □

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Open Problems.

- How to characterize graphs with connectivity dimension $n - 2$?
- How to characterize graphs with connectivity dimension 2?
- How to characterize graphs with connectivity dimension k?

Graphs with Given Connectivity Dimension

Example. Consider the threshold graph G generated by the sequence

$$\epsilon, 1, 0, 1, 0, 1.$$

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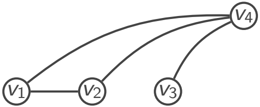

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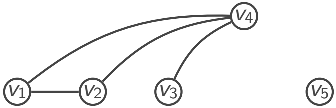

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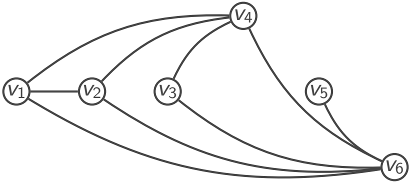

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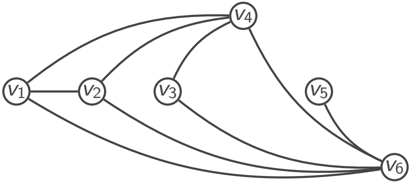

- threshold graphs are maximally local connected[1], i. e. they satisfy

$$\kappa(x, y) = \min\{\deg(x), \deg(y)\} \quad \text{for all vertices } x, y \in V(G).$$

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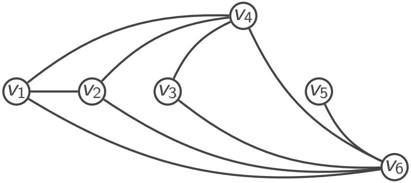

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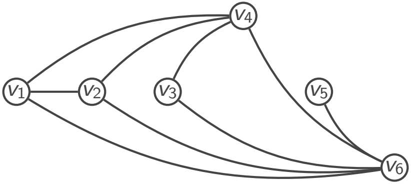

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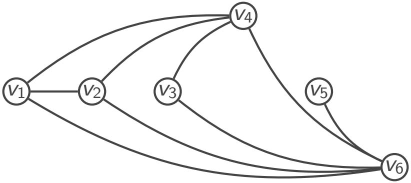

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- there is a graph with connectivity dimension k for any $n \geq k+1$

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Proposition.

Let G be the threshold graph of order n generated by the sequence

$$x_1^{k_1} x_2^{k_2} \dots x_m^{k_m}, \quad x_i \in \{0,1\}, x_m = 1, k_1 \geq 2.$$

Then

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Determining a Graph's Connectivity Dimension

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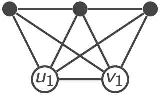

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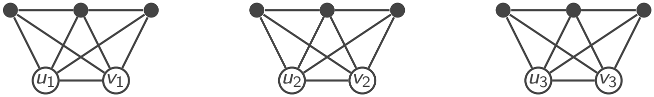

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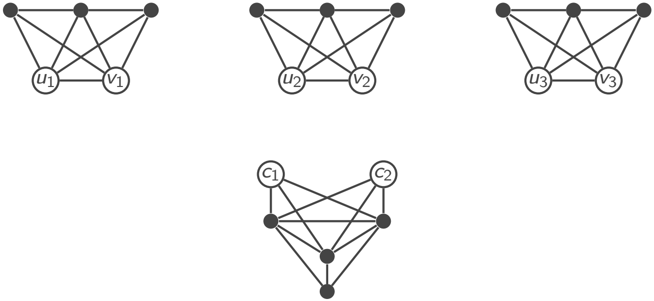

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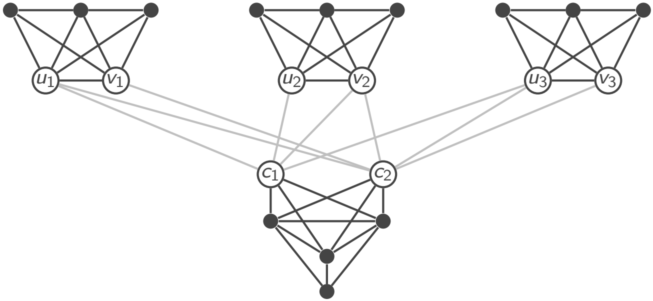

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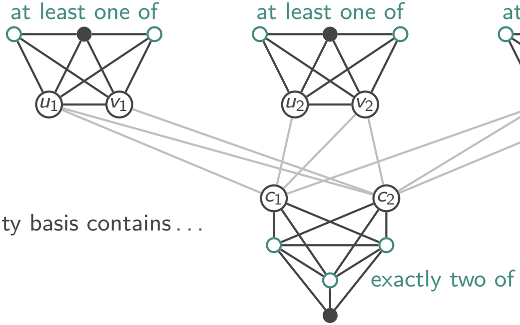

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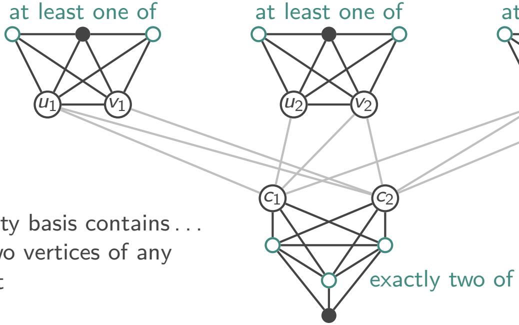

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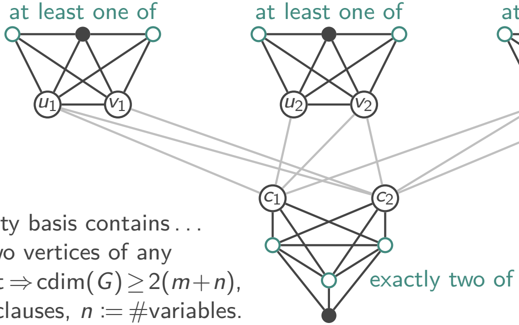

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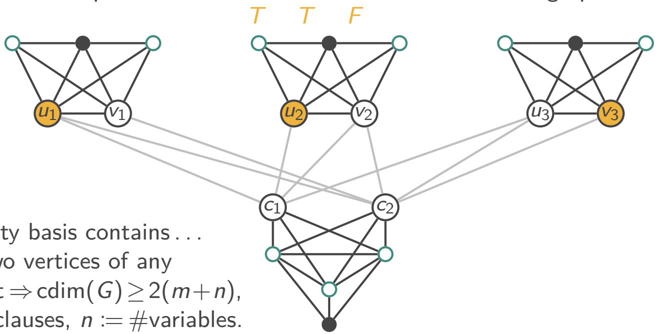

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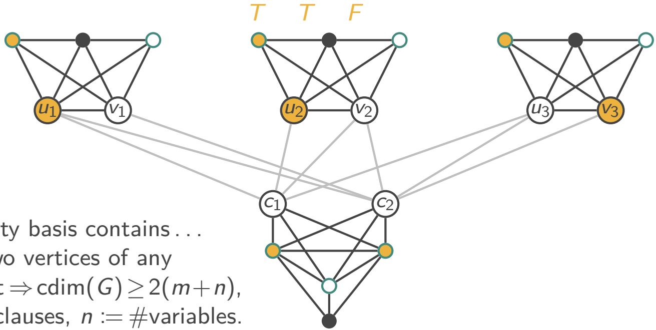

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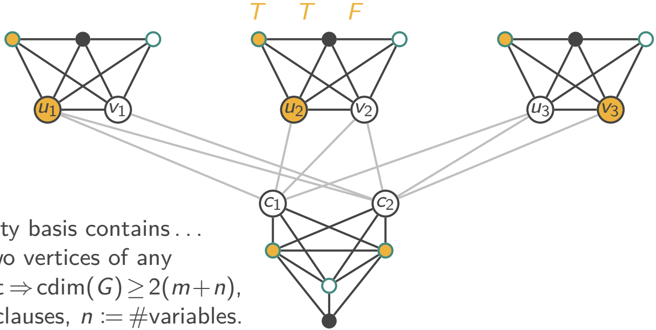

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- How is the connectivity dimension related to other graph parameters?

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