D_n-Specht Ideals

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Motivation

- **Symmetric group** S_n acts on $\mathbb{R}[\underline{X}] = \mathbb{R}[X_1, \dots, X_n]$ via permutation

$$(1,3) \cdot X_1X_2(X_3^2 - 1) = X_3X_2(X_1^2 - 1)$$

- **Invariant system** of polynomial equations: $\mathcal{F} = \{f_1, \dots, f_m\} \subset \mathbb{R}[\underline{X}]$ & $\sigma \cdot f_i \in \mathcal{F} \,\forall\, \sigma \in S_n$

$$0 = X_1X_2(X_3^2 - 1) = X_2X_3(X_1^2 - 1) = X_1X_3(X_2^2 - 1)$$

- $x \in \mathbb{R}^n$ is a solution $\implies$ $(x_{\sigma(1)}, \dots, x_{\sigma(n)})$ is solution, for all $\sigma \in S_n$.

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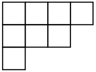
- $x \in \mathbb{R}^n$ is a solution $\implies$ $(x_{\sigma(1)}, \ldots, x_{\sigma(n)})$ is solution, for all $\sigma \in S_n$.

- What are possible **orbit types** of real solutions?

$$x = (\underbrace{x_1, \ldots, x_1}_{\lambda_1}, \underbrace{x_2, \ldots, x_2}_{\lambda_2}, \ldots, \underbrace{x_\ell, \ldots, x_\ell}_{\lambda_\ell}) \in \mathbb{R}^n$$

Partitions, tableaux and Specht ideals

- **Partition** $\lambda = (\lambda_1 \geqslant \lambda_2 \geqslant \ldots \geqslant \lambda_\ell \geqslant 0) \vdash n$ can be represented by a **diagram** of shape λ: For $(4, 3, 1) \vdash 8$,



- **Young tableau** of shape $\lambda \vdash n$ is a diagram T of shape λ filled-in with all integers 1 to n.

4	2	6	1
8	7	5	
3			

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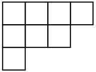

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- **Specht polynomial** sp_T^S associated with T is the product of the column Vandermonde determinants

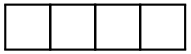
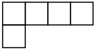
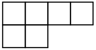
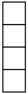
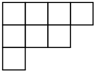
$$\mathrm{sp}_T^S = (X_4 - X_8)(X_4 - X_3)(X_8 - X_3)(X_2 - X_7)(X_6 - X_5) \cdot 1$$

- **Specht ideal** and **Specht variety**associated with λ:

$$I_\lambda^S = \langle \mathrm{sp}_T^S \,|\, \mathrm{shape}(T) = \lambda \rangle \subset \mathbb{R}[\underline{X}]$$

$$V_\lambda^S = \{x \in \mathbb{R}^n : \mathrm{sp}_T^S(x) = 0 \;\forall\, T \text{ of shape } \lambda\} \subset \mathbb{R}^n$$

Specht varieties

λ	sp_T^S	V_λ^S
	1	$\varnothing$
	$X_i - X_j$	$\{(a,\ldots,a) : a \in \mathbb{R}\}$
	$(X_i - X_j)(X_k - X_l)$	$S_n \cdot \{(a,\ldots,a,b) : a,b \in \mathbb{R}\}$
	$\prod_{1\leqslant i<j\leqslant 4}(X_i - X_j)$	$S_n \cdot \{(a,a,b,c)\}$
	$\ldots$	?

Orbit types

- Up to permutation, every $x \in \mathbb{R}^n$ is of form

$$x = (\underbrace{x_1, \ldots, x_1}_{\lambda_1}, \underbrace{x_2, \ldots, x_2}_{\lambda_2}, \ldots, \underbrace{x_k, \ldots, x_k}_{\lambda_k})$$

  with $x_i \neq x_j$ and $\lambda_1 \geqslant \lambda_2 \geqslant \ldots \geqslant \lambda_k$

- $\Lambda(x) = (\lambda_1, \lambda_2, \ldots, \lambda_k) \vdash n$ is the **orbit type** of x and
  $O(\lambda) := \{x \in \mathbb{R}^n \mid \Lambda(x) = \lambda\} \neq \varnothing$

- $\Lambda(\sqrt{2}, 0, \pi, \sqrt{2}, -\pi, 0) = (2, 2, 1, 1)$

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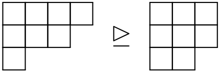
- λ **dominates** $\mu \iff \sum_{j=1}^{k} \lambda_j \geqslant \sum_{j=1}^{k} \mu_j, \; \forall\, k \geqslant 1$:



Poset and Variety decomposition

- Theorem: [Haiman, Woo'05], [Moustrou, Riener, Verdure'21]
  Let $\lambda, \mu \vdash n$ be partitions. Then,

$$\mu \trianglelefteq \lambda \Leftrightarrow I_\mu^S \subseteq I_\lambda^S \Leftrightarrow V_\mu^S \supseteq V_\lambda^S .$$

- Theorem: [Haiman, Woo'05], [Moustrou, Riener, Verdure'21]

$$V_\lambda^S = \biguplus_{\mu \ntrianglelefteq \lambda} O(\mu),$$

The hyperoctahedral group B_n

- $B_n = \{\pm 1\} \wr S_n$ the group of symmetries of the hypercube
- As *permutation group* $B_n \subset \text{Sym}(\{-n, \ldots, -1, 1, \ldots, n\})$ contains all permutations σ with

$$\sigma(i) + \sigma(-i) = 0, \ \forall\, 1 \leqslant i \leqslant n$$

- Generated by permutation matrices whose nonzero entries are ± 1
- Acts on $\mathbb{R}[\underline{X}]$ by permutation of variables and switching of signs

$$((-1, +1), (1, 2)) \cdot (X_1 + X_2^2) = -X_2 + X_1^2$$

Bipartitions, bitableaux and Specht ideals

- **Bipartition** can be represented by a bidiagram, i.e., a pair of diagrams: For $((4,3),(1,1))$

$$\left(\begin{array}{|c|c|c|c|}\hline & & & \\ \hline & & \\ \cline{1-3}\end{array}\ ,\ \begin{array}{|c|}\hline \\ \hline \\ \hline\end{array}\right)$$

- **Bitableau** is a filling of the bidiagram with all the integers 1 to n.

$$\left(\begin{array}{|c|c|c|c|}\hline 4 & 10 & 6 & 1 \\ \hline 8 & 7 & 5 \\ \cline{1-3}\end{array}\ ,\ \begin{array}{|c|c|}\hline 2 & 3 \\ \hline 9 \\ \cline{1-1}\end{array}\right)$$

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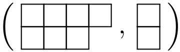

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- **$(B_n$-)Specht polynomial** $\mathrm{sp}^B_{(T_1,T_2)}$ associated with (T_1,T_2)

$$\mathrm{sp}^B_{(T_1,T_2)} = \mathrm{sp}^S_{T_1}(X_1^2,\dots,X_n^2)\cdot \mathrm{sp}^S_{T_2}(X_1^2,\dots,X_n^2)\prod_{j\in T_2} X_j$$

For instance, $(X_4^2 - X_8^2)(X_{10}^2 - X_7^2)(X_6^2 - X_5^2)(X_2^2 - X_9^2)X_2X_3X_9$.

- **(B_n)-Specht ideals** and **Specht varieties** $I^B_{(\lambda,\mu)}\mathrm{sp}, V^B_{(\lambda,\mu)}$

B_n Specht varieties

(λ, μ)	$\mathrm{sp}^B_{(T_1, T_2)}$	$\mathsf{V}^B_{(\lambda,\mu)}$
$(\ydiagram{4}, \varnothing)$	1	$\varnothing$
$(\varnothing, \ydiagram{4})$	$\Delta := X_1X_2X_3X_4$	$B_n \cdot \{(a,b,c,0)\}$
$\left(\varnothing, \ydiagram{1,1,1,1}\right)$	$\prod_{1\leqslant i<j\leqslant 4}(X_i^2 - X_j^2) \cdot \Delta$	$B_n \cdot \{(a,a,b,c)\} \cup \{(a,b,c,0)\}$
$\left(\ydiagram{1,1}, \ydiagram{2}\right)$	$(X_1^2 - X_2^2)(X_3X_4)$	$B_n \cdot \{(a,b,0,0)\} \cup \{(a,a,a,b)\}$
$\left(\ydiagram{4,3}, \ydiagram{2,1}\right)$	$\ldots$	?

B_n-Orbit types

- Up to sign switching and permutation, every $x \in \mathbb{R}^n$ is of the form

$$x = (\underbrace{x_1, \dots, x_1}_{\lambda_1 + \mu_1}, \dots, \underbrace{x_m, \dots, x_1 m}_{\lambda_1 + \mu_m}, \underbrace{0, \dots, 0}_{\lambda_1}, \underbrace{x_{m+1}, \dots, x_{m+1}}_{\lambda_{m+2}}, \dots, \underbrace{x_l, \dots, x_l}_{\lambda_{l+1}})$$

with $x_i > 0$ pairwise distinct.

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  with $x_i > 0$ pairwise distinct.

- $(\lambda, \mu), (\lambda', \mu')$ bipartitions of n

$$(\lambda', \mu') \trianglelefteq (\lambda, \mu) :\Leftrightarrow \begin{cases} \sum_{j \leqslant k}(\lambda'_j + \mu'_j) \leqslant \sum_{j \leqslant k}(\lambda_j + \mu_j), & \forall\, k, \\ \lambda'_k + \sum_{j \leqslant k-1}(\lambda'_j + \mu'_j) \leqslant \lambda_k + \sum_{j \leqslant k-1}(\lambda_j + \mu_j), & \forall\, k. \end{cases}$$

- $((2,1,1),(3,1)) \trianglelefteq ((3,2),(2,1))$ are comparable:

$$3 \geqslant 2, \quad 5 \geqslant 5, \quad 7 \geqslant 6, \quad 8 \geqslant 7, \quad 8 \geqslant 8.$$

- $((2),(1,1))$ and $(\varnothing,(4))$ are not comparable, since $2 > 0$ but $3 < 4$.

Poset and Variety decomposition

- Theorem: [Debus, Moustrou, Riener, Verdure'23]
  Let $(\lambda, \mu), (\lambda', \mu')$ be bipartitions of n. Then,

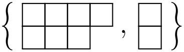
$$(\lambda', \mu') \trianglelefteq (\lambda, \mu) \Leftrightarrow I^B_{(\lambda', \mu')} \subseteq I^B_{(\lambda, \mu)} \Leftrightarrow V^B_{(\lambda', \mu')} \supseteq V^B_{(\lambda, \mu)}.$$

- Theorem: [Debus, Moustrou, Riener, Verdure'23]

$$V^B_{(\lambda, \mu)} = \biguplus_{(\lambda', \mu') \ntrianglelefteq (\lambda, \mu)} O(\lambda', \mu')$$

The hyperoctahedral group D_n

- D_n the subgroup of B_n of index 2
- consists of elements with an even amount of sign changes
- reflection group, irreducible representations correpond to **dipartitions**, i.e. unordered bipartitions
- Dipartitions can be represented by an unordered pair of diagrams: For $\{(4,3),(1,1)\}$



- Two dipartitions for bipartitions of the form (λ, λ): For $\{(3,1),(3,1)\}$

Dipartitions, ditableaux and Specht ideals

- **Ditableau** is a filling of the diagrams with all integers 1 to n

$$\left\{ \begin{array}{|c|c|c|} \hline 4 & 5 & 6 \\ \hline 8 \\ \cline{1-1} \end{array} \; , \; \begin{array}{|c|c|c|} \hline 2 & 3 & 1 \\ \hline 7 \\ \cline{1-1} \end{array} \right\}$$

- For $\{\lambda, \mu\}$ with $\lambda \neq \mu$ the **(D_n)-Specht ideal**

$$I^D_{\{\lambda,\mu\}} = \langle \mathrm{sp}^B_{(T_1,T_2)} \,|\, \mathrm{shape}((T_1, T_2)) \in \{(\lambda, \mu), (\mu, \lambda)\} \rangle$$

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- **$(D_n$-)Specht polynomials** $\mathrm{sp}^D_{(T_1,T_2)}$ for (T_1, T_2) of shape (λ, λ)

$$\mathrm{sp}^D_{\{T_1,T_2,+\}} = \mathrm{sp}^S_{T_1}(X_1^2,\ldots,X_n^2)\cdot \mathrm{sp}^S_{T_2}(X_1^2,\ldots,X_n^2)\left(\prod_{j\in T_2} X_j + \prod_{j\in T_1} X_j\right)$$

$$\mathrm{sp}^D_{\{T_1,T_2,-\}} = \mathrm{sp}^S_{T_1}(X_1^2,\ldots,X_n^2)\cdot \mathrm{sp}^S_{T_2}(X_1^2,\ldots,X_n^2)\left(\prod_{j\in T_2} X_j - \prod_{j\in T_1} X_j\right)$$

- For instance, $(X_4^2 - X_8^2)(X_2^2 - X_7^2)(X_2X_3X_1X_7 \pm X_4X_5X_6X_8)$.

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D_n Specht varieties

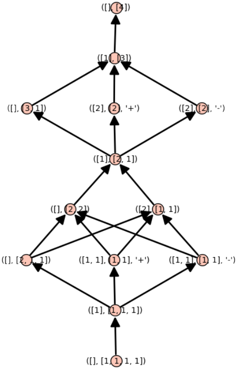
$\{\lambda, \mu\}$	$\mathrm{sp}^D_{\{T_1,T_2\}}$	$\mathsf{V}^D_{\{\lambda,\mu\}}$
$\{\ydiagram{4}\,,\ \varnothing\}$	1, $\Delta = X_1X_2X_3X_4$	$\varnothing$
$\left\{\varnothing\,,\ \ydiagram{1,1,1,1}\right\}$	$\prod_{1\leqslant i<j\leqslant 4}(X_i^2 - X_j^2)\cdot\Delta$, $\prod_{1\leqslant i<j\leqslant 4}(X_i^2 - X_j^2)$	$D_n \cdot \{(a,a,b,c)\} \cup \{(-a,a,b,c)\}$
$\left\{\ydiagram{1,1}\,,\ \ydiagram{1,1}\,,+\right\}$	$(x_1^2 - x_2^2)(x_3^2 - x_4^2)(x_1x_2 + x_3x_4)$	$D_n \cdot \{(a,-a,b,b)\} \subset V^D_{\{(1,1),(1,1),+\}}$
$\left\{\ydiagram{1,1}\,,\ \ydiagram{1,1}\,,-\right\}$	$(x_1^2 - x_2^2)(x_3^2 - x_4^2)(x_1x_2 - x_3x_4)$	$D_n \cdot \{(a,a,b,b)\} \subset V^D_{\{(1,1),(1,1),-\}}$

D_n-Poset

- $\{\lambda, \mu\}$, $\{\lambda', \mu'\}$ dipartitions of n

$$\{\lambda', \mu'\} \trianglelefteq_D \{\lambda, \mu\} :\Leftrightarrow \begin{cases} (\lambda', \mu') \trianglelefteq_B (\lambda, \mu) \text{ or } (\lambda', \mu') \trianglelefteq_B (\mu, \lambda), \\ (\mu', \lambda') \trianglelefteq_B (\lambda, \mu) \text{ or } (\mu', \lambda') \trianglelefteq_B (\mu, \lambda) \end{cases}$$

- $\{\lambda, \lambda, +\}$ and $\{\lambda, \lambda, -\}$ are incomparable for all λ



Poset and Variety decomposition

- Theorem:
  Let $\{\lambda, \mu\}, \{\lambda', \mu'\}$ be dipartitions of n. Then,

$$\{\lambda', \mu'\} \trianglelefteq_D \{\lambda, \mu\} \Leftrightarrow I^D_{\{\lambda',\mu'\}} \subseteq I^D_{\{\lambda,\mu\}} \Leftrightarrow V^D_{\{\lambda',\mu'\}} \supseteq V^D_{\{\lambda,\mu\}}.$$

- Theorem: For $\lambda \neq \mu$ we have

$$V^D_{\{\lambda,\mu\}} = \biguplus_{\substack{(\lambda',\mu') \trianglelefteq_B (\lambda,\mu), \\ (\lambda',\mu') \trianglelefteq_B (\mu,\lambda)}} O(\lambda', \mu').$$

Furthermore, we have

$$V^D_{\{\lambda,\lambda,+\}} = V^B_{(\lambda,\lambda)} \cup \{x \in (\mathbb{R}^*)^n \mid \mathrm{Stab}_{S_n}(x^2) \cong S_{\lambda \uplus \lambda}, 2 \nmid \#\{i : x_i < 0\}\}$$
$$V^D_{\{\lambda,\lambda,-\}} = V^B_{(\lambda,\lambda)} \cup \{x \in (\mathbb{R}^*)^n \mid \mathrm{Stab}_{S_n}(x^2) \cong S_{\lambda \uplus \lambda}, 2 \mid \{i : x_i < 0\}\}$$

Outlook

- Are the Specht ideals $I^{\mathrm{sp}}_{(\lambda,\mu)}$ radical and do the Specht polynomials $\mathcal{G}_{(\lambda,\mu)} := \{\mathrm{sp}_T : shape(T) \in \bigcup_{(\lambda',\mu')\trianglelefteq(\lambda,\mu)}(\lambda',\mu')\}$ form a universal Gröbner basis?
  - True for S_n [Haiman, Woo'05] and [Murai, Ohsugi, Kohji'21]
  - Proofs do not seem to transfer to type B and D.
- What about other reflection groups?
  - $S_n \cong G(1,1,n)$, $B_n \cong G(2,1,n)$ and $D_n \cong G(2,2,n)$. What about $G(r,p,n)$?
  - What about dihedral groups?

Thank you for your attention!